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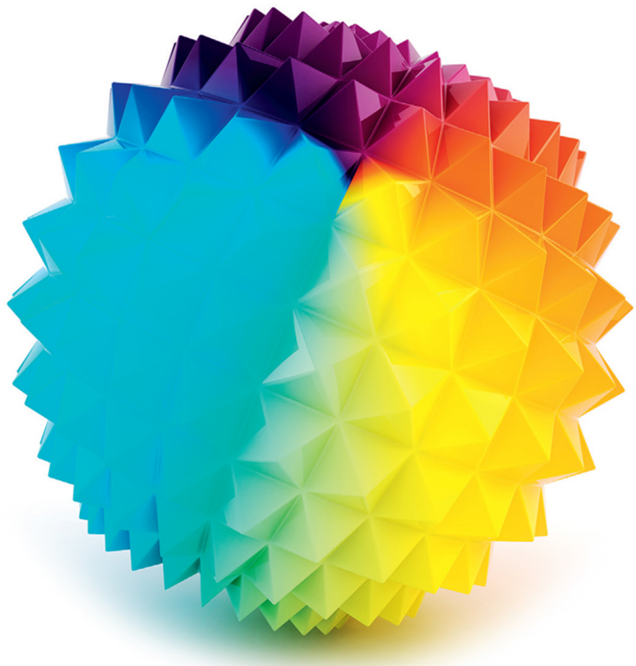

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Robert Barclay

Contents

Part 1 Calculus

Chapter 1
Differentiation 1: Rules and standard derivatives

Key points

Differentiate exponential and natural logarithmic functions
- differentiate functions involving e^x and $\ln x$ ☐

Differentiate functions using the chain rule
- apply the chain rule to differentiate the composition of at most three functions ☐

Differentiate functions given in the form of a product or a quotient
- differentiate functions of the form $f(x)g(x)$ and $\dfrac{f(x)}{g(x)}$ ☐
- know the definitions and apply the derivatives of $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ ☐
- derive and use the derivatives of $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ ☐
- differentiate functions that require more than one application or a combination of applications of the chain rule, product rule and quotient rule ☐
- apply $\dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{dx}{dy}\right)}$ where appropriate ☐

Differentiate inverse trigonometric functions
- differentiate expressions of the form $\sin^{-1}[f(x)]$, $\cos^{-1}[f(x)]$ and $\tan^{-1}[f(x)]$ ☐

Apply differentiation to problems in context
- apply differentiation to problems in context ☐
- apply differentiation to optimisation ☐

Key links

Differentiate implicit functions and parametric equations (Chapter 2), investigate stationary points and points of inflection (Chapter 6) and find Maclaurin expansions (Chapter 8) using the content of this chapter.

The Chain rule

The chain rule for differentiation states that $(f(g(x))' = f'(g(x)).g'(x)$.

Alternatively, using Leibniz notation, treat $y = f(g(x))$ as $y = f(u)$ where $u = g(x)$

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

Example

$$f(x) = \sqrt{x^2+5} = (x^2+5)^{\frac{1}{2}} \Rightarrow f'(x) = \tfrac{1}{2}(x^2+5)^{-\frac{1}{2}}.2x$$
$$= x(x^2+5)^{-\frac{1}{2}} = \frac{x}{\sqrt{x^2+5}}$$

Example

$$y = \cos^3(1-x) \Rightarrow \frac{dy}{dx} = 3\cos^2(1-x).-\sin(1-x).-1$$
$$= 3\cos^2(1-x)\sin(1-x)$$

Hints & tips

Remember $(f(g))' = f'(g) \times g'$. Give final answers in their simplest form.

The Product rule

The product rule is a formula used to find the derivative of a product of functions.

