Sophie Goldie
Rose Jewell

A-LEVEL

MATHEMATICS

FIRST AID KIT

- Support for challenging topics
- Extra practice for essential skills
- Suitable for all A-level maths specifications

HODDER EDUCATION
LEARN MORE

Contents

Algebra review 1

Year 1

THE LOWDOWN

① In **algebra**, symbols (called **variables**) are used to represent numbers.

A number on its own is a called a **constant.**

A **term** is made from numbers and variables alone or multiplied together or on their own.

An **expression** is made by adding or subtracting one or more terms. It has no =.

> **Hint:** You can use letters (such as n, x, y or θ) to represent:
> - any number (a **variable**)
> - an **unknown** number that you want to work out.

Example $5 - \frac{3xy}{2} + x^2$ is an expression with three terms.

② You can **expand brackets** by multiplying them out.

Example Expand and simplify $(5x + 4)(3x - 2)$

> **Hint:** Multiply each term in the second bracket by each term in the first.

	$3x$	-2
$5x$		
$+4$		

→

	$3x$	-2
$5x$	$15x^2$	$-10x$
$+4$	$12x$	-8

So $(5x + 4)(3x - 2) = 15x^2 + 2x - 8$

③ You can **factorise** an expression by rewriting it with brackets.

Example Factorising $8x^2y^2 - 12xy^3$ gives $4xy^2(2x - 3y)$

> **Hint:** 8 and 12 have a **common factor** of 4
> x and x^2 have a common factor of x
> y^3 and y^2 have a common factor of y^2.

④ You can put one expression equal to another to make an **equation.**

An equation has an = and is usually only true for certain values.

Example Solve

$$\begin{aligned} 4x + 3 &= 2 - 5x \\ +5x \quad 9x + 3 &= 2 \quad +5x \\ -3 \quad 9x &= -1 \quad -3 \\ \div 9 \quad x &= -\frac{1}{9} \quad \div 9 \end{aligned}$$

> **Watch out!** Keep the equation balanced by doing the **same** to both sides!

> **Hint:** Check you are right by substituting your answer back into the equation.

⑤ An **identity** is an equation that is true for all values of x.

The symbol ≡ is used to show that an equation is an identity.

Example $3(x + 1) \equiv 3x + 3$

GET IT RIGHT

a) Solve $\frac{3}{x-2} = \frac{4}{2x}$.

b) Factorise $15x^2 - 10xy + 12x - 8y$.

Solution:

a) Clear fractions: $\frac{3}{x-2} = \frac{4}{2x}$

Expand brackets: $3(2x) = 4(x-2)$

$6x = 4x - 8 \Rightarrow 2x = -8 \Rightarrow x = -4$

b) There is no common factor… $15x^2 - 10xy + 12x - 8y$

so factorise pairs of terms: $= 5x(3x - 2y) + 4(3x - 2y)$

Now factorise fully: $= (5x + 4)(3x - 2y)$

▶▶ See page 8 for using the **grid method to factorise**.

> **Hint:** You can check you are right by expanding the brackets. Use a grid to help you.

YOU ARE THE EXAMINER

Sam and Lilia have both made some mistakes in their maths homework.
Which questions have they got right?
Where have they gone wrong?

SAM'S SOLUTION

1 Simplify $(5x + 1) - (x - 2)$

$(5x + 1) - (x - 2) = 5x + 1 - x - 2$
$= 4x - 1$

2 Simplify $3(ab)$

$3(ab) = 3ab$

3 Factorise $12x^2y - 8x^4y^3 + 4xy$

$12x^2y - 8x^4y^3 + 4xy = 4xy(3x - 2x^2y^2)$

4 Expand $(3x - 2)(x - 5)$

$(3x - 2)(x - 5) = 3x^2 - 2x - 15x + 10$
$= 3x^2 - 17x + 10$

5 Solve $\frac{x+3}{2} - \frac{2x+5}{3} = 4$

$6 \times \frac{x+3}{2} - 6 \times \frac{2x+5}{3} = 6 \times 4$

$3(x + 3) - 2(2x + 5) = 24$

$3x + 9 - 4x - 10 = 24$

$-x = 25$

$x = -25$

LILIA'S SOLUTION

1 Simplify $(5x + 1) - (x - 2)$

$(5x + 1) - (x - 2) = 5x + 1 - x + 2$
$= 4x + 3$

2 Simplify $3(ab)$

$3(ab) = 3a \times 3b$
$= 9b$

3 Factorise $12x^2y - 8x^4y^3 + 4xy$

$12x^2y - 8x^4y^3 + 4xy =$
$4xy(3x - 2x^3y^2 + 1)$

4 Expand $(3x - 2)(x - 5)$

$(3x - 2)(x - 5) = 3x^2 - 2x + 15x - 10$
$= 3x^2 + 13x - 10$

5 Solve $\frac{x+3}{2} - \frac{2x+5}{3} = 4$

$3(x + 3) - 2(2x + 5) = 4$

$3x + 9 - 4x - 10 = 4$

$-x = 5$

$x = -5$

SKILL BUILDER

1 Simplify these expressions.

a) $5x(4xy - 3) + 3y(2x - y)$ **b)** $2x(3x - 3) - (2 - 5x)$ **c)** $3x(1 - 2x) - 3x(4x - 2)$

Hint: Be careful when there is a negative outside the bracket.

2 Expand and simplify.

a) $(x + 5)(x - 4)$ **b)** $(2x + 3)(3x - 5)$ **c)** $(4x - 2y)(2y - 3x)$

Hint: Use the grid method to help you.

3 Factorise each expression fully.

a) $12a^2 + 8a$ **b)** $8b^2c - 4b$ **c)** $3bc - 2b + 6c^2 - 4c$

Hint: Make sure the terms inside the brackets in your answer don't have a common factor.

4 Solve these equations.

a) $5(2x - 4) = 13$ **b)** $4(x + 3) = 5(3 - x)$ **c)** $3(2x + 1) - 2(1 - x) = 2(9x + 4)$

Hint: Expand any brackets first.

5 Solve these equations.

a) $\frac{x}{4} = \frac{2x - 1}{3}$ **b)** $\frac{1}{4x + 3} = \frac{2}{5 - x}$ **c)** $\frac{x}{5} - 6 = \frac{2x + 3}{3}$

Hint: Clear any fractions first.

Algebra review 2

Year 1

THE LOWDOWN

① **Simultaneous equations** are two equations connecting two unknowns.

② To solve simultaneous equations, **substitute** one equation into the other.

Example $2x + y = 10$ and $y = 3 - x$

Substituting for y gives $2x + 3 - x = 10 \Rightarrow x = 7$

Since $y = 3 - x$ then $y = 3 - 7 = -4$

Or you can add/subtract one equation to **eliminate** one of the unknowns

Example

$$\begin{aligned} 2x + 3y &= 27 \\ + \quad x - 3y &= 9 \quad ① \\ \hline 3x \quad &= 36 \Rightarrow x = 12 \end{aligned}$$

From ① $12 - 3y = 9$ so $y = 1$

③ A **formula** is a rule connecting two or more variables.

The subject of a formula is the variable calculated from the rest of the formula.

The subject is a letter on its own on one side and is not on both sides of the =.

Example $v = u + at$ or $s = ut + \frac{1}{2}at^2$

④ You can **rearrange** a formula to make a different variable the subject.

Rearrange the formula in the same way you would solve an equation.

Example $v = u + at$

Swap sides: $u + at = v$

Subtract u from both sides: $at = v - u$

Divide both sides by t: $a = \dfrac{v - u}{t}$

Watch out! Your calculator may solve equations for you but in the exam you may lose marks if you don't show enough working!

Hint: Don't forget to find the values of both unknowns.
Check your answer works for both equations.

GET IT RIGHT

a) Solve $3a + 2b = -5$ and $4a + 3b = -4.5$.

b) Make x the subject of $y = \dfrac{2x + b}{x - 3c}$.

Solution:

a) To eliminate b, multiply each equation to get $6b$ in both.
Then subtract one equation from the other.

$3a + 2b = -5$ ① ① × by 3 $9a + 6b = -15$

$4a + 3b = -4.5$ ② ② × by 2 $-\; 8a + 6b = -9$

$a = -6$

Use equation ① to find b: $3 \times -6 + 2b = -5$

$2b = 13 \Rightarrow b = 6.5$

b) Clear the fraction: $y(x - 3c) = 2x + b$

Expand the brackets: $xy - 3cy = 2x + b$

Gather x terms on one side: $xy - 2x = b + 3cy$

Factorise: $x(y - 2) = b + 3cy$

Divide by $(y - 2)$: $x = \dfrac{b + 3cy}{y - 2}$

▶▶ See page 18 for **inverse functions** and page 38 for **more on simultaneous equations**.

