Sample

WJEC LEVEL 2 CERTIFICATE IN ADDITIONAL MATHEMATICS

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Contents

Contents

Introduction

This book has been written to support the WJEC Level 2 Certificate in Additional Mathematics, but you may also use it independently as an introduction to Mathematics beyond GCSE. It is expected that many of the students using this book could be working with little day-today teacher support, and with this in mind, the text has been written in an interactive way and the answers are fuller than is often the case in books of this nature.

The qualification is designed for high-achieving students who have already acquired, or are expected to achieve, grades B to A* in GCSE Mathematics. It is hoped that many of these students will progress to study Mathematics at Advanced Level and beyond.

Higher-order mathematical skills are studied in greater depth with an emphasis on algebraic reasoning, rigorous argument and problem-solving skills. Students following this course will be well prepared to tackle a Level 3 Mathematics qualification.

The content is split into algebra, geometry (including mensuration and trigonometry) and calculus with each section containing work that stretches and challenges, and which goes beyond the GCSE Programme of Study. The topics are frequently linked together as progress is made through the book, highlighting the beauty and inter-connectedness of mathematics.

Each chapter begins with a quote, designed to engage and bring the topic to life and/or provide an alternative viewpoint. The chapters are then broken down into sub-sections, each with a short introduction followed by a number of worked examples (with solutions) covering important techniques and question styles, and finally, one or more sets of exercise questions. Coloured boxes, hints and notes help to clarify some of the key points.

In addition, each chapter includes a number of activities. These are often used to introduce a new concept, or to reinforce the examples in the text. Throughout the book, the emphasis is on understanding the mathematics being used, rather than merely being able to perform the calculations, but the exercises do, nonetheless, provide plenty of scope for practising basic techniques.

Four symbols are used throughout the book:

! This 'warning sign' alerts you either to restrictions that need to be imposed or to possible pitfalls.

SA This indicates a question requiring selection and application of mathematical methods. These questions will sometimes involve more than one topic area.

IR This indicates a question requiring interpretation and reasoning to solve a mathematical problem. These questions will also sometimes involve more than one topic area.

RWC This indicates a question that relates to real-world contexts.

Numerous 'Discussion points' are used throughout as prompts to help you understand the theory that has been, or is about to be, introduced. Answers to these are also included.

'Prior knowledge' boxes highlight the GCSE Mathematics, or content earlier in the book, that you should be familiar with before you tackle a topic.

'Future uses' sections explain how the mathematics covered in a chapter can be used for further study, including at later points in the book, while 'Real-world contexts' explain the applications of the mathematics covered in each chapter. Also at the end of each chapter you will find a list of learning outcomes and key points.

A short glossary of key words is provided following the two practice question papers. Answers to these, all exercise questions, activities and discussion points are then given at the back of the book.

It is hoped that students who use this book will develop a fascination for mathematics, be inspired and challenged by the rigorous nature of the course and be able to appreciate the power of mathematics for its own sake, as well as a problem-solving tool.

6 Geometry I

The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds.

John Maynard Keynes

Prior knowledge

Much of the work in this chapter will already have been covered in your GCSE studies.

Knowledge of geometry topics will be needed within many sections of the specification.

This section provides a summary of the main facts that are required.

1 Pythagoras' theorem

$c^2 = a^2 + b^2$

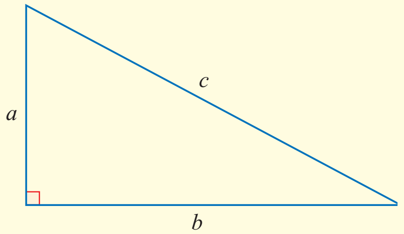


Figure 6.1

Links to GCSE content, to familarise students with concepts and topics

Thought-provoking and challenging activities throughout, to encourage a deeper understanding of each topic

ACTIVITY 6.1

Write down the square of all the integers from 1 to 25 inclusive.
Check that $5^2 = 3^2 + 4^2$.

Write down as many other examples of $c^2 = a^2 + b^2$ as you can find.

How is each set of a, b and c linked to a right-angled triangle?

Pythagorean triples

The following are all Pythagorean triples as each set of three numbers satisfies $c^2 = a^2 + b^2$.

3, 4, 5   5, 12, 13   8, 15, 17   7, 24, 25

Using similar triangles, any multiple or fraction of each set will also be a Pythagorean triple.

