

Answers

1 Further algebra

1.1 The general binomial expansion (Page 2)

1 (i) $(1+x)^{-2}$

$$= 1 - 2x + \frac{-2(-2-1)}{2!}x^2 + \frac{-2(-2-1)(-2-2)}{3!}x^3 + \dots$$

$$= 1 - 2x + \frac{-2(-3)}{2!}x^2 + \frac{-2(-3)(-4)}{3!}x^3 + \dots$$

$$= 1 - 2x + 3x^2 - 4x^3 + \dots$$

Valid for $|x| < 1$

(ii) $(1+4x)^{\frac{1}{2}}$

$$= 1 + \frac{1}{2}(4x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(4x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(4x)^3 + \dots$$

$$= 1 + 2x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(16x^2) + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(64x^3) + \dots$$

$$= 1 + 2x - 2x^2 + 4x^3 - \dots$$

$$= 1 + 2x - 2x^2 + 4x^3 - \dots$$

Valid for $|4x| < 1 \Rightarrow |x| < \frac{1}{4}$

2 (i) $(2+x)^{-4}$

$$= \left[2\left(1 + \frac{1}{2}x\right) \right]^{-4}$$

$$= 2^{-4} \left(1 + \frac{1}{2}x\right)^{-4}$$

$$= \frac{1}{16} \left(1 + \frac{1}{2}x\right)^{-4}$$

$$= \frac{1}{16} \left(1 - 4\left(\frac{1}{2}x\right) + \frac{-4(-4-1)}{2!}\left(\frac{1}{2}x\right)^2 + \dots\right)$$

$$= \frac{1}{16} \left(1 - 2x + \frac{-4(-5)}{2!}\left(\frac{1}{4}x^2\right) + \dots\right)$$

$$= \frac{1}{16} \left(1 - 2x + \frac{5}{2}x^2 + \dots\right)$$

$$= \frac{1}{16} - \frac{1}{8}x + \frac{5}{32}x^2 - \dots$$

Valid for $\left|\frac{1}{2}x\right| < 1 \Rightarrow |x| < 2$

(ii) $\frac{1}{\sqrt{4-x}}$

$$= (4-x)^{-\frac{1}{2}}$$

$$= \left[4\left(1 - \frac{1}{4}x\right) \right]^{-\frac{1}{2}}$$

$$= 4^{-\frac{1}{2}} \left(1 - \frac{1}{4}x\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2} \left(1 - \frac{1}{4}x\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2} \left(1 + -\frac{1}{2}\left(-\frac{1}{4}x\right) + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!}\left(-\frac{1}{4}x\right)^2 + \dots\right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{8}x + \frac{-\frac{1}{2}(-\frac{3}{2})}{2!}\left(\frac{1}{16}x^2\right) + \dots\right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{8}x + \frac{3}{128}x^2 + \dots\right)$$