It may be stated as $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

Example

$$f(x) = x^2\cos x \Rightarrow f'(x) = 2x.\cos x + x^2.-\sin x = 2x\cos x - x^2\sin x$$

Example

$$y = x^3(2-x)^4 \Rightarrow \frac{dy}{dx} = 3x^2.(2-x)^4 + x^3.4(2-x)^3.-1$$
$$= 3x^2(2-x)^4 - 4x^3(2-x)^3$$
$$= x^2(2-x)^3[3(2-x)-4x]$$
$$= x^2(2-x)^3(6-7x)$$

Hints & tips

Remember $(fg)' = f'g + fg'$. Give final answers in their simplest form.

The Quotient rule

The quotient rule is a formula used to find the derivative of a quotient of functions.

It may be stated as $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

Example

$$f(x) = \frac{1+\sin x}{1-\sin x} \Rightarrow f'(x) = \frac{\cos x.(1-\sin x)-(1+\sin x).-\cos x}{(1-\sin x)^2} = \frac{2\cos x}{(1-\sin x)^2}$$

Example

$$y = \frac{2x+1}{\sqrt{3-x^2}} = \frac{2x+1}{(3-x^2)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2.(3-x^2)^{\frac{1}{2}} - (2x+1).\frac{1}{2}(3-x^2)^{-\frac{1}{2}}.-2x}{3-x^2}$$

$$= \frac{(3-x^2)^{-\frac{1}{2}}\left[2(3-x^2)+x(2x+1)\right]}{3-x^2}$$

$$= \frac{6+x}{(3-x^2)^{\frac{3}{2}}} = \frac{6+x}{\sqrt{(3-x^2)^3}}$$

Hints & tips

Remember $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Look for a common factor on the numerator when simplifying.

Check-up

1 Differentiate $f(x) = \frac{1-x}{x^2+1}$. **3**

Derivatives of exponential and natural logarithmic functions

The derivatives of e^x and $\ln x$ are given in the Formulae list.

$f(x)$	$f'(x)$
e^x (or $\exp(x)$)	e^x
$\ln x$ (or $\log_e x$)	$\frac{1}{x}$

Example

$$f(x) = x^3e^{2x+1} \Rightarrow f'(x) = 3x^2.e^{2x+1} + x^3.e^{2x+1}.2 = x^2e^{2x+1}(3+2x)$$

Example

$$y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \frac{1}{\cos x}.-\sin x = -\frac{\sin x}{\cos x} = -\tan x$$

Hints & tips

Remember

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

is only valid for the natural logarithmic function.

Check-up

2 Differentiate

a) $f(x) = \sqrt{e^{4x}+1}$ **b)** $y = x^3\ln 2x$. **2, 2**

Derivatives of trigonometric functions

The reciprocal trigonometric functions secant, cosecant and tangent are defined by:

$$\cot x = \frac{1}{\tan x},\ \sec x = \frac{1}{\cos x} \text{ and } \operatorname{cosec} x = \frac{1}{\sin x}.$$

The derivatives of these functions and of $\tan x$ can be obtained by expressing each function in terms of $\sin x$ and/or $\cos x$ then using the quotient rule.

Example

$$y = \tan x = \frac{\sin x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{\cos x.\cos x - \sin x.-\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Check-up

3 Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$. **3**

In the examination use the derivatives of $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ which are given in the Formulae list, unless you are specifically asked to derive them.

$f(x)$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
$f'(x)$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x \tan x$	$-\operatorname{cosec} x \cot x$

Example

$f(x) = \sec^2 3x$

$\Rightarrow f'(x) = 2\sec 3x.\sec 3x \tan 3x.3$

$= 6\sec^2 3x \tan 3x$

Check-up

4 Given that $f(x) = \tan^2\left(\frac{x}{2}\right)$, find $f'(x)$. **2**

5 Given that $f(x) = e^{\sin x}\sec x$ evaluate $f'\left(\frac{\pi}{4}\right)$. **3**

Derivatives of inverse functions

When a function $y = f(x)$ has an inverse denoted by $y = f^{-1}(x)$, then the derivative of the inverse function can be obtained as follows.

$$y = f^{-1}(x) \Rightarrow x = f(y) \Rightarrow \frac{dx}{dy} = f'(y) \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}$$

Then express the derivative in simplest form in terms of x.