For example,   9, 12, 15   2.5, 6, 6.5   16, 30, 34   1.4, 4.8, 5

2 Trigonometry in two dimensions

You have met definitions of the three trigonometric functions, sin, cos and tan, using the sides of a right-angled triangle.

sin is an abbreviation of sine, cos of cosine and tan of tangent.

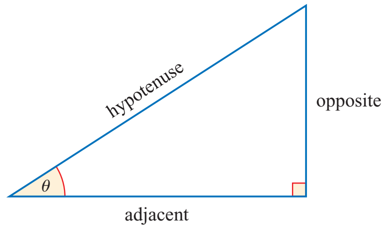


Figure 6.2

Discussion point

→ Do these definitions work for angles of any size?

In Figure 6.2

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

ACTIVITY 6.2

(i) Using only a pencil, ruler and protractor, estimate $\sin 62°$.
(ii) Use your calculator to check your percentage error.
(iii) Suggest a way of reducing the percentage error when using this method.

Example 6.1

Work out the length of the side marked a in the triangle in Figure 6.3.

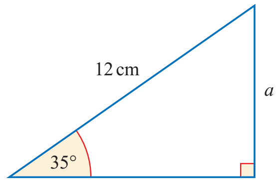


Figure 6.3

Discussion points to help stretch and challenge students

Solution

Side a is *opposite* the angle of 35°, and the *hypotenuse* is 12 cm, so we use sin 35°.

$$\sin 35° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{a}{12}$$

$$\Rightarrow \quad a = 12 \sin 35°$$

$$\Rightarrow \quad a = 6.9\,\text{cm (1 d.p.)}$$

Example 6.2

RWC

The diagram represents a ladder leaning against a wall.

Work out the length of the ladder.

Give your answer to 3 significant figures.

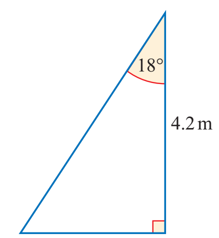


Figure 6.4

Questions that relate to real-world contexts, easily highlighted with the RWC callout

Lots of examples and step-by-step worked solutions to help develop problem-solving skills

Solution

The side of length 4.2 m is *adjacent* to the angle of 18°, and we want the *hypotenuse* so use cos 18°.

$$\cos 18° = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{4.2}{\text{hypotenuse}}$$

$$\text{hypotenuse} = \frac{4.2}{\cos 18°}$$

$$= 4.42\,\text{m (3 s.f.)}$$

Example 6.3

Work out the size of the angle marked θ in the triangle in Figure 6.5.

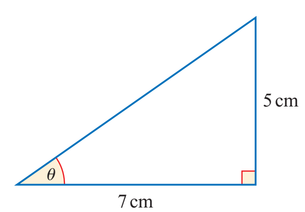


Figure 6.5

Discussion point

➜ The full calculator value for $\frac{5}{7}$ has been used to work out the value of θ. What is the least number of decimal places that you could use to give the same value for the angle (to 1 d.p.) in this example?

Solution

The sides whose lengths are known are those *opposite* and *adjacent* to θ so we use tan θ.

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{7}$$

$$\Rightarrow \quad \theta = 35.5° \text{ (1 d.p.)}$$

Note panels provide additional useful information

Key things to remember highlighted throughout for easy reference

Example 6.4

RWC

A bird flies straight from the top of a 15 m tall tree, at an angle of depression of 27°, to catch a worm on the ground.

(i) How far does the bird fly?

(ii) How far was the worm from the bottom of the tree?

Note

Questions involving right-angled triangles will often entail applying trigonometry in a context. Examples and exercises include some questions without a context to provide practice of the skills needed in applications questions.

Solution

First draw a sketch, labelling the information given and using letters to mark what you want to find.

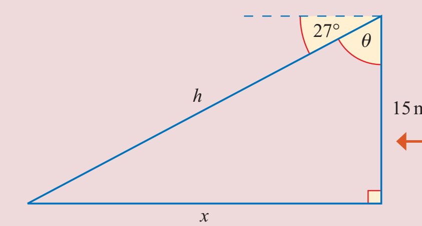


Figure 6.6

Remember, *angles of depression* are measured <u>down</u> from the horizontal and angles of elevation are measured <u>up</u> from the horizontal.

(i) $\theta + 27° = 90°$

$$\Rightarrow \quad \theta = 63°$$

$$\cos 63° = \frac{15}{h}$$

$$\Rightarrow \quad h = \frac{15}{\cos 63°} = 33.040\,338\,97$$

The bird flies 33 m.