$$= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \dots$$

Valid for $\left|\frac{1}{4}x\right| < 1 \Rightarrow |x| < 4$

3 (i) $(1+ax)^b$

$$= 1 + b(ax) + \frac{b(b-1)}{2!}(ax)^2 + \dots$$

$$= 1 + abx + \frac{b(b-1)}{2!}(a^2)x^2 + \dots$$

$$ab = -10 \Rightarrow a = -\frac{10}{b}$$

$$\frac{a^2b(b-1)}{2} = 75 \Rightarrow a^2b(b-1) = 150$$

Substituting for a ,

$$\left(-\frac{10}{b}\right)^2 b(b-1) = 150$$

$$\left(\frac{100}{b^2}\right) b(b-1) = 150$$

$$100b(b-1) = 150b^2$$

$$100b^2 - 100b = 150b^2$$

$$0 = 50b^2 + 100b$$

$$0 = 50b(b+2)$$

$$b \neq 0 \text{ so } b = -2$$

$$a = -\frac{10}{b} = -\frac{10}{-2} = 5$$

(ii) Valid for $|5x| < 1 \Rightarrow |x| < \frac{1}{5}$

$$\begin{aligned}
 4 \quad (i) \quad & \frac{1}{\sqrt{1-2x^2}} \\
 &= (1-2x^2)^{-\frac{1}{2}} \\
 &= 1 + \frac{-1}{2}(-2x^2) + \frac{-1\left(-\frac{1}{2}-1\right)}{2!}(-2x^2)^2 + \dots \\
 &= 1 + x^2 + \frac{3}{2}x^4 + \dots
 \end{aligned}$$

$$\text{Valid for } |2x^2| < 1 \Rightarrow |x| < \sqrt{\frac{1}{2}}$$

$$\begin{aligned}
 (ii) \quad & \frac{1+x}{\sqrt{1-2x^2}} \\
 &= (1+x)\left(1+x^2+\frac{3}{2}x^4+\dots\right) \\
 &= 1+x^2+\frac{3}{2}x^4+x+x^3+\frac{3}{2}x^5+\dots \\
 &= 1+x+x^2+x^3+\frac{3}{2}x^4+\frac{3}{2}x^5+\dots
 \end{aligned}$$

$$\begin{aligned}
 5 \quad (i) \quad & (1-3x)^n \\
 &= 1+n(-3x)+\frac{n(n-1)}{2!}(-3x)^2+\dots \\
 &= 1-3nx+\frac{n^2-n}{2}(9x^2)+\dots \\
 &= 1-3nx+\frac{9(n^2-n)}{2}x^2+\dots \\
 &-3n=\frac{9(n^2-n)}{2} \\
 &-6n=9(n^2-n) \\
 &-6n=9n^2-9n \\
 &0=9n^2-3n \\
 &0=3n(3n-1) \\
 &n \neq 0 \text{ so } n=\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad & (1-3x)^{\frac{1}{3}} \\
 &= 1+\frac{1}{3}(-3x)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-3x)^2 \\
 &\quad +\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-3x)^3+\dots \\
 &= 1-x+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(9x^2)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(-27x^3)+\dots \\
 &= 1-x-x^2-\frac{5}{3}x^3-\dots
 \end{aligned}$$

$$\begin{aligned}
 6 \quad (i) \quad & (1-3x)(1+ax)^{-2} \\
 &= (1-3x)\left[1-2(ax)+\frac{-2(-2-1)}{2!}(ax)^2+\dots\right] \\
 &= (1-3x)\left[1-2ax+3a^2x^2+\dots\right] \\
 &\text{Just considering the } x^2 \text{ terms,} \\
 &3a^2x^2-3x \times -2ax \\
 &= 3a^2x^2+6ax^2 \\
 &= (3a^2+6a)x^2
 \end{aligned}$$

$$\text{Coefficient of } x^2 = 0$$

$$\Rightarrow 3a^2+6a=0$$

$$3a(a+2)=0$$

$$a \neq 0 \text{ so } a=-2$$

$$\begin{aligned}
 (ii) \quad & (1-3x)(1-2x)^{-2} \\
 &= (1-3x)\left[1-2(-2x)+\frac{-2(-2-1)}{2!}(-2x)^2\right. \\
 &\quad \left.+\frac{-2(-2-1)(-2-2)}{3!}(-2x)^3+\dots\right] \\
 &= (1-3x)\left[1+4x+12x^2+32x^3+\dots\right]
 \end{aligned}$$