Hint: You can **eliminate** an unknown if both equations have the **same** amount of that unknown.

Look at the signs of that unknown:
Different signs: ADD Same signs: SUBTRACT

YOU ARE THE EXAMINER

Mo and Lilia have both missed out some work in their maths homework. Complete their working.

MO'S SOLUTION

Make a the subject of $s = ut + \frac{1}{2}at^2$

Swap sides: $ut + \frac{1}{2}at^2 = s$

Subtract ut from both sides:

$\frac{1}{2}at^2 = \boxed{}$

Multiply both sides by $\boxed{}$:

$at^2 = \boxed{}$

Divide both sides by $\boxed{}$:

$a = \boxed{}$

LILIA'S SOLUTION

Solve $3x + 4y = -1$ ①

$6x - 5y = -8.5$ ②

Multiply equation ① by 2: $\boxed{}$ ③

Use ③ and ① to eliminate x:

$\boxed{}$

$\boxed{}$ $6x - 5y = -8.5$

$\boxed{} = \boxed{}$

So $y = \boxed{}$

Using equation ①:

$3x + 4 \times \boxed{} = -1$

which gives $x = \boxed{}$

SKILL BUILDER

1 Which one of the following is the correct x-value for the linear simultaneous equations $5x - 3y = 1$ and $3x - 4y = 4$?

A $x = \frac{16}{29}$ **B** $x = \frac{16}{11}$ **C** $x = -\frac{8}{11}$ **D** $x = -\frac{3}{11}$ **E** $x = \frac{5}{29}$

2 Which one of the following is the correct y-value for the linear simultaneous equations $5x - 2y = 3$ and $y = 1 - 2x$?

A $y = -9$ **B** $y = \frac{5}{9}$ **C** $y = \frac{7}{9}$ **D** $y = -\frac{1}{9}$ **E** $y = -\frac{7}{3}$

3 Solve **a)** $2a + b = 14$, $a = 2b - 3$ **b)** $2c + 3d = 14$, $4c + 3d = 10$ **c)** $5e - 3f = 11$, $4e + 2f = 18$

4 Decide whether each statement is true (T) or false (F).

a) $y = x + \frac{5}{2} \Rightarrow 2y = x + 5$ **Hint:** Substitute in values for x to check your answers.

b) $y = 9x^2 \Rightarrow y = (3x)^2$

c) $\frac{2}{3}(x - 4) = 6 \Rightarrow 2(3x - 12) = 18$

5 Make y the subject of $ax + by = c$.

6 Make x the subject of $y = \frac{ax + 1}{x + by}$. **Hint:** Look at the 'Get it right' box.

7 Make x the subject of $y = \frac{5}{\sqrt{a^2 - x^2}}$.

Hint: Follow these steps:
1. Multiply both sides by $\sqrt{a^2 - x^2}$.
2. Divide both sides by y.
3. Square both sides to remove the square root.
4. Make x^2 the subject and then square root.

Indices and surds

GCSE Review/Year 1

THE LOWDOWN

① 3^4 means $\underbrace{3 \times 3 \times 3 \times 3}_{4 \text{ of them}}$; 4 is the **index** or **power** and 3 is the **base**.

② **Laws of indices**

$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ of them}}$ $a^0 = 1$ $a^1 = a$

$a^m \times a^n = a^{m+n}$ $\dfrac{a^m}{a^n} = a^{m-n}$ $(a^m)^n = a^{m \times n}$

③ **Fractional and negative indices**

$a^{-n} = \dfrac{1}{a^n}$ $a^{\frac{1}{n}} = \sqrt[n]{a}$ $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Remember $\sqrt{\ }$ means the positive square root only.

④ You can use **roots** to solve some equations involving powers.

Example $x^4 = 20 \Rightarrow x = \pm\sqrt[4]{20} = \pm 2.11$ (to 3 s.f.)

⑤ $\sqrt{2} = 1.414213\ldots$ the decimal carries on forever and never repeats.
$\sqrt{2}$ is a **surd**; a surd is a number that is left in square root form.
When you are asked for an **exact answer** then you often need to use surds.

⑥ To **simplify** a surd use square factors of the number under the root.

Example $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$

⑦ **Useful rules:**

$\sqrt{xy} = \sqrt{x}\sqrt{y}$ $\sqrt{x} \times \sqrt{x} = x$ $(x + y)(x - y) = x^2 - y^2$

⑧ A fraction isn't in its simplest form when there is a surd in the bottom line.
To simplify it you need to **rationalise the denominator**.

Example $\dfrac{3}{\sqrt{5}} = \dfrac{3 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \dfrac{3\sqrt{5}}{5}$

Hint: You might like this memory aid: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.
A fractional index is like a flower, the bottom's the root and the top's the power!

Watch out!
When the power is even then there are two roots: one positive and one negative.
When the power is odd, there is only one root.

▶▶ See page 8 for the **difference of two squares**.

GET IT RIGHT

Rationalise the denominator of each fraction: **a)** $\dfrac{14}{3+\sqrt{2}}$ **b)** $\dfrac{1}{5-2\sqrt{3}}$

Solution:

Step 1 a) $\dfrac{14}{(3+\sqrt{2})}$

$= \dfrac{14(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$

Step 2 $= \dfrac{14(3-\sqrt{2})}{3^2-(\sqrt{2})^2}$

Step 3 $= \dfrac{14(3-\sqrt{2})}{9-2}$

$= \dfrac{14(3-\sqrt{2})}{7}$

$= 2(3-\sqrt{2})$

b) $\dfrac{1}{(5-2\sqrt{3})}$

$= \dfrac{1(5+2\sqrt{3})}{(5-2\sqrt{3})(5+2\sqrt{3})}$

$= \dfrac{(5+2\sqrt{3})}{5^2-(2\sqrt{3})^2}$

$= \dfrac{(5+2\sqrt{3})}{25-4\times 3}$

$= \dfrac{5+2\sqrt{3}}{13}$

Hint: Step 1: Multiply 'top' and 'bottom' lines by the 'bottom line with the opposite sign'. Don't forget brackets!
Step 2: Multiply out the brackets. Remember $(x + y)(x - y) = x^2 - y^2$
Step 3: Simplify.

YOU ARE THE EXAMINER

Sam and Nasreen have both made some mistakes in their maths homework. Which questions have they got right? Where have they gone wrong?

SAM'S SOLUTION

1 Solve $x^3 + 1 = 28$

$x^3 = 27$

Cube root: $x = \pm 3$

2 Simplify $\dfrac{4ab^4}{(2a^2b)^3}$

$= \dfrac{4ab^4}{8a^6b^3} = \dfrac{b}{2a^5}$

3 Simplify $\sqrt{147} - \sqrt{27}$

$\sqrt{147} - \sqrt{27} = \sqrt{120} = \sqrt{4 \times 30} = 2\sqrt{30}$

4 Evaluate $64^{-\frac{3}{2}}$

$64^{-\frac{3}{2}} = \dfrac{1}{64^{\frac{3}{2}}} = \dfrac{1}{(\sqrt{64})^3} = \dfrac{1}{8^3} = \dfrac{1}{512}$

NASREEN'S SOLUTION

1 Solve $x^3 + 1 = 28$

$x^3 = 27$

Cube root: $x = 3$

2 Simplify $\dfrac{4ab^4}{(2a^2b)^3}$

$= \dfrac{4ab^4}{2a^6b^3} = 2a^5b$

3 Simplify $\sqrt{147} - \sqrt{27}$

$\sqrt{147} - \sqrt{27} = \sqrt{49 \times 3} - \sqrt{9 \times 3}$

$= 7\sqrt{3} - 3\sqrt{3} = 4\sqrt{3}$

4 Evaluate $64^{-\frac{3}{2}} = 64^{\frac{2}{3}} = (\sqrt[3]{64})^2 = 4^2 = 16$

SKILL BUILDER

Don't use your calculator; only use it to ***check*** *your answers!*

1 Find the value of $\left(\dfrac{1}{3}\right)^{-2}$

A -9 **B** $\dfrac{1}{9}$ **C** 9 **D** $-\dfrac{1}{9}$ **E** $-\dfrac{2}{3}$

2 Find the value of $\dfrac{(2x^4y^2)^3}{10\left(x^3\sqrt{y^5}\right)^2}$, giving the answer in its simplest form.

A $\dfrac{1}{5}x^6y$ **B** $\dfrac{2}{25}x^6y$ **C** $\dfrac{4x^6}{5y^4}$ **D** $\dfrac{4}{5}x^6y$ **E** $\dfrac{8x^{12}y^6}{10x^6y^5}$