! Make sure that you record the full calculator value of h for future use.

(ii) Using Pythagoras' theorem

$$h^2 = x^2 + 15^2$$

$$\Rightarrow \quad x^2 = 33.040\,338\,97^2 - 15^2 = 866.663\,999$$

$$\Rightarrow \quad x = 29.439\,157\,58$$

The worm is 29.4 m from the bottom of the tree.

Discussion point

➜ If you used trigonometry for part (ii) of this question, which would be the best function to use? Why?

Warning signs that alert students to either restrictions that need to be imposed, or possible pitfalls

Historical note

The word for trigonometry is derived from three Greek words.

Tria: *three* gonia: *angle* metron: *measure*

(τρια) (γονια) (μετρον)

This shows how trigonometry developed from studying angles, often in connection with astronomy, although the subject was probably discovered independently by a number of people. Hipparchus (150 BC) is believed to have produced the first trigonometric tables which gave lengths of chords of a circle of unit radius. His work was further developed by Ptolemy in AD100.

Interesting facts and historical notes, to help place mathematical concepts into a wider, real-world context

Exercises on each topic to help develop skills and boost confidence

Exercise 6A

① Work out the length marked x in each of these triangles. Give your answers correct to 1 decimal place.

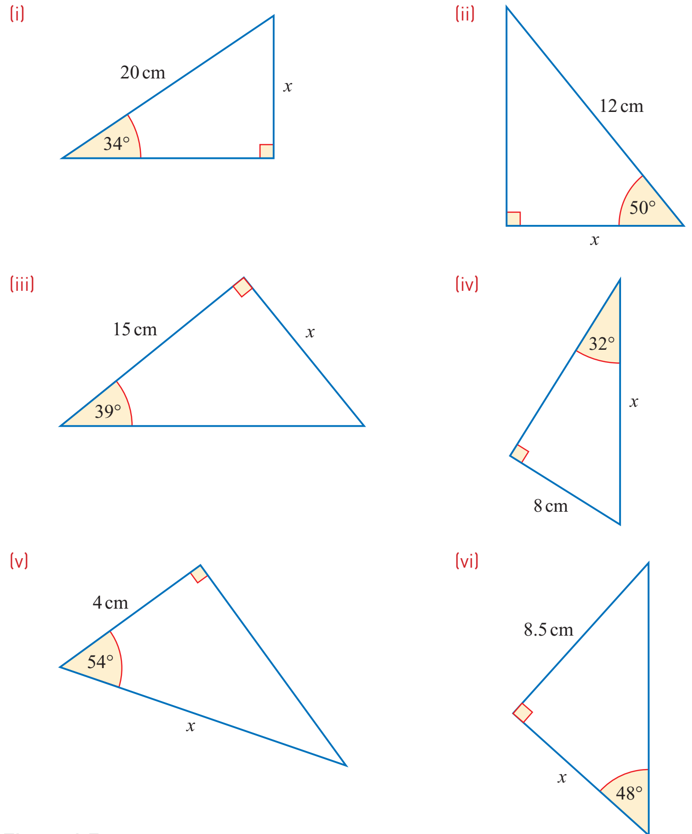


Figure 6.7

② Work out the size of the angle marked θ in each of these triangles. Give your answers correct to 1 decimal place.

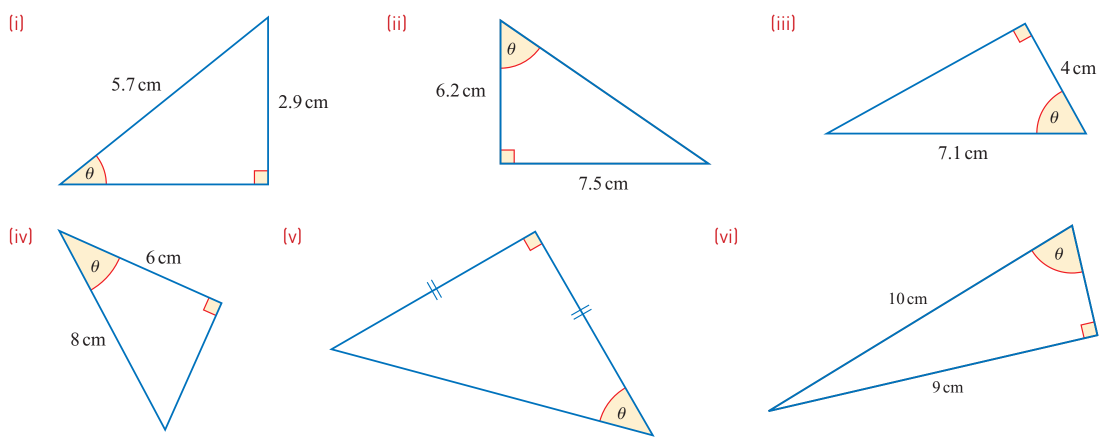


Figure 6.8

Exercise 8G

If you have access to a graphic calculator you will find it helpful to use it to check your answers.