The x^3 term is given by

$$\begin{aligned}
 &1 \times 32x^3 - 3x \times 12x^2 \\
 &= -4x^3
 \end{aligned}$$

$$\begin{aligned}
 7 \quad (i) \quad & (a+x)^{-2} \\
 &= \left[a\left(1+\frac{1}{a}x\right)\right]^{-2} \\
 &= a^{-2}\left(1+\frac{1}{a}x\right)^{-2} \\
 &= \frac{1}{a^2}\left(1+\frac{1}{a}x\right)^{-2} \\
 &= \frac{1}{a^2}\left[1-2\left(\frac{1}{a}x\right)+\frac{-2(-2-1)}{2!}\left(\frac{1}{a}x\right)^2+\dots\right] \\
 &= \frac{1}{a^2}\left[1-\frac{2}{a}x+\frac{3}{a^2}x^2+\dots\right] \\
 &= \frac{1}{a^2}-\frac{2}{a^3}x+\frac{3}{a^4}x^2+\dots
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad & (1-x)(a+x)^{-2} \\
 &= (1-x)\left(\frac{1}{a^2}-\frac{2}{a^3}x+\frac{3}{a^4}x^2+\dots\right)
 \end{aligned}$$

Considering the x^2 terms,

$$\begin{aligned}
 &1 \times \frac{3}{a^4}x^2 - x \times -\frac{2}{a^3}x \\
 &= \frac{3}{a^4}x^2 + \frac{2}{a^3}x^2 \\
 &= \left(\frac{3}{a^4} + \frac{2}{a^3}\right)x^2 \\
 &\frac{3}{a^4} + \frac{2}{a^3} = 0 \\
 &3+2a=0 \\
 &a=-\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 8 \quad (i) \quad & y = \frac{1}{\sqrt{1-2x}-\sqrt{1-x}} \\
 &= \frac{1}{\sqrt{1-2x}-\sqrt{1-x}} \times \frac{\sqrt{1-2x}+\sqrt{1-x}}{\sqrt{1-2x}+\sqrt{1-x}} \\
 &= \frac{\sqrt{1-2x}+\sqrt{1-x}}{1-2x-(1-x)} \\
 &= \frac{\sqrt{1-2x}+\sqrt{1-x}}{-x} \\
 &= -\frac{1}{x}(\sqrt{1-2x}+\sqrt{1-x})
 \end{aligned}$$

$$(ii) y = -\frac{1}{x}(\sqrt{1-2x} + \sqrt{1-x})$$

$$\sqrt{1-2x}$$

$$= (1-2x)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-2x)^2 + \dots$$

$$= 1 - x - \frac{1}{2}x^2 - \dots$$

$$\sqrt{1-x}$$

$$= (1-x)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-x)^2 + \dots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \dots$$

$$y = -\frac{1}{x}\left(1 - x - \frac{1}{2}x^2 + 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \dots\right)$$

$$= -\frac{1}{x}\left(2 - \frac{3}{2}x - \frac{5}{8}x^2 - \dots\right)$$

$$= -\frac{2}{x} + \frac{3}{2} + \frac{5}{8}x + \dots$$

Coefficient of the x term is $\frac{5}{8}$

1.2 Review of algebraic fractions (Page 6)

$$1 \quad (i) \quad \frac{3a}{4b} \times \frac{8b^2}{9a^2} = \frac{2b}{3a}$$

$$(ii) \quad \frac{f^2 - 16}{f^2 - 6f + 8} = \frac{(f-4)(f+4)}{(f-4)(f-2)} = \frac{f+4}{f-2}$$

$$(iii) \quad \frac{h^3 - 4h}{4} \div \frac{h^2 - 4h + 4}{8}$$

$$= \frac{h^3 - 4h}{4} \times \frac{8}{h^2 - 4h + 4}$$

$$= \frac{h(h^2 - 4)}{4} \times \frac{8}{(h-2)(h-2)}$$

$$= \frac{h(h-2)(h+2)}{4} \times \frac{8}{(h-2)(h-2)}$$

$$= \frac{2h(h+2)}{h-2}$$

$$2 \quad (i) \quad \frac{2}{m} + \frac{3}{4m} = \frac{8}{4m} + \frac{3}{4m} = \frac{11}{4m}$$