3 Simplify $(2 - 2\sqrt{3})^2$, giving your answer in factorised form.

A 16 **B** $8(2 - \sqrt{3})$ **C** $-8(1 + \sqrt{3})$ **D** $16 - 8\sqrt{3}$ **E** $4(4 - \sqrt{3})$

4 Decide whether each statement is true or false.

Write down the correct statement, where possible, when the statement is false.

a) $-4^2 = 16$ **b)** $\sqrt{9} = \pm 3$ **c)** $\sqrt{a} + \sqrt{b} \equiv \sqrt{a + b}$

d) $\sqrt{16a^2b^4} \equiv 4ab^2$ **e)** $\sqrt{a^2 + b^2} \equiv a + b$ **f)** $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) \equiv a - b$

5 Rationalise the denominator:

a) $\dfrac{4}{\sqrt{2}}$ **b)** $\dfrac{4}{3 + \sqrt{5}}$ **c)** $\dfrac{2}{6 - 3\sqrt{2}}$

6 Solve these equations.

a) $x^2 - 49 = 15$ **b)** $5x^3 - 27 = 13$ **c)** $10x(x - 4) = 4(1 - 10x)$

Quadratic equations

Year 1

THE LOWDOWN

① A **quadratic equation** can be written in the form $ax^2 + bx + c = 0$.

② You can solve some quadratic equations by factorising.

Example Solve $x^2 - 10x + 16 = 0$

$ac = 1 \times 16 = 16$ and $b = -10$, so $-10x$ splits to $-2x - 8x$

x^2	$-2x$
$-8x$	16

➡

x	x^2	$-2x$
-8	$-8x$	16

➡

	x	-2
x	x^2	$-2x$
-8	$-8x$	16

x is the HCF of x^2 and $-8x$

-8 is the HCF of $-8x$ and 16

From the grid: $(x - 2)(x - 8) = 0$

So $x = 2$ or $x = 8$

③ Make sure you can recognise these special forms:

- The difference of two squares: $x^2 - a^2 = (x + a)(x - a)$.
- Perfect squares: $x^2 + 2ax + a^2 = (x + a)^2$.

④ Watch out for quadratic equations in disguise!

Rewrite them as a quadratic equation first.

Example Solve $2^{2y} - 10 \times 2^y + 16 = 0$

Let $x = 2^y$, so $x^2 - 10x + 16 = 0$

Since $x = 2$ or $x = 8$ then $2^y = 2$ or $2^y = 8$

So $y = 1$ or $y = 3$

Hint: Use a grid and split the middle term. Look for two numbers that multiply to give ac and add to give b. Take out the common factors.

Hint: You can write 2^{2y} as $(2^y)^2$.

▶▶ See page 12 for **completing the square** and page 20 for **factorising cubic equations**.

GET IT RIGHT

Show that there is only one solution to $2y + 5\sqrt{y} - 3 = 0$.

Solution:

Let $x = \sqrt{y}$, so the equation is $2x^2 + 5x - 3 = 0$

$ac = 2 \times -3 = -6$ and $b = 5$, so $5x$ splits to $-x + 6x$

$2x^2$	$6x$
$-x$	-3

➡

$2x$	$2x^2$	$6x$
-1	$-x$	-3

➡

	x	$+3$
$2x$	$2x^2$	$6x$
-1	$-x$	-3

From the grid: $(x + 3)(2x - 1) = 0$

So $x = -3$ or $x = \frac{1}{2}$

So $\sqrt{y} = -3$ which is impossible,

or $\sqrt{y} = \frac{1}{2} \Rightarrow y = \frac{1}{4}$, so there is only one solution.

Hint: Since $y = (\sqrt{y})^2$,

$2(\sqrt{y})^2 + 5\sqrt{y} - 3 = 0$

$2x^2 \quad + 5x \quad - 3 = 0$

Watch out: Make sure you find the solutions for y and not just for x.

Watch out: $\sqrt{\ }$ means the **positive square root**, so it is never negative.

YOU ARE THE EXAMINER

Which one of these solutions is correct? Where are the errors in the other solution?

Solve $5x^4 = 10x^2$

LILIA'S SOLUTION

$5x^4 = 10x^2$

Divide by $5x^2$: $x^2 = 2$

Take the square root: $x = \sqrt{2}$

PETER'S SOLUTION

Let $y = x^2$, so $5y^2 = 10y$

Rearrange: $5y^2 - 10y = 0$

Factorise: $5y(y - 2) = 0$

Solve: $y = 0$ or $y = 2$

So $x^2 = 0$ or $x^2 = 2$

Take the square root: $x = 0$ or $x = \pm\sqrt{2}$

SKILL BUILDER

1 Which of the following is the solution of the quadratic equation $x^2 - 5x - 6 = 0$?

A $x = -6$ or $x = 1$ **B** $x = -2$ or $x = 3$ **C** $x = 2$ or $x = 3$

D $x = -3$ or $x = 2$ **E** $x = -1$ or $x = 6$

2 Factorise $6x^2 + 19x - 20$

A $(x + 4)(6x - 5)$ **B** $(3x + 10)(2x - 2)$ **C** $(x + 20)(6x - 1)$

D $(3x + 4)(2x - 5)$ **E** $(6x + 10)(x - 2)$

3 Which of the following is the solution of the quadratic equation $2x^2 - 9x - 18 = 0$?

A $x = \frac{3}{2}$ or $x = -6$ **B** $x = \frac{9}{2}$ or $x = -2$ **C** $x = 6$ or $x = 3$

D $x = -\frac{9}{2}$ or $x = 2$ **E** $x = -\frac{3}{2}$ or $x = 6$

4 Factorise.

a) $x^2 + 7x + 12$ **b)** $x^2 - 2x - 15$ **c)** $x^2 + 6x + 9$

d) $x^2 - 12x + 36$ **e)** $x^2 - 49$ **f)** $4x^2 - 100$

g) $2x^2 + 7x + 3$ **h)** $6x^2 + x - 15$ **i)** $9x^2 - 12x + 4$

5 Solve these equations by factorising.

a) $x^2 + 3x - 10 = 0$ **b)** $2x^2 - x - 1 = 0$ **c)** $2x^2 = 11x - 12$

d) $4x^2 = 5x$ **e)** $4x^2 + 25 = 20x$ **f)** $9x^2 = 25$

6 Solve these equations by factorising.

a) **i)** $x^2 - 8x + 12 = 0$ **ii)** $y^4 - 8y^2 + 12 = 0$ **iii)** $z - 8\sqrt{z} + 12 = 0$

b) **i)** $x^2 - 10x + 9 = 0$ **ii)** $3^{2y} - 10 \times 3^y + 9 = 0$ **iii)** $z - 10\sqrt{z} + 9 = 0$

c) **i)** $4x^2 + 3x - 1 = 0$ **ii)** $4y^4 + 3y^2 - 1 = 0$ **iii)** $4z + 3\sqrt{z} - 1 = 0$

7 Daisy factorises $f(x) = 5x^2 + 25x + 20$ by dividing through by 5 to give $f(x) = x^2 + 5x + 4$.

a) Explain why Daisy is wrong.

b) Ahmed says $f(x) = 5x^2 + 25x + 20$ and $g(x) = x^2 + 5x + 4$ have the same roots. Is Ahmed right?

Inequalities

Year 1

THE LOWDOWN

① An **inequality** states that two expressions are not equal.

You can solve an inequality in a similar way to solving an equation.

Remember:

- keep the inequality sign instead of =
- when you multiply or divide by a negative number, **reverse** the inequality
- the solution to an inequality is a **range of values**.

Example $10 - 2x < 4$

Subtract 10 from both sides: $-2x < -6$

Divide both sides by -2: $x > 3$

② To solve a quadratic inequality:

- replace the inequality sign with = solve to find the critical values a and b
- the solution is either between the critical values: $a < x < b$.

OR at the extremes: $x < a$ or $x > b$.