① For each of the curves given below

(a) work out $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$

(b) work out the value(s) of $\frac{d^2y}{dx^2}$ at those points

(c) classify the point(s) on the curve with these x-values

(d) work out the corresponding y-value(s)

(e) sketch the curve.

(i) $y = 1 + x - 2x^2$ (ii) $y = 12x + 3x^2 - 2x^3$

(iii) $y = x^3 - 4x^2 + 9$ (iv) $y = x(x-1)^2$

(v) $y = x^2(x-1)^2$ (vi) $y = x^3 - 48x$

(vii) $y = x^3 + 6x^2 - 36x + 25$ (viii) $y = 2x^3 - 15x^2 + 24x + 8$

② The graph of $y = px + qx^2$ passes through the point $(3, -15)$ and its gradient at that point is -14.

(i) Work out the values of p and q.

(ii) Calculate the maximum value of y and state the value of x at which it occurs.

③ (i) Identify the stationary points of the function $f(x) = x^2(3x^2 - 2x - 3)$ and distinguish between them.

(ii) Sketch the curve $y = f(x)$.

④ The curve $y = ax^2 + bx + c$ crosses the y-axis at the point $(0, 2)$ and has a minimum point at $(3, 1)$.

(i) Work out the equation of the curve.

(ii) Check that the stationary point is a minimum.

FUTURE USES

- This work will be extended if you study Mathematics at a higher level.
- At A-Level you will learn additional formulae to deal with more complex algebraic products and quotients.
- There are also applications in other subjects, for example, Kinematics, Physics and Economics.

REAL-WORLD CONTEXT

Differentiation is used in the study of motion.
It is also the basis of differential equations which can be used to solve problems involving growth and decay.
Navier-Stokes equations, which are a particular form of differential equation, are vital to video-gaming and also help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution and many other things.

The 'future uses' note explains how the mathematics covered in this chapter can be used for further study

'Real-world contexts' explain the applications of the mathematics covered in each chapter

Example 4.15

(i) Find the remainder when $(x^2 - 4x + 7)$ is divided by $(x + 6)$.

(ii) Find the remainder when $(x^3 + 5x - 7)$ is divided by $(2x - 1)$.

Solution

(i) Substitute $x = -6$ into $x^2 - 4x + 7$.

The remainder is $(-6)^2 - 4 \times (-6) + 7 = 36 + 24 + 7 = 67$.

(ii) Substitute $x = \frac{1}{2}$ into $x^3 + 5x - 7$.

The remainder is $\left(\frac{1}{2}\right)^3 + 5 \times \left(\frac{1}{2}\right) - 7 = \frac{1}{8} + \frac{5}{2} - 7 = -\frac{35}{8}$.

This value of x is the solution of $2x - 1 = 0$.

Exercise 4C

① Find the remainder when $(x^2 + 4x - 9)$ is divided by $(x - 2)$.

② Find the remainders when each of these polynomials, $\mathrm{P}(x)$, is divided by the corresponding linear expression.

(i)	$\mathrm{P}(x) = x^2 - 2x + 2$	$(x - 5)$
(ii)	$\mathrm{P}(x) = x^3 + x + 6$	$(x + 1)$
(iii)	$\mathrm{P}(x) = x^3 + x^2 - 7x - 6$	$(x - 4)$
(iv)	$\mathrm{P}(x) = x^2 - 3x - 2$	$(2x - 1)$
(v)	$\mathrm{P}(t) = t^4 - 2t^3 - 1$	$(2t + 3)$
(vi)	$\mathrm{P}(c) = c^5 + 3c^4 - 2c + 1$	$(3c - 1)$

SA ③ When $(x^3 + ax - 3)$ is divided by $(x + 2)$ the remainder is 7.
Find the value of a.

SA ④ When $(x^4 - bx^2 + x - 2)$ is divided by $(x + 3)$ the remainder is -5.
Find the remainder when $(x^4 - bx^2 + x - 2)$ is divided by $(x - 2)$.