$$(ii) \quad \frac{p-2}{p} + \frac{p+2}{3}$$

$$= \frac{3(p-2)}{3p} + \frac{p(p+2)}{3p}$$

$$= \frac{3p-6}{3p} + \frac{p^2+2p}{3p}$$

$$= \frac{p^2+5p-6}{3p}$$

$$(iii) \quad \frac{r+5}{4r^2} + \frac{3}{5r} - \frac{2}{r^3}$$

$$= \frac{5r(r+5) + 3(4r^2) - 2(20)}{20r^3}$$

$$= \frac{5r^2 + 25r + 12r^2 - 40}{20r^3}$$

$$= \frac{17r^2 + 25r - 40}{20r^3}$$

1.3 Partial fractions (Page 7)

$$1 \quad (i) \quad \frac{5x+7}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$$

$$5x+7 \equiv A(x+2) + B(x+1)$$

$$x = -1 \Rightarrow 2 = A$$

$$x = -2 \Rightarrow -3 = -B \Rightarrow B = 3$$

$$\text{So } \frac{5x+7}{(x+1)(x+2)} \equiv \frac{2}{x+1} + \frac{3}{x+2}$$

$$(ii) \quad \frac{3x-2}{x^2-4} = \frac{3x-2}{(x-2)(x+2)}$$

$$\frac{3x-2}{(x-2)(x+2)} \equiv \frac{A}{x-2} + \frac{B}{x+2}$$

$$3x-2 \equiv A(x+2) + B(x-2)$$

$$x = 2 \Rightarrow 4 = 4A \Rightarrow A = 1$$

$$x = -2 \Rightarrow -8 = -4B \Rightarrow B = 2$$

$$\text{So } \frac{3x-2}{(x-2)(x+2)} \equiv \frac{1}{x-2} + \frac{2}{x+2}$$

$$(iii) \quad \frac{4}{x^2-x} = \frac{4}{x(x-1)}$$

$$\frac{4}{x(x-1)} \equiv \frac{A}{x} + \frac{B}{x-1}$$

$$4 \equiv A(x-1) + Bx$$

$$x = 1 \Rightarrow 4 = B$$

$$x = 0 \Rightarrow 4 = -A \Rightarrow A = -4$$

$$\text{So } \frac{4}{x(x-1)} \equiv -\frac{4}{x} + \frac{4}{x-1}$$

$$(iv) \quad \frac{42-18x}{(x+1)(x-2)(x-4)} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-4}$$

$$42-18x \equiv A(x-2)(x-4) + B(x+1)(x-4) + C(x+1)(x-2)$$

$$x = 2 \Rightarrow 6 = -6B \Rightarrow B = -1$$

$$x = -1 \Rightarrow 60 = 15A \Rightarrow A = 4$$

$$x = 4 \Rightarrow -30 = 10C \Rightarrow C = -3$$

$$\text{So } \frac{42-18x}{(x+1)(x-2)(x-4)} \equiv \frac{4}{x+1} - \frac{1}{x-2} - \frac{3}{x-4}$$

$$(v) \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$= 2x + \frac{5x - 3}{(x - 3)(x + 1)}$$

$$\frac{5x - 3}{(x - 3)(x + 1)} \equiv \frac{A}{x - 3} + \frac{B}{x + 1}$$

$$5x - 3 \equiv A(x + 1) + B(x - 3)$$

$$x = 3 \Rightarrow 12 = 4A \Rightarrow A = 3$$

$$x = -1 \Rightarrow -8 = -4B \Rightarrow B = 2$$

$$\text{So } \frac{5x - 3}{(x - 3)(x + 1)} \equiv \frac{3}{x - 3} + \frac{2}{x + 1}$$

$$\text{and } \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{3}{x - 3} + \frac{2}{x + 1}$$

$$2 \quad (i) \quad \frac{x^2 + 3x - 1}{x^2(x - 1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}$$