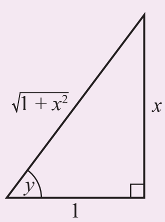
Hints & tips

$\tan^{-1} x$ is 'the angle whose tangent is x', i.e. $\tan^{-1} x = \theta \Leftrightarrow \tan\theta = x$.

Note that $\tan^{-1} x \neq \frac{1}{\tan x}$.

Example

Obtain the derivative of $y = \tan^{-1} x$.



Solution

$$y = \tan^{-1} x \Rightarrow x = \tan y \Rightarrow \frac{dx}{dy} = \sec^2 y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

Check-up

6 Show that

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$. **3**

In the examination use the derivatives of $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ which are given in the Formulae list, unless you are specifically asked to derive them.

$f(x)$	$\sin^{-1}x$	$\cos^{-1}x$	$\tan^{-1}x$
$f'(x)$	$\frac{1}{\sqrt{1-x^2}}$	$-\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

Example

$$f(x)=\sin^{-1}5x \Rightarrow f'(x)=\frac{1}{\sqrt{1-(5x)^2}}.5=\frac{5}{\sqrt{1-25x^2}}$$

Example

$$y=x\cos^{-1}\frac{x}{2} \Rightarrow \frac{dy}{dx}=1.\cos^{-1}\frac{x}{2}+x.-\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}.\frac{1}{2}$$

$$=\cos^{-1}\frac{x}{2}-\frac{x}{2\sqrt{\frac{4-x^2}{4}}}=\cos^{-1}\frac{x}{2}-\frac{x}{\sqrt{4-x^2}}$$

Check-up ?

7 Differentiate

$y=\cos^{-1}\left(\frac{1}{x}\right)$. **3**

Higher derivatives

The derivative of $f'(x)$ is denoted by $f''(x)$ and is called the *second derivative*.

The derivative of $f''(x)$ is denoted by $f'''(x)$ and is called the *third derivative*, and so on.

In Leibniz notation the higher derivatives are denoted by $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and so on.

Example

$$f(x)=\cos 3x \Rightarrow f'(x)=-\sin 3x.3=-3\sin 3x$$

$$f''(x)=-3\cos 3x.3=-9\cos 3x$$

$$f'''(x)=9\sin 3x.3=27\cos 3x$$

Check-up ?

8 Given that $f(x)=\ln(3x+2)$, find $f'(x)$ and $f''(x)$. **3**

Example

$$y=\frac{1}{2x+1}=(2x+1)^{-1} \Rightarrow \frac{dy}{dx}=-1(2x+1)^{-2}.2=-2(2x+1)^{-2}$$

$$\frac{d^2y}{dx^2}=4(2x+1)^{-3}.2=8(2x+1)^{-3}$$

$$\frac{d^3y}{dx^3}=-24(2x+1)^{-4}.2=-48(2x+1)^{-4}$$

Applying differentiation in context

Example

The distance, s metres, travelled by a particle moving along a straight line after t seconds is given by $s = \frac{2}{3}t^3 - \frac{7}{2}t^2 + t + 4$. Find the acceleration of the particle after 3 seconds.

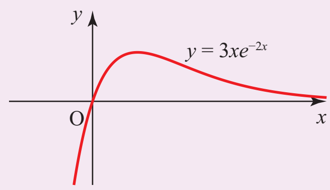
Solution

$$a = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}\left(2t^2 - 7t + 1\right) = 4t - 7 \Rightarrow a = 5 \text{ when } t = 3$$

So the acceleration of the particle is $5\,\text{m}\,\text{s}^{-2}$ after 3 seconds.

Example

Part of the graph of the curve with equation $y = 3xe^{-2x}$ is shown in the diagram.



Find the coordinates of the maximum turning point on the graph.