IR ⑤ $\mathrm{P}(x) = x^3 + cx + d$

When $\mathrm{P}(x)$ is divided by $(x - 1)$ the remainder is -3.

When $\mathrm{P}(x)$ is divided by $(x + 2)$ the remainder is 3.

Find the values of c and d.

The SA symbols indicates a question that requires selection and application of mathematical methods

The IR symbol indicates a question that requires interpretation and reasoning to solve the problem

4 The factor theorem

Prior knowledge

You should be familiar with function notation from your GCSE work.

The highest power in a quadratic is 2. Cubic expressions go up to 3, quartics to 4, quintics to 5, and so on. Such expressions are collectively referred to as polynomials. The degree of a polynomial is its highest power.

Note: a polynomial does not have negative or non-integer powers.

Just like the quadratic formula, there are formulae for solving cubic equations and quartic equations.

Extra notes to reassure students or highlight key information

You should not attempt to learn this formula. It is included here for interest only.

The formula for solving the cubic equation $ax^3 + bx^2 + cx + d = 0$ is

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

Clearly this is not a practical formula to use without a pre-programmed calculator, or a computer. The quartic formula is even more complicated, and too long to include here. Interestingly, it has been proved to be impossible to write a quintic formula.

In this course, you will only be asked to solve polynomial equations which can be reduced to linear and/or quadratic factors.

FUTURE USES

Some calculators can solve complicated equations. Such calculators use the Newton–Raphson method, which is a technique that students of Mathematics at A-Level will learn. The above formula sometimes involves the use of imaginary numbers even if the final answers are not themselves imaginary. Students of A-Level Further Mathematics will learn about imaginary numbers (square roots of negative numbers).

Solving polynomial equations first involves use of the factor theorem.

Look at this quadratic equation.

$$x^2 - 5x - 6 = 0$$

Factorising $\Rightarrow \quad (x - 6)(x + 1) = 0$

$\Rightarrow \quad (x - 6) = 0$ or $(x + 1) = 0$

$\Rightarrow \quad x = 6$ or $x = -1$

Discussion points

→ What happens if you substitute $x = 6$ into $x^2 - 5x - 6$?

→ What about $x = -1$?

The *factor theorem* states this result in a general form:

If $(x - a)$ is a factor of the polynomial $\mathrm{f}(x)$, then

- $\mathrm{f}(a) = 0$
- $x = a$ is a root of the equation $\mathrm{f}(x) = 0$.

Conversely, if $\mathrm{f}(a) = 0$, then $(x - a)$ is a factor of $\mathrm{f}(x)$.

Example 4.16

Given that

$\mathrm{f}(x) = x^3 + 2x^2 - x - 2$

(i) find $\mathrm{f}(1)$, $\mathrm{f}(-1)$, $\mathrm{f}(2)$, $\mathrm{f}(-2)$

(ii) and hence factorise $x^3 + 2x^2 - x - 2$.

Check progress using a checklist of learning outcomes and key points at the end of each chapter

LEARNING OUTCOMES

Now you have finished the chapter, you should be able to

- differentiate positive and negative powers of a variable such as x
- differentiate sums and differences of functions of x
- differentiate functions of x that first need expanding or dividing
- use differentiation to work out the gradient of a curve
- use this information to identify stationary points on a curve
- derive the equation of a tangent to a curve
- identify when a function is increasing and when it is decreasing
- calculate the position of any stationary points on the curve
- use the second derivative to determine the nature of any stationary points.

KEY POINTS

1 $y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}$

$y = c \quad \Rightarrow \quad \frac{dy}{dx} = 0$

where n is a positive integer and k and c are constants.

2 $y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x)$.

3 For the tangent and normal at (x_1, y_1)

- the gradient of the tangent, m_1 = the value of $\frac{dy}{dx}$
- the equation of the tangent is $y - y_1 = m_1(x - x_1)$.

4 A function $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$.

A function $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$.

5 The second derivative is obtained by differentiating $\frac{dy}{dx}$ and is denoted by $\frac{d^2y}{dx^2}$.

6 At a stationary point, $\frac{dy}{dx} = 0$.

The nature of the stationary point can be determined by looking at the sign of the gradient just either side of it.