$$x^2 + 3x - 1 \equiv Ax(x - 1) + B(x - 1) + Cx^2$$

$$x = 1 \Rightarrow 3 = C$$

$$x = 0 \Rightarrow -1 = -B \Rightarrow B = 1$$

$$x^2 \text{ terms} \Rightarrow 1 = A + C \Rightarrow A = -2$$

$$\text{So } \frac{x^2 + 3x - 1}{x^2(x - 1)} \equiv -\frac{2}{x} + \frac{1}{x^2} + \frac{3}{x - 1}$$

$$(ii) \quad \frac{1}{x(x - 1)^2} \equiv \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}$$

$$1 \equiv A(x - 1)^2 + Bx(x - 1) + Cx$$

$$x = 1 \Rightarrow 1 = C$$

$$x = 0 \Rightarrow 1 = A$$

$$x^2 \text{ terms} \Rightarrow 1 = A + B \Rightarrow B = 0$$

$$\text{So } \frac{1}{x(x - 1)^2} \equiv \frac{1}{x} + \frac{0}{x - 1} + \frac{1}{(x - 1)^2}$$

$$= \frac{1}{x} + \frac{1}{(x - 1)^2}$$

$$3 \quad (i) \quad \frac{1}{(x + 1)(x^2 + 1)} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

$$1 \equiv A(x^2 + 1) + (Bx + C)(x + 1)$$

$$x = -1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$x^2 \text{ terms} \Rightarrow 0 = A + B \Rightarrow B = -\frac{1}{2}$$

$$x = 0 \Rightarrow 1 = A + C \Rightarrow C = \frac{1}{2}$$

$$\text{So } \frac{1}{(x + 1)(x^2 + 1)} \equiv \frac{\frac{1}{2}}{x + 1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2 + 1}$$

$$= \frac{1}{2(x + 1)} + \frac{1 - x}{2(x^2 + 1)}$$

$$(ii) \quad \frac{10}{(x - 4)(x^2 + 4)} \equiv \frac{A}{x - 4} + \frac{Bx + C}{x^2 + 4}$$

$$10 \equiv A(x^2 + 4) + (Bx + C)(x - 4)$$

$$x = 4 \Rightarrow 10 = 20A \Rightarrow A = \frac{1}{2}$$

$$x^2 \text{ terms} \Rightarrow 0 = A + B \Rightarrow B = -\frac{1}{2}$$

$$x = 0 \Rightarrow 10 = 4A - 4C \Rightarrow C = -2$$

$$\text{So } \frac{10}{(x - 4)(x^2 + 4)} \equiv \frac{\frac{1}{2}}{x - 4} + \frac{-\frac{1}{2}x - 2}{x^2 + 4}$$

$$= \frac{1}{2(x - 4)} - \frac{x + 4}{2(x^2 + 4)}$$

1.4 Using partial fractions with the binomial expansion (Page 8)

$$1 \quad (i) \quad \frac{2 - 11x}{(1 + 2x)(2 - x)} = \frac{A}{1 + 2x} + \frac{B}{2 - x}$$

$$2 - 11x \equiv A(2 - x) + B(1 + 2x)$$

$$x = 2 \Rightarrow -20 = 5B \Rightarrow B = -4$$

$$x = -\frac{1}{2} \Rightarrow \frac{15}{2} = \frac{5}{2}A \Rightarrow A = 3$$

$$\text{So } \frac{2 - 11x}{(1 + 2x)(2 - x)} = \frac{3}{1 + 2x} - \frac{4}{2 - x}$$