Solution

At maximum turning point

$$\frac{dy}{dx} = 0 \Rightarrow 3e^{-2x} - 6xe^{-2x} = 0 \Rightarrow 3e^{-2x}(1 - 2x) = 0 \Rightarrow x = \tfrac{1}{2}$$

$$\Rightarrow y = \tfrac{3}{2}e^{-1} = \frac{3}{2e}, \text{ so the maximum turning}$$

point is $\left(\frac{1}{2}, \frac{3}{2e}\right)$

Hints & tips

$e^x > 0, \forall x \in \mathbb{R}$

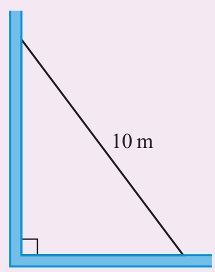
so $e^{-2x} \neq 0$.

Check-up

9 A function f is defined on the domain $0 \leq x \leq 2$ by $f(x) = e^{x-1} - x + 3$. Find the maximum and minimum values of f. **4**

Example

Ten metres of decorative edging is used to section off a flower bed against two perpendicular walls in the corner of a garden. Find the maximum possible area of the flower bed.



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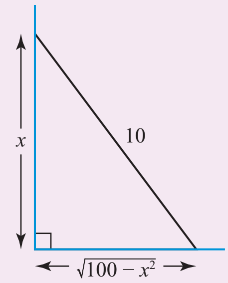
Hints & tips

In your answers give exact values rather than decimal approximations.

Solution

Let the length of the flower bed be x m, then using Pythagoras' theorem the width is $\sqrt{100-x^2}$.



Hence the area is given by $A = \frac{1}{2}x\sqrt{100-x^2}$.

$$\frac{dA}{dx} = \tfrac{1}{2}(100-x^2)^{\frac{1}{2}} + \tfrac{1}{2}x.\tfrac{1}{2}(100-x^2)^{-\frac{1}{2}}.-2x$$

$$= \tfrac{1}{2}(100-x^2)^{-\frac{1}{2}}\left[(100-x^2)-x^2\right] = \frac{100-2x^2}{2\sqrt{100-x^2}}$$

At SP $\frac{dA}{dx} = 0 \Rightarrow 100-2x^2 = 0 \;\; \Rightarrow x = \sqrt{50} = 5\sqrt{2}$

x	←	$5\sqrt{2}$	→
$\frac{dA}{dx}$	+	0	–
slope	↗	→	↘

so area is maximised when $x = 5\sqrt{2}$ and maximum area

$$= \tfrac{1}{2} \times 5\sqrt{2} \times \sqrt{100-\left(5\sqrt{2}\right)^2} = 25\,\text{m}^2$$

Practice questions ?

Marks

1 Differentiate the following. **3, 3, 3**

a) $y = \cos^4(\pi - x)$ **b)** $f(x) = (x+1)(2x-1)^3$ **c)** $y = \ln\left(\dfrac{x}{1+x}\right)$.

2 Differentiate these. **2, 3**

a) $f(x) = e^{3x}\cot 4x$ **b)** $y = \tan^{-1}\sqrt{2x-1}$.

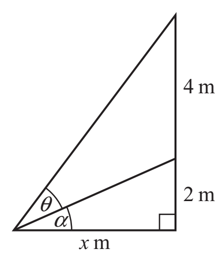
3 Given that $y = e^{2x}\sin x$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. **4**

4 Given that $f(x) = \sin^{-1} 3x$ evaluate $f'\left(\dfrac{\sqrt{5}}{9}\right)$. **3**

5 The distance, s metres, travelled by a particle moving along a straight line after t seconds is given by $s = 5\sqrt{(2t+1)^3}$.
Find the acceleration of the particle after 4 seconds. **3**

6 A function f is defined on the domain $0 \le x \le 5$ by $f(x) = \dfrac{x+1}{x^2+3}$.
Find the maximum and minimum values of f. **5**

7 A movie screen on a wall is 4 metres high and 2 metres above the floor.



a) Show that, from a position x metres from the front of the room, the viewing angle, θ, of the screen is given by

$$\theta = \tan^{-1}\left(\frac{6}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right).$$ **2**

b) At what distance from the front of the room is the viewing angle of the screen as large as possible and what is the largest viewing angle? **5**

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£14.99
ISBN 978-1-398-31221-0
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