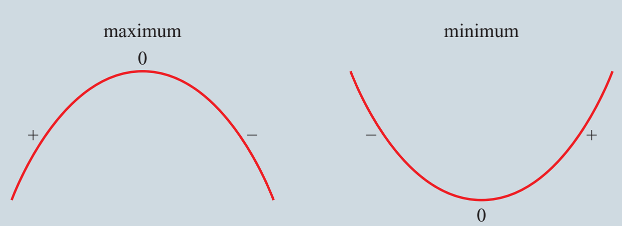


Figure 8.26

Help students to achieve their potential and prepare for assessment with two practice papers at the end of the book

Practice questions Paper 1

① Find $\frac{dy}{dx}$ for each of the following.

(a) $y = 3x^2 - 4x + 8$ **[3 marks]**

(b) $y = \frac{5}{x^3}$ **[1 mark]**

(c) $y = x^{\frac{3}{7}}$ **[1 mark]**

② Showing all your working, simplify each of the following.

(a) $\frac{3x^{\frac{3}{5}} \times 2x^{\frac{4}{5}}}{x^{\frac{3}{10}}}$ **[2 marks]**

(b) $\frac{4y^{\frac{1}{6}} + y^{\frac{5}{6}}}{6y^{\frac{1}{6}}}$ **[2 marks]**

③ Without using a calculator, find the value of $(9^{\frac{1}{3}})^6$.
Show all your working. **[1 mark]**

④ Prove that $\frac{x+1}{6} + \frac{3x}{4} - \frac{2x-3}{18} \equiv \frac{29x+12}{36}$. **[4 marks]**

⑤ (i) Factorise $16x^2 + 2x - 5$. **[2 marks]**

(ii) **Hence** solve the equation $16x^2 + 2x - 5 = 0$. **[2 marks]**

⑥ (i) Write $x^2 + 8x + 13$ in the form $(x + a)^2 + b$. **[2 marks]**

(ii) **Hence** write down the least value of $x^2 + 8x + 13$. **[1 mark]**

(iii) What is the value of x when $x^2 + 8x + 13$ takes its least value? **[1 mark]**

⑦ Given that $y = x^2 + 5x$, find $\frac{dy}{dx}$ from first principles. **[5 marks]**

⑧ (a) Find the remainder when $x^3 + 9x^2 + 23x + 15$ is divided by $x - 2$. **[2 marks]**

(b) (i) Show that $x + 1$ is a factor of $x^3 + 9x^2 + 23x + 15$. **[2 marks]**

(ii) **Hence** factorise $x^3 + 9x^2 + 23x + 15$. **[4 marks]**

⑨ Find the coordinates and nature of each of the stationary points on the curve $y = x^3 - 27x + 8$.
You must show all your working. **[7 marks]**

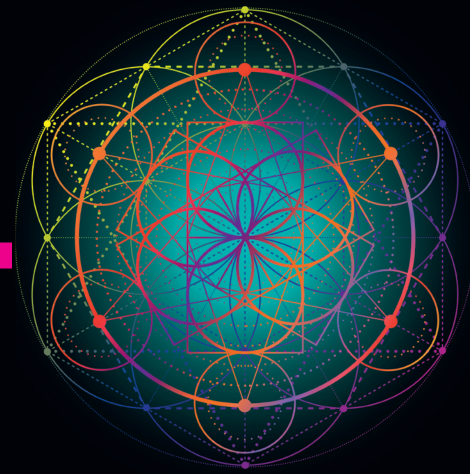
⑩ **Without using a calculator**, write $\frac{2}{5 - \sqrt{3}}$ in the form $\frac{a + \sqrt{b}}{c}$ where a, b and c are integers.
You **must** show all your working. **[3 marks]**

⑪ Showing all your working, find the coordinates of the points of intersection of the curve $y = x^2 - 6x + 3$ and the line $y = x + 11$. **[4 marks]**

Practice questions **Paper 1**

WJEC LEVEL 2

CERTIFICATE IN ADDITIONAL MATHEMATICS



Stretch and challenge students with this introduction to higher level mathematics.

Including plenty of practice activities and worked examples, this book bridges the gap from GCSE to A-level, building reasoning and problem-solving skills in preparation for the next step.

- Develop understanding of mathematics with discussion points, thought-provoking activities and rigorous exercise questions
- Tackle new concepts confidently using the notes on how prior knowledge can be applied
- Develop problem-solving skills with step-by-step worked examples and practice questions
- Be aware of potential misunderstandings with common pitfalls noted throughout the text
- Check progress using a checklist of key points and learning outcomes at the end of each chapter
- Get ready for the exams with two practice papers at the end of the book

Answers to all exercise questions and full worked solutions to practice paper questions will be at the back of the book. They will also be available online

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