$$(ii) \quad \frac{1}{1 + 2x} = (1 + 2x)^{-1}$$

$$= 1 + -1(2x) + \frac{-1(-1-1)}{2!}(2x)^2 + \dots$$

$$= 1 - 2x + 4x^2 - \dots$$

$$\text{Valid for } |2x| < 1 \Rightarrow |x| < \frac{1}{2}.$$

$$(iii) \quad \frac{1}{2 - x} = (2 - x)^{-1}$$

$$= \left[2 \left(1 - \frac{1}{2}x \right) \right]^{-1}$$

$$= 2^{-1} \left(1 - \frac{1}{2}x \right)^{-1}$$

$$= \frac{1}{2} \left(1 - \frac{1}{2}x \right)^{-1}$$

$$= \frac{1}{2} \left(1 + -1 \left(-\frac{1}{2}x \right) + \frac{-1(-1-1)}{2!} \left(-\frac{1}{2}x \right)^2 + \dots \right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots \right)$$

$$= \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \dots$$

$$\text{Valid for } |x| < 1$$

$$(iv) \quad \frac{2 - 11x}{(1 + 2x)(2 - x)} = \frac{3}{1 + 2x} - \frac{4}{2 - x}$$

$$= 3 \left(1 - 2x + 4x^2 - \dots \right) - 4 \left(\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \dots \right)$$

$$= 3 - 6x + 12x^2 - 2 - x - \frac{1}{2}x^2 + \dots$$

$$= 1 - 7x + \frac{23}{2}x^2 \dots$$

$$\text{Valid for } |x| < \frac{1}{2}$$

$$2 \text{ (i)} \quad \frac{8-x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$8-x \equiv A(x+1) + B(x-2)$$

$$x=2 \Rightarrow 6=3A \Rightarrow A=2$$

$$x=-1 \Rightarrow 9=-3B \Rightarrow B=-3$$

$$\text{So } \frac{8-x}{(x-2)(x+1)} = \frac{2}{x-2} - \frac{3}{x+1}$$

$$\frac{2}{x-2}$$

$$= 2(x-2)^{-1}$$

$$= 2 \left[-2 \left(1 - \frac{1}{2}x \right) \right]^{-1}$$

$$= 2(-2)^{-1} \left(1 - \frac{1}{2}x \right)^{-1}$$

$$= - \left(1 + -1 \left(-\frac{1}{2}x \right) + \frac{-1(-1-1)}{2!} \left(-\frac{1}{2}x \right)^2 + \dots \right)$$

$$= - \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots \right)$$

$$= -1 - \frac{1}{2}x - \frac{1}{4}x^2 - \dots$$

$$\frac{3}{x+1}$$

$$= 3(1+x)^{-1}$$

$$= 3 \left(1 + -1x + \frac{-1(-1-1)}{2!}x^2 + \dots \right)$$

$$= 3(1-x+x^2-\dots)$$

$$= 3-3x+3x^2-\dots$$

$$\frac{2}{x-2} - \frac{3}{x+1} = \left(-1 - \frac{1}{2}x - \frac{1}{4}x^2 - \dots \right)$$

$$- (3-3x+3x^2-\dots)$$

$$= -4 + \frac{5}{2}x - \frac{13}{4}x^2 + \dots$$

Valid for $|x| < 1$

$$\text{(ii)} \quad \frac{x^2+9x+2}{(1-2x)(1+x)^2} \equiv \frac{A}{1-2x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

$$x^2+9x+2 \equiv A(1+x)^2 + B(1-2x)(1+x) + C(1-2x)$$

$$x=-1 \Rightarrow -6=3C \Rightarrow C=-2$$

$$x=\frac{1}{2} \Rightarrow \frac{27}{4} = \frac{9}{4}A \Rightarrow A=3$$

$$x^2 \text{ terms} \Rightarrow 1 = A - 2B \Rightarrow B=1$$

$$\text{So } \frac{x^2+9x+2}{(1-2x)(1+x)^2} \equiv \frac{3}{1-2x} + \frac{1}{1+x} - \frac{2}{(1+x)^2}$$