To decide which solution is correct you can:

- **test** a value between a and b see if it satisfies the inequality
- **look** at a sketch of the graph of the quadratic.

Example Solve $2x^2 - 4 \geqslant 14$

$2x^2 - 4 = 14$

$x^2 = 9$

$\Rightarrow x = -3$ or $x = 3$

Since $x^2 \geqslant 9$

then $x \leqslant -3$ or $x \geqslant 3$

③ You can use **set notation** to write the solutions to inequalities

- $x < a$ or $x > b$ is written as $\{x : x < a\} \cup \{x : x > b\}$
- $a \leqslant x \leqslant b$ is written as $\{x : a \leqslant x\} \cap \{x : x \leqslant b\}$

Watch out! Two regions need two inequalities to describe them!

Watch out! Note the word 'or'; x can't be both less than -3 and greater than 3!

Hint: $\cup$ means **union**.

$A \cup B$ means anything in set A or set B (or both).

$\cap$ means **intersection**.

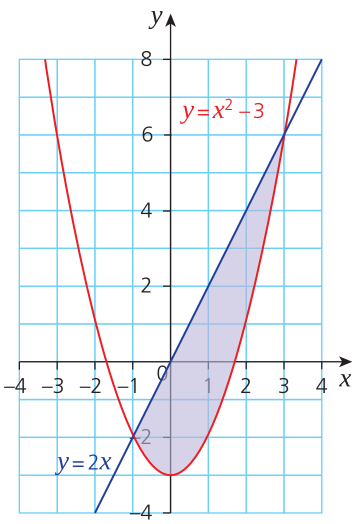
$A \cap B$ means anything that is in set A and in set B.

GET IT RIGHT

a) Draw the graphs of $y = x^2 - 3$ and $y = 2x$ on the same axes.

b) Solve $x^2 - 3 = 2x$.

c) Solve the inequality $x^2 - 3 \leqslant 2x$.



Solution:

a) $y = x^2 - 3$ is the same shape as $y = x^2$ but translated 3 squares down.

$y = 2x$ is a straight line through the origin, with gradient 2.

b) $x^2 - 3 = 2x$

Subtract $2x$: $x^2 - 2x - 3 = 0$

Factorise: $(x + 1)(x - 3) = 0$

Solve: $x = -1$ and $x = 3$

c) So critical values are $x = -1$ and $x = 3$.

You want $y = x^2 - 3$ to be below $y = 2x$

so from the graph you can see the solution is $-1 \leqslant x \leqslant 3$.

Hint: Use a solid line, ——, to show the line **is** included.

Use a dashed line, - - -, to show the line **is not** included.

▶▶ See page 12 for the **discriminant** and section 3 for **graphs**.

▶▶ See pages 94–97 for **Venn diagrams**.

YOU ARE THE EXAMINER

Which one of these solutions is correct? Where are the errors in the other solution?

Solve $2x^2 + 5x \geqslant 12$

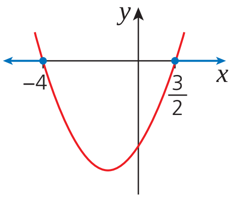
SAM'S SOLUTION

Rearrange $2x^2 + 5x - 12 \geqslant 0$

Find critical values: $2x^2 + 5x - 12 = 0$

$(2x - 3)(x + 4) = 0$

Critical values: $x = -4$ or $x = \frac{3}{2}$



So the solution is $x \leqslant -4$ or $x \geqslant \frac{3}{2}$

LILIA'S SOLUTION

Find critical values: $2x^2 + 5x = 12$

$x(2x + 5) = 12$

$2 \times 6 = 12$, so $x = 2$ or $2x + 5 = 6$

So critical values are $x = 2$ or $x = \frac{1}{2}$

Test a value between $x = \frac{1}{2}$ and $x = 2$:

Try $x = 1.8$

$2 \times 1.8^2 + 5 \times 1.8 = 15.48 \geqslant 12$

So the solution is $\frac{1}{2} \leqslant x \leqslant 2$

SKILL BUILDER

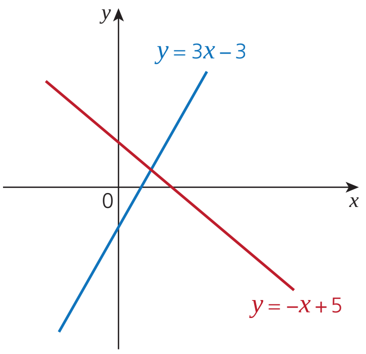
1 Solve $x + 7 < 3x - 5$.

A $x > 6$ **B** $x < 1$ **C** $x > 1$ **D** $x < 6$ **E** $x > 4$

2 Solve $\frac{2(2x + 1)}{3} \geq 6$.

A $x \geqslant 5$ **B** $x \geqslant \frac{3}{2}$ **C** $x > 4$ **D** $x \geqslant 4$ **E** $x > 5$

3 The diagram shows the lines $y = 3x - 3$ and $y = -x + 5$.



For what values of x is the line $y = 3x - 3$ above the line $y = -x + 5$?

A $x < 4$ **B** $x < 2$ **C** $x > \frac{1}{2}$ **D** $x > 2$ **E** $x > 4$

4 Solve the inequality $x^2 + 2x - 15 \leqslant 0$.

A $-5 \leqslant x \leqslant 3$ **B** $-3 \leqslant x \leqslant 5$ **C** $3 \leqslant x \leqslant 5$

D $x = -5$ or $x = 3$ **E** $x \leqslant -5$ or $x \geqslant 3$

5 Solve the inequality $6x - 6 < x^2 - 1$.

A $x < -1$ or $x > 7$ **B** $1 < x < 5$ **C** $x < 1$ or $x > 7$

D $x < -5$ or $x > -1$ **E** $x < 1$ or $x > 5$

6 Use set notation to write the solutions to the inequalities in **questions 4** and **5**.

7 A ball is thrown up in the air.

The height h metres of the ball at time t seconds is given by $h = 2 + 15t - 5t^2$.

a) Find the times when the ball reaches a height of 12 metres.

b) Find when the height of the ball is:

i) above 12 metres **ii)** below 12 metres.

Completing the square

Year 1

THE LOWDOWN

① You can write a quadratic expression in the form $a(x + p)^2 + q$.
This is called **completing the square**.
Follow the steps to complete the square for a quadratic in the form $ax^2 + bx + c$.

Example		$2x^2 - 12x + 7$
Step 1	Factor a from first 2 terms:	$2(x^2 - 6x) + 7$
Step 2	Take the coefficient of x:	-6
Step 3	Halve it:	-3
Step 4	Square the result:	$+9$
Step 5	Add and subtract this square:	$2\left[\underbrace{x^2 - 6x + 9}_{\text{this is a perfect square}} - 9\right] + 7$
Step 6	Factorise:	$= 2[(x - 3)^2 - 9] + 7$
Step 7	Simplify:	$= 2(x - 3)^2 - 11$

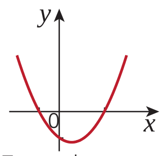
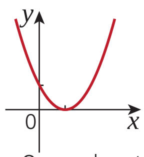
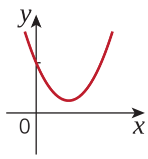
② The **turning point** of the graph of $y = a(x + p)^2 + q$ is at $(-p, q)$.
It is symmetrical about the line $x = -p$.

Example $y = 2x^2 - 12x + 7$ has a turning point at $(3, -11)$.

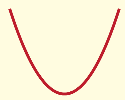
③ You can use the **quadratic formula** to solve a quadratic equation in the form $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

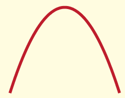
The square root of a negative number isn't real.
So when $b^2 - 4ac$ is negative the quadratic equation has no real roots.
$b^2 - 4ac$ is called the **discriminant**, it tells you how many roots to expect.

When the discriminant is…	*Positive* $b^2 - 4ac > 0$	*Zero* $b^2 - 4ac = 0$	*Negative* $b^2 - 4ac < 0$
The number of real roots is…	2	1 (repeated)	0
When a is positive graph is: Note when a is negative the curve is 'upside down'  Positive a Negative a	 Two real roots	 One real root	 No real roots

Hint: The graph of a quadratic equation $y = ax^2 + bx + c$ is a curve called a **parabola.**
The curve is



Positive shaped when a is +ve and



Negative shaped when a is −ve.

Hint: When a question asks for **exact solutions** then you should leave your answer as a surd (with the $\sqrt{\ }$).