$$\frac{3}{1-2x} = 3(1-2x)^{-1}$$

$$= 3(1+2x+4x^2+\dots)$$

$$= 3+6x+12x^2+\dots$$

$$\frac{1}{1+x} = (1+x)^{-1}$$

$$= 1-x+x^2+\dots$$

$$\frac{2}{(1+x)^2} = 2(1+x)^{-2}$$

$$= 2(1-2x+3x^2-\dots)$$

$$= 2-4x+6x^2-\dots$$

$$\frac{3}{1-2x} + \frac{1}{1+x} - \frac{2}{(1+x)^2}$$

$$= (3+6x+12x^2+\dots) + (1-x+x^2+\dots)$$

$$- (2-4x+6x^2-\dots)$$

$$= 2+9x+7x^2+\dots$$

$$\text{Valid for } |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

$$3 \text{ (i)} \quad \frac{4x+14}{(1-x)(2+x)(1+x)} = \frac{A}{1-x} + \frac{B}{2+x} + \frac{C}{1+x}$$

$$4x+14 \equiv A(2+x)(1+x) + B(1-x)(1+x) + C(1-x)(2+x)$$

$$x=1 \Rightarrow 18=6A \Rightarrow A=3$$

$$x=-2 \Rightarrow 6=-3B \Rightarrow B=-2$$

$$x=-1 \Rightarrow 10=2C \Rightarrow C=5$$

$$\text{So } \frac{4x+14}{(1-x)(2+x)(1+x)} = \frac{3}{1-x} - \frac{2}{2+x} + \frac{5}{1+x}$$

$$\frac{3}{1-x} = 3(1-x)^{-1}$$

$$= 3(1+x+x^2+\dots)$$

$$= 3+3x+3x^2+\dots$$

$$\frac{2}{2+x} = 2(2+x)^{-1}$$

$$= 2 \left[2 \left(1 + \frac{1}{2}x \right) \right]^{-1}$$

$$= 2 \left(2^{-1} \left(1 + \frac{1}{2}x \right)^{-1} \right)$$

$$= 2 \left(\frac{1}{2} \left(1 + \frac{1}{2}x \right)^{-1} \right)$$

$$= \left(1 + \frac{1}{2}x \right)^{-1}$$

$$= 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \dots$$

$$\frac{5}{1+x} = 5(1+x)^{-1}$$

$$= 5(1-x+x^2-\dots)$$

$$= 5-5x+5x^2-\dots$$

$$\text{So } \frac{3}{1-x} - \frac{2}{2+x} + \frac{5}{1+x}$$

$$= (3+3x+3x^2+\dots) - (1-\frac{1}{2}x+\frac{1}{4}x^2-\dots)$$

$$+ (5-5x+5x^2-\dots)$$

$$= 7 - \frac{3}{2}x + \frac{31}{4}x^2 - \dots$$

$$4 \text{ (i)} \quad \frac{3+2x^2}{(1+x)^2(1-4x)} \equiv \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-4x}$$

$$3+2x^2 \equiv A(1+x)(1-4x) + B(1-4x) + C(1+x)^2$$

$$x = -1 \Rightarrow 5 = 5B \Rightarrow B = 1$$

$$x = \frac{1}{4} \Rightarrow \frac{25}{8} = \frac{25}{16}C \Rightarrow C = 2$$

$$x^2 \text{ terms} \Rightarrow 2 = -4A + C \Rightarrow A = 0$$

$$\text{So } \frac{3+2x^2}{(1+x)^2(1-4x)} \equiv \frac{1}{(1+x)^2} + \frac{2}{1-4x}$$

$$\text{(ii)} \quad \frac{3+2x^2}{(1+x)^2(1-4x)} = \frac{1}{(1+x)^2} + \frac{2}{1-4x}$$

$$\begin{aligned} \frac{1}{(1+x)^2} &= (1+x)^{-2} \\ &= 1 - 2x + 3x^2 - \dots \end{aligned}$$

$$\begin{aligned} \frac{2}{1-4x} &= 2(1-4x)^{-1} \\ &= 2(1+4x+16x^2+\dots) \\ &= 2+8x+32x^2+\dots \end{aligned}$$

$$\begin{aligned} \text{So } \frac{1}{(1+x)^2} + \frac{2}{1-4x} &= (1-2x+3x^2-\dots) + (2+8x+32x^2+\dots) \\ &= 3+6x+35x^2+\dots \end{aligned}$$