▶▶ See page 38 for **simultaneous equations**.
◀◀ See page 8 for **quadratic equations**.
◀◀ See page 10 for **inequalities**.

GET IT RIGHT

The equation $16x^2 + kx + 9 = 0$ has no real roots.
Work out the possible values of k.

Solution:

Use the discriminant: $b^2 - 4ac < 0$

$a = 16$, $b = k$ and $c = 9$: $k^2 - 4 \times 16 \times 9 < 0$

$k^2 - 576 < 0 \Rightarrow k^2 < 576$

Find the critical values: $k^2 = 576 \Rightarrow k = \pm 24$

So $-24 < k < 24$

Watch out! Use the **discriminant** when a question asks about the number of roots.

Hint: When $k = \pm 24$ then $b^2 - ac = 0$ so the equation has one repeated root.

YOU ARE THE EXAMINER

Mo and Lilia have both missed out some work in their maths homework. Complete their working.

MO'S SOLUTION

a) Write $y = 5x^2 + 10x + 8$ in the form

$y = a(x + b)^2 + c$

$y = 5x^2 + 10x + 8$

$= 5(x^2 + \boxed{}x) + 8$

$= 5\left(x^2 + \boxed{}x + \boxed{} - \boxed{}\right) + 8$

$= 5\left[\left(x + \boxed{}\right)^2 - \boxed{}\right] + 8$

$= 5\left(x + \boxed{}\right)^2 + \boxed{}$

b) $y = 5x^2 + 10x + 8$ has $\boxed{}$ real roots because $b^2 - 4ac \boxed{} 0$.

LILIA'S SOLUTION

a) Find the exact solutions of $3(x - 4)^2 - 15 = 0$

$3(x - 4)^2 = \boxed{}$

$(x - 4)^2 = \boxed{}$

$x - 4 = \pm\sqrt{\boxed{}}$

$x = \boxed{} \pm \sqrt{\boxed{}}$

b) $y = 3(x - 4)^2 - 15$

The coordinates of the turning point are $\left(\boxed{}, \boxed{}\right)$.

The equation of the line of symmetry is $x = \boxed{}$.

SKILL BUILDER

1 Which of the following is the exact solution of the quadratic equation $2x^2 - 3x - 4 = 0$?

A $x = \dfrac{-3 \pm \sqrt{41}}{4}$ **B** $x = \dfrac{3 \pm \sqrt{41}}{4}$ **C** $x = \dfrac{3 \pm \sqrt{23}}{4}$ **D** $x = \dfrac{-3 \pm \sqrt{23}}{4}$

E There are no real solutions.

Hint: Use the quadratic formula and use your calculator to check.

2 Write $x^2 - 12x + 3$ in completed square form.

A $(x - 6)^2 + 3$ **B** $(x - 12)^2 - 141$

C $(x + 6)^2 - 33$ **D** $(x - 6 - \sqrt{33})(x - 6 + \sqrt{33})$

E $(x - 6)^2 - 33$

3 The curve $y = -2(x - 5)^2 + 3$ meets the y-axis at A and has a maximum point at B. What are the coordinates of A and B?

A A(0, 3) and B(5, 3) **B** A(0, −47) and B(5, 3)

C A(0, −47) and B(5, −6) **D** A(0, 3) and B(−10, −47)

E A(0, −47) and B(−10, 3)

4 Four of the following statements are true and one is false. Which one is false?

A $3x^2 - 2x + 1 = 0$ has no real roots.

B $2x^2 - 5x + 1 = 0$ has two distinct real roots.

C $9x^2 - 6x + 1 = 0$ has one repeated real root.

D $x^2 + 2x - 5 = 0$ has two distinct real roots.

E $4x^2 - 9 = 0$ has one repeated real root.

5 Find the exact solutions of $3(x + 5)^2 - 9 = 0$.

Hint: You don't need to expand the bracket!

6 The equation $3x^2 + 7x + k = 0$ has two real roots. Work out the possible values of k.

Hint: Look at the 'Get it right' box.

7 The equation $5x^2 + kx + 5 = 0$ has no real roots. Work out the possible values of k.

Algebraic fractions

Year 1

THE LOWDOWN

① An algebraic fraction is a fraction with a letter symbol in the denominator. You can **simplify** an algebraic fraction by **cancelling common factors**. Factorise the expressions on the top and bottom of the fraction first.

Example $\dfrac{x^2-1}{x^2-2x-3} = \dfrac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x-3)} = \dfrac{x-1}{x-3}$

You can combine algebraic fractions in the same way as you would ordinary fractions.

② **Addition and subtraction** – find a common denominator.

Example
$$\frac{2}{x+4}+\frac{3}{x} = \frac{2\times x}{(x+4)\times x}+\frac{3\times(x+4)}{x\times(x+4)}$$
$$= \frac{2x}{x(x+4)}+\frac{3(x+4)}{x(x+4)}$$
$$= \frac{2x+3(x+4)}{x(x+4)} = \frac{5x+12}{x(x+4)}$$

Sometimes you need to rewrite a term as a fraction.

Example $\dfrac{5}{x^2}-2x = \dfrac{5}{x^2}-\dfrac{2x}{1} = \dfrac{5}{x^2}-\dfrac{2x\times x^2}{1\times x^2} = \dfrac{5-2x^3}{x^2}$

③ **Multiplication** – multiply numerators (tops) and multiply denominators (bottoms).

Example $\dfrac{4}{3-2x}\times\dfrac{5}{x+2} = \dfrac{20}{(3-2x)(x+2)}$

④ **Division** – flip the second fraction and multiply.

Example
$$\frac{4x}{x-5}\div\frac{x}{2x+1} = \frac{4x}{x-5}\times\frac{2x+1}{x}$$
$$= \frac{4\cancel{x}(2x+1)}{\cancel{x}(x-5)} = \frac{4(2x+1)}{x-5}$$

Watch out! You can only cancel common factors. For example, $\dfrac{x}{x+4}$ doesn't simplify to $\dfrac{1}{4}$ as x isn't a factor of the bottom line.

Hint To rewrite with a common denominator, multiply the top **and** bottom of each fraction by the bottom of the other fraction.

Hint When you 'flip' a fraction you are finding the **reciprocal**. The reciprocal of $\dfrac{2}{x+1}$ is $\dfrac{x+1}{2}$. The reciprocal of x is $\dfrac{1}{x}$.

GET IT RIGHT

Simplify $\dfrac{\left(\frac{x^2-25}{6x}\right)}{\left(\frac{x-5}{2x^3}\right)}$.

Solution:

Step 1 Rewrite as a division: $\dfrac{\left(\frac{x^2-25}{6x}\right)}{\left(\frac{x-5}{2x^3}\right)} = \dfrac{x^2-25}{6x}\div\dfrac{x-5}{2x^3}$

Step 2 Flip the second fraction and multiply: $= \dfrac{x^2-25}{6x}\times\dfrac{2x^3}{x-5}$

Step 3 Simplify: $= \dfrac{2x^3(x^2-25)}{6x(x-5)}$

$$= \frac{2x^3(x-5)(x+5)}{6x(x-5)} = \frac{x^2(x+5)}{3}$$

◀◀ **Factorising quadratics** on page 8 and **simplifying expressions** on page 2.
▶▶ **Partial fractions** on page 30.

YOU ARE THE EXAMINER

Which one of these solutions is correct? Where is the error in the other solution?

Simplify $\dfrac{x+1}{x-\dfrac{1}{x}}$

SAM'S SOLUTION

$$\frac{x+1}{\dfrac{x^2-1}{x}} = \frac{(x+1)(x^2-1)}{x}$$

$$= \frac{(x+1)(x+1)(x-1)}{x}$$

$$= \frac{(x+1)^2(x-1)}{x}$$

LILIA'S SOLUTION

$$\frac{x+1}{\dfrac{x^2-1}{x}} = (x+1) \div \frac{x^2-1}{x}$$

$$= (x+1) \times \frac{x}{x^2-1}$$

$$= \frac{x(x+1)}{x^2-1}$$

$$= \frac{x(x+1)}{(x+1)(x-1)}$$

$$= \frac{x}{x-1}$$

SKILL BUILDER

1 Simplify the expression $\dfrac{x^2-9}{x^2-x-12}$ as far as possible.

A $\dfrac{9}{x-12}$ **B** $\dfrac{3}{4}$ **C** $\dfrac{x+3}{x-4}$ **D** $\dfrac{x-3}{x-4}$ **E** $\dfrac{x+3}{x+4}$

2 Simplify $\dfrac{6x^3}{(x+1)^2} \times \dfrac{3x+3}{2x}$ as far as possible.

A $9x$ **B** $\dfrac{6x^3(x+3)}{(x+1)^2}$ **C** $\dfrac{3x^2(3x+3)}{(x+1)^2}$ **D** $\dfrac{12x^2}{x+1}$ **E** $\dfrac{9x^2}{x+1}$

3 Decide whether each of the following statements is true or false.

a) $\dfrac{1}{x+4} \equiv \dfrac{1}{x} + \dfrac{1}{4}$ **b)** $\dfrac{2x}{x+y} \equiv \dfrac{2}{y}$ **c)** $\dfrac{1}{xy} \equiv \dfrac{1}{x} \times \dfrac{1}{y}$

d) $\dfrac{1}{4} \div x \equiv \dfrac{1}{4x}$ **e)** $\dfrac{x^2}{y^2} \equiv \dfrac{x}{y}$ **f)** $\dfrac{x+4}{y} \equiv \dfrac{x}{y} + \dfrac{4}{y}$

Hint: Substitute in values of x and y to check that you get the same answer for both sides.

4 Write as a single fraction.

a) $\dfrac{x+3}{2x^2} - \dfrac{5}{x}$ **b)** $\dfrac{1}{x} + \dfrac{1}{y}$ **c)** $\dfrac{x}{2} - \dfrac{x-3}{x}$

Hint: Start by finding a common denominator.

5 Write as a single fraction.

a) $\dfrac{x^2+y}{2x} \div \dfrac{x}{y^2}$ **b)** $\dfrac{x^2-25}{2x} \times \dfrac{x^2}{x^2+6x+5}$ **c)** $\dfrac{x^2}{3+\dfrac{2}{x}}$

Proof

Year 1 and 2

THE LOWDOWN

① When you are asked to **prove** or show a statement or **conjecture** is true you must **show all your working.**

You can use:

a) **Proof by direction argument (deduction)**
You start from a known result and then use logical argument – you usually need to use algebra.

b) **Proof by exhaustion**
You test every possible case.

c) **Proof by contradiction**
Start by assuming the conjecture is false and then use logical argument to show this leads to a contradiction.

d) **Disproof by counter example**
You only need one counter example to disprove a conjecture.

② Here are some useful starting points for questions on proof.

An **integer** is a whole number.

A **rational number** can be written as a fraction.

Example $9 = \frac{9}{1}$, $0.75 = \frac{3}{4}$ and $0.\dot{4}2857\dot{1} = \frac{3}{7}$

An **irrational number** can't be written as a fraction – its decimal part carries on for ever and never repeats.

Example $\pi = 3.14159....$, $e = 2.71828...$ and $\sqrt{2} = 1.41421...$

Consecutive numbers are numbers that follow one after another other in order.

Examples:
3 consecutive integers: $n, n + 1, n + 2$ or $n - 1, n, n + 1$
3 consecutive even numbers: $2n, 2n + 2, 2n + 4$
3 consecutive odd numbers: $2n + 1, 2n + 3, 2n + 5$
3 consecutive square numbers: $n^2, (n + 1)^2, (n + 2)^2$

▶▶ **Proofs in trigonometry** on pages 56 and 58.

Hint: Often proofs use the following symbols:

⇒ implies or 'leads to'
A shape is a square ⇒ the shape is a quadrilateral.

⇐ implied by or 'follows from'
The shape is a polygon ⇐ a shape has 5 sides.

⇔ implies and is implied by or 'leads to and follows from'
A shape has 3 straight sides ⇔ the shape is a triangle.

GET IT RIGHT

a) Use proof by exhaustion to show that, with the exceptions of 2 and 3, every prime number is either 1 more or 1 less than a multiple of 6.

b) Find a counter example to prove that the following statement is not true.
$n^2 > n$ for all values of n

c) Prove that the sum of two consecutive odd numbers is a multiple of 4.

Solution:

a) A multiple of 6 is not prime as 1, 2, 3 and 6 are factors.
1 more than a multiple of 6 is odd, so could be prime.
2 more than a multiple of 6 is even so is not prime.
3 more than a multiple of 6 is a multiple of 3 so is not prime.
4 more than a multiple of 6 is even so is not prime.
5 more than a multiple of 6 is odd, so could be prime.
5 more than a multiple of 6 is the same as 1 less than a multiple of 6.
So all prime numbers are either 1 more or 1 less than a multiple of 6.

b) When $n = 0.5$, $0.5^2 = 0.25 < 0.5$, so $n^2 < n$ when $0 < n < 1$

c) $2n + 1 + 2n + 3 = 4n + 4 = 4(n + 1)$
Since 4 is a factor then the sum of two consecutive odd numbers is a multiple of 4.

Hint: When you are asked to find a counter example it is often a good idea to test negative numbers and numbers less than 1.

Watch out! You must write a conclusion.

Hint: To show a number is a multiple of 4, you need to show 4 is a factor.

YOU ARE THE EXAMINER

Sam's proof is muddled up and Lilia's proof is incomplete.
Correct both proofs.

SAM'S SOLUTION

Proof that there are an infinite number of primes:

Assume that there are a finite number of primes and list all the primes.

A: Either way there is a new prime.

B: None of the primes on the list is a factor of Q (there is always a remainder of 1).

C: So Q is either prime, or it has a prime factor not on the list of all primes.

D: Multiply all the primes together, call the result P.

E: Let $Q = P + 1$

This contradicts my assumption that there is a finite list of primes.

LILIA'S SOLUTION

Proof that $\sqrt{2}$ is irrational:

Assume $\sqrt{2}$ is rational, so it can be written as $\sqrt{2} = \frac{a}{b}$ where $\frac{a}{b}$ is a fraction in its lowest terms.

Squaring both sides gives $\boxed{} = \frac{a^2}{b^2}$

Rearranging gives $2b^2 = \boxed{}$

a^2 is ___ since $2 \times$ any number is ____.

So a must be ____.

Let $a = 2n$, so $2b^2 = (2n)^2 = \boxed{}$

Dividing by 2 gives $b^2 = \boxed{}$

So b^2 is ___ and so b is ____.

But if a and b are both ____ then $\frac{a}{b}$ can't be in its ____ terms.

This contradicts my original assumption so $\sqrt{2}$ must be ______.

SKILL BUILDER

1 'For all values of n greater than or equal to 1, $n^2 + 3n + 1$ is a prime number.'
Which value of n gives a counter-example that disproves this conjecture?

A $n = 7$ **B** $n = 2$ **C** $n = 8$ **D** $n = 6$

2 Below is a proof that appears to show that $2 = 0$.
The proof must contain an error.
At which line does the error occur?

Let $a = b = 1$

[Line 1] $\Rightarrow a^2 = b^2$

[Line 2] $\Rightarrow a^2 - b^2 = 0$

[Line 3] $\Rightarrow (a + b)(a - b) = 0$

[Line 4] $\Rightarrow a + b = 0$

[Line 5] $\Rightarrow 2 = 0$

A line 1 **B** line 2 **C** line 3 **D** line 4 **E** line 5

3 Look at the following statements about non-zero numbers, x and y. Which of them are true?

(1) $xy = 1 \Rightarrow x = 1, y = 1$

(2) $xy = 1 \Leftarrow x = \frac{1}{y}$

(3) $xy = 1 \Leftarrow x = \frac{1}{y}$ or $y = \frac{1}{x}$

A (1) only **B** (2) only **C** (3) only **D** (1) and (3) **E** (2) and (3)

Functions

THE LOWDOWN

① A **function** is a rule. It maps a number in one set to a number in another set.
A function is usually written using letters such as f, g or h.

Example $f(x) = x^2$ and $g(x) = 4x - 1$ are functions.
$f(3) = 3^2 = 9$ and $g(2) = 4 \times 2 - 1 = 7$

② The set of inputs (x-values) to a function is the **domain**.
The set of outputs from the function is the range.
Every input has **exactly one** output, but some inputs may map to the same output. You say a function is **one-to-one** or **many-to-one**.

Example $f(x) = x^2$, $-3 \leqslant x \leqslant 3$ has a domain of $-3 \leqslant x \leqslant 3$
and a range of $0 \leqslant f(x) \leqslant 9$ since $f(0) = 0$ and
$f(3) = f(-3) = 9$

③ For a composite function $gf(x)$ you apply g to the output of f.
Make sure you apply the right function first – start with x and work outwards.

Example Given $f(x) = x^2$ and $g(x) = 4x - 1$, find i) $g(f(x))$ ii) $fg(x)$.

i) $x \xrightarrow{f} x^2 \xrightarrow{g} 4x^2 - 1$ so $gf(x) = 4x^2 - 1$

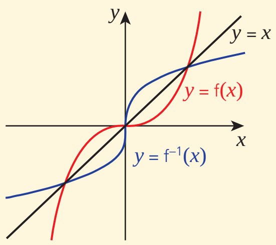
ii) $x \xrightarrow{g} 4x - 1 \xrightarrow{f} (4x - 1)^2$

④ The inverse function f^{-1} reverses the effect of the function.
It maps the range back onto the domain.
Note: $ff^{-1}(x) = f^{-1}f(x) = x$

Example $f(x) = x^3$ so

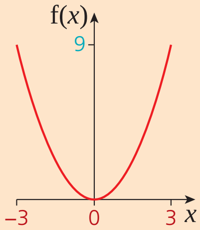
$2 \xrightarrow{f(2)} 8 \xrightarrow{f^{-1}(8)} 2$

The graphs of a function and its inverse function are reflections of each other in the line $y = x$.



Hint: $f(x) = x^2$ means square the input number.
$g(x) = 4x - 1$ means multiply the input by 4 and subtract 1.

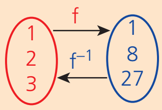
Watch out! Draw a graph to help you find the range.



Hint:
$3 \xrightarrow{f} 9 \xrightarrow{g} 35$
so $gf(3) = 35$

Watch out! In general, $gf(x)$ is not the same as $fg(x)$.

Watch out! A function can only have an inverse function if it is a one-to-one function.



GET IT RIGHT

You are given $f(x) = 3x - 2$, $x \in \mathbb{R}$ and $g(x) = \sqrt{x - 3}$, $x \geqslant 3, x \in \mathbb{R}$

a) Write down the range of each function.

b) Find $fg(12)$ and $gf(x)$.

c) Find $f^{-1}(x)$.

Hint: $\mathbb{R}$ is the set of real numbers – this means all numbers, positive negative, rational, irrational and 0.

Solution:

a) $f(x) \in \mathbb{R}$ and $g(x) \geqslant 0$, $g(x) \in \mathbb{R}$

b) $12 \xrightarrow{g} 3 \xrightarrow{f} 7$ so $fg(12) = 7$

$x \xrightarrow{f} 3x - 2 \xrightarrow{g} \sqrt{3x - 2 - 3}$ so $gf(x) = \sqrt{3x - 5}$

c) **Step 1** Replace $f(x)$ with y: $y = 3x - 2$

Step 2 Interchange y and x: $x = 3y - 2$

Step 3 Rearrange to make y the subject: $\frac{x + 2}{3} = y$

Step 4 Replace y with $f^{-1}(x)$: $f^{-1}(x) = \frac{x + 2}{3}$

Watch out! The square root of a negative number isn't real, so $g(x)$ is only valid when $x \geqslant 3$. This is called **restricting the domain.**

YOU ARE THE EXAMINER

Parts of each of these solutions are wrong. Where are the mistakes?

You are given $f(x) = \dfrac{1}{2x+3}$.

a) What value of x must be excluded from the domain of $f(x)$?

b) Solve $f^{-1}(x) = 1$.

SAM'S SOLUTION

a) $x \neq -\dfrac{3}{2}$

b) $f(x) = \dfrac{1}{2x+3}$

$\Rightarrow f^{-1}(x) = 2x+3$

Solve $f^{-1}(x) = 1$

$\Rightarrow \quad 2x+3 = 1$

$\Rightarrow \quad 2x = -2$

$\Rightarrow \quad x = -1$

LILIA'S SOLUTION

a) $x \neq 0$

b) $\dfrac{1}{2x+3} = y$

Swap x and y: $\dfrac{1}{2y+3} = x$

Make y the subject: $1 = 2xy + 3x$

$\Rightarrow 2xy = 1 - 3x \Rightarrow y = \dfrac{1-3x}{2x}$

Solve $f^{-1}(x) = \dfrac{1-3x}{2x} = 1$

$1 - 3x = 2x \Rightarrow 5x = 1 \Rightarrow x = \dfrac{1}{5}$

SKILL BUILDER

1 The function f is defined as $f: x \rightarrow \sqrt{2x-3}$. Write down a suitable domain for f.

A $x \in \mathbb{R}$ **B** $x \geqslant 0$ **C** $x \geqslant \dfrac{3}{2}$ **D** $x \geqslant 3$

2 The function g is defined as $g(x) = x^2 - 2x - 1$ for $-2 \leqslant x \leqslant 2$. What is the range of the function?

A $-1 \leqslant g(x) \leqslant 7$ **B** $g(x) \leqslant 7$ **C** $g(x) \geqslant -2$ **D** $-2 \leqslant g(x) \leqslant 7$

Hint: Sketch the graph.

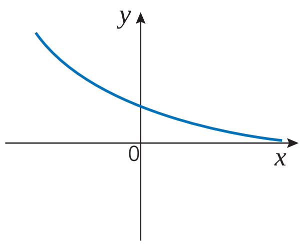
3 The functions f and g are defined for all real numbers x as $f(x) = x^2 - 2$ and $g(x) = 3 - 2x$.
Find an expression for the function $fg(x)$ (for all real numbers x).

A $fg(x) = 7 - 2x^2$ **B** $fg(x) = -2x^3 + 3x^2 + 4x - 6$ **C** $fg(x) = 4x^2 - 12x + 7$
D $fg(x) = 7 + 4x^2$ **E** $fg(x) = 1 - 2x^2$

4 The function f is defined by $f(x) = 2x^3 - 1$. Find the inverse function $f^{-1}(x)$.

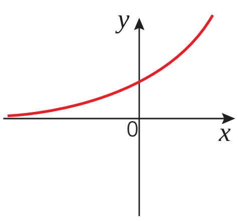
A $f^{-1}(x) = \dfrac{\sqrt[3]{x+1}}{2}$ **B** $f^{-1}(x) = \dfrac{1}{2x^3 - 1}$ **C** $f^{-1}(x) = \sqrt[3]{\dfrac{x+1}{2}}$ **D** $f^{-1}(x) = \dfrac{\sqrt[3]{x}+1}{2}$

5 The diagram shows the graph of $y = g(x)$.

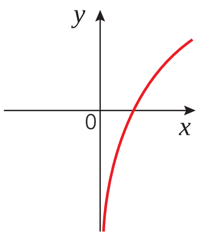


Which of the diagrams below shows the graph of $y = g^{-1}(x)$?

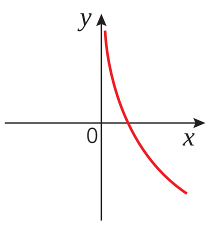
A



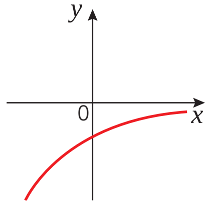
B



C



D



6 The functions p and q are defined as $p(x) = \dfrac{1}{x}$, $x \neq 0$ and $q(x) = 2x - 1$, $x \in \mathbb{R}$.

a) Find an expression for the function $pq(x)$. **b)** Solve $pq(x) = x$.

Polynomials

Year 1

THE LOWDOWN

① A **polynomial** equation only has positive integer powers of x.
The **order** of a polynomial is the highest power of x.

Example $f(x) = 2x^3 + 3x^2 - 18x + 8$ has order 3 (a cubic).

② The **factor theorem** says that if $(x - a)$ is a factor of $f(x)$ then $f(a) = 0$ and $x = a$ is a root of the equation $f(x) = 0$.

Example $f(2) = 2 \times 2^3 + 3 \times 2^2 - 18 \times 2 + 8 = 16 + 12 - 36 + 8 = 0$
So $(x - 2)$ is a factor of $f(x) = 2x^3 + 3x^2 - 18x + 8$

③ Use the factor theorem and the grid method to factorise a polynomial.

Example $2x^3 + 3x^2 - 18x + 8 = (x - 2) \times$ (some quadratic)

Step 1 Draw a grid
Write x and -2 on one side and fill in $2x^3$ and 8.

×			
x	$2x^3$		
-2			8

Step 2 What do you multiply x by to get $2x^3$? Answer: $x \times 2x^2 = 2x^3$.
Now complete the 1st and last columns.

×	$2x^2$		-4
x	$2x^3$		$-4x$
-2	$-4x^2$		8

Step 3 You have $-4x^2$ and the equation says $+3x^2$ so you need $+7x^2$.

	$2x^2$		-4
x	$2x^3$	$+7x^2$	$-4x$
-2	$-4x^2$		8

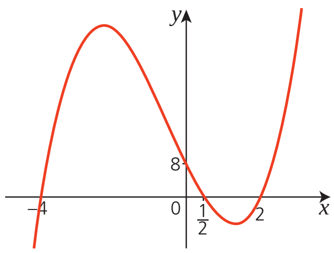
Step 4 What do you multiply x by to get $+7x^2$? Answer: $x \times 7x = 7x^2$.
Now you can complete the grid.

	$2x^2$	$+7x$	-4
x	$2x^3$	$+7x^2$	$-4x$
-2	$-4x^2$	$-14x^2$	$+8$

So $2x^3 + 3x^2 - 18x + 8 = (x - 2)(2x^2 + 7x - 4)$
Factorising the 2nd bracket gives $(x - 2)(2x - 1)(x + 4)$

⑤ Now you can sketch the graph of $y = f(x)$.

Example $y = 2x^3 + 3x^2 - 18x + 8$
When $x = 0$: $y = 8$
Since $y = (x - 2)(2x - 1)(x + 4)$
when $y = 0$: $x = -2$, $x = \frac{1}{2}$, $x = -4$

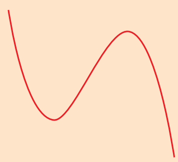


Hint: Any 'whole number' roots will be factors of the constant term.
So you can spot roots by checking factors of +8.

Watch out! The quadratic factor often factorises.

◀◀ See page 8 for **factorising quadratics** and page 4 for **solving simultaneous equations**.

Watch out! A cubic has up to 2 turning points.
If the sign of x^3 is negative the curve is 'upside down' like this:



GET IT RIGHT

$f(x) = 2x^3 + ax^2 + bx - 3$ has a factor of $(x + 3)$ and a root $x = 1$.
Find the values of a and of b.

Solution:
$(x + 3)$ is a factor $\Rightarrow f(-3) = 0 \Rightarrow 2 \times (-3)^3 + a \times (-3)^2 + b \times (-3) - 3 = 0$
$\Rightarrow -54 + 9a - 3b - 3 = 0 \Rightarrow 9a - 3b = 57$
$x = 1$ is a root $\Rightarrow f(1) = 0 \Rightarrow 2 \times 1^3 + a \times 1^2 + b \times 1 - 3 = 0$
$\Rightarrow 2 + a + b - 3 = 0 \Rightarrow a + b = 1$
Using your calculator to solve gives $a = 5$ and $b = -4$

Hint: There are 2 unknowns so you need to form 2 equations and solve them simultaneously.

YOU ARE THE EXAMINER

Which one of these solutions is correct? Where are the errors in the other solution?

a) Show that $(3x - 2)$ is a factor of $f(x) = 3x^3 - 5x^2 - 4x + 4$.

b) Use the factor theorem to find two more factors of $f(x)$.

SAM'S SOLUTION

a) $f\left(\frac{2}{3}\right) = 3 \times \left(\frac{2}{3}\right)^3 - 5 \times \left(\frac{2}{3}\right)^2 - 4 \times \frac{2}{3} + 4$

$= \frac{24}{27} - \frac{20}{9} - \frac{8}{3} + 4 = 0$

So $(3x - 2)$ is a factor.

b) Check factors of 4: $\pm 1, \pm 2, \pm 4$

$f(1) = 3 \times 1 - 5 \times 1 - 4 \times 1 + 4 = -2$

$f(2) = 3 \times 2^3 - 5 \times 2^2 - 4 \times 2 + 4 = 0$

$f(4) = 3 \times 4^3 - 5 \times 4^2 - 4 \times 4 + 4 = 100$

$f(-1) = 3(-1)^3 - 5(-1)^2 - 4(-1) + 4 = 0$

Factors are $(x - 2)$ and $(x + 1)$

LILIA'S SOLUTION

a) $f\left(\frac{2}{3}\right) = 0$

b) Check factors of 4:

$1, 2, 4, -1, -2, -4$

$f(1) = -2$

$f(2) = 0$

$f(4) = 100$

Check negative factors of 4:

$f(-1) = 0$

So the factors are $x = 2$ and $x = -1$

SKILL BUILDER

1 You are given that $f(x) = 4x^3 - x + 3$ and $g(x) = 2x^2 - 3x + 4$.
Which of the following polynomials is $f(x) + g(x)$?

A $6x^3 - 4x + 7$ **B** $4x^3 + 2x^2 - 2x + 7$ **C** $6x^2 - 4x + 7$
D $4x^3 + 2x^2 + 2x + 7$ **E** $4x^3 + 2x^2 - 4x + 7$

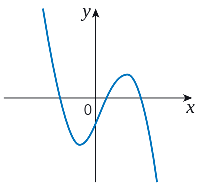
2 You are given $f(x) = 2x^3 - 3$ and $g(x) = 3x^2 + x - 2$. Which of the following is $f(x) \times g(x)$?

A $5x^5 + 2x^4 - 4x^3 - 9x^2 - 3x + 6$ **B** $6x^5 + x^4 - 2x^3 - 9x^2 - 3x + 6$
C $6x^5 + 2x^4 - 4x^3 + 9x^2 + 3x - 6$ **D** $6x^5 + 2x^4 - 4x^3 - 9x^2 - 3x + 6$
E $6x^5 + 2x^4 - 4x^3 - 9x^2 - x - 2$

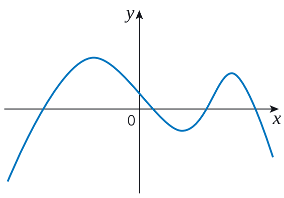
Hint: Use a grid to help you.

3 Which of the following graphs represents the shape of the curve $y = -x^4 + 5x^3 - 5x^2 - 5x + 6$?

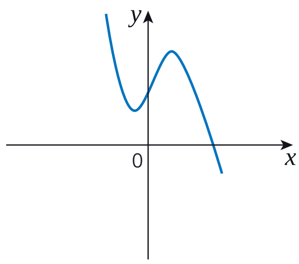
A



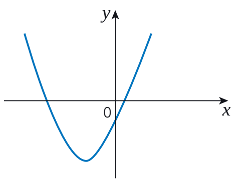
B



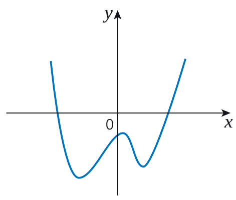
C



D



E



Hint: A polynomial of order n has at most $n - 1$ turning points.

4 You are given that $2x^3 + x^2 - 7x - 6 = (x - 2)(2x^2 + bx + 3)$. The value of b is:

A -3 **B** 5 **C** 3 **D** -5 **E** $-\frac{1}{4}$

5 A polynomial is given by $f(x) = x^3 - 3x^2 - 7x - 15$. Which one of these is a factor of $f(x)$?

A $(x - 1)$ **B** $(x - 3)$ **C** $(x - 5)$ **D** $(x + 1)$ **E** $(x + 3)$

6 **a)** Given $(2x - 1)$ is a factor of $f(x) = 2x^3 - 3x^2 + kx + 6$, find the value of k.
b) Factorise $f(x)$ completely.
c) Sketch the graph of $y = 2x^3 - 3x^2 + kx + 6$.

Master essential skills and boost progress in A-level Maths with extra support and practice for every topic.

This book will guide you through the course with the 'lowdown' on each topic, followed by 'get it right' worked examples and 'skills builder' practice questions, so that you can tackle A-level Maths with confidence.

- Learn to apply GCSE knowledge to A-level concepts by recapping essential points, studying worked examples and applying prior knowledge to practice questions
- Check and consolidate understanding using skills-focused and problem-solving questions as well as online multiple-choice questions with fully explained solutions
- Develop awareness of potential misunderstandings with 'be the examiner' exercises
- Identify and avoid common mistakes using the worked solutions in the back of the book, together with details on misconceptions
- Develop advanced calculator skills with links to extra online material designed to enhance understanding

HODDER EDUCATION
t: 01235 827827
e: education@bookpoint.co.uk
w: hoddereducation.co.uk

ISBN 978-1-5104-8240-1