Further practice (Page 11)

$$1 \text{ (i)} \quad (1-2x)^{-1}$$

$$\begin{aligned} &= 1 + -1(-2x) + \frac{-1(-1-1)}{2!}(-2x)^2 \\ &\quad + \frac{-1(-1-1)(-1-2)}{3!}(-2x)^3 + \dots \\ &= 1 + 2x + \frac{-1(-2)}{2!}(4x^2) + \frac{-1(-2)(-3)}{3!}(-8x^3) + \dots \\ &= 1 + 2x + 4x^2 + 8x^3 + \dots \end{aligned}$$

$$\text{Valid for } |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

$$\text{(ii)} \quad (1-9x)^{-\frac{1}{3}}$$

$$\begin{aligned} &= 1 + -\frac{1}{3}(-9x) + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}(-9x)^2 \\ &\quad + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}(-9x)^3 + \dots \\ &= 1 + 3x + \frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2!}(81x^2) \\ &\quad + \frac{-\frac{1}{3}\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}(-729x^3) + \dots \\ &= 1 + 3x + 18x^2 + 126x^3 + \dots \end{aligned}$$

$$\text{Valid for } |9x| < 1 \Rightarrow |x| < \frac{1}{9}$$

$$2 \text{ (i)} \quad (9-3x)^{\frac{1}{2}}$$

$$\begin{aligned} &= \left[9\left(1-\frac{1}{3}x\right)\right]^{\frac{1}{2}} \\ &= 9^{\frac{1}{2}}\left(1-\frac{1}{3}x\right)^{\frac{1}{2}} \\ &= 3\left(1-\frac{1}{3}x\right)^{\frac{1}{2}} \\ &= 3\left(1+\frac{1}{2}\left(-\frac{1}{3}x\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{3}x\right)^2 + \dots\right) \\ &= 3\left(1-\frac{1}{6}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{1}{9}x^2\right) + \dots\right) \\ &= 3\left(1-\frac{1}{6}x - \frac{1}{72}x^2 + \dots\right) \\ &= 3 - \frac{1}{2}x - \frac{1}{24}x^2 - \dots \end{aligned}$$

$$\text{Valid for } \left|\frac{1}{3}x\right| < 1 \Rightarrow |x| < 3$$

$$\text{(ii)} \quad \frac{3x}{2+x}$$

$$\begin{aligned} &= 3x(2+x)^{-1} \\ &= 3x\left[2\left(1+\frac{1}{2}x\right)\right]^{-1} \\ &= 3x\left[2^{-1}\left(1+\frac{1}{2}x\right)^{-1}\right] \\ &= 3x\left[\frac{1}{2}\left(1+\frac{1}{2}x\right)^{-1}\right] \\ &= \frac{3x}{2}\left(1+\frac{1}{2}x\right)^{-1} \\ &= \frac{3x}{2}\left(1 + -1\left(\frac{1}{2}x\right) + \frac{-1(-1-1)}{2!}\left(\frac{1}{2}x\right)^2 + \dots\right) \\ &= \frac{3x}{2}\left(1 - \frac{1}{2}x + \frac{-1(-2)}{2!}\left(\frac{1}{4}x^2\right) + \dots\right) \\ &= \frac{3x}{2}\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 + \dots\right) \\ &= \frac{3}{2}x - \frac{3}{4}x^2 + \dots \end{aligned}$$

$$\text{Valid for } \left|\frac{1}{2}x\right| < 1 \Rightarrow |x| < 2$$

$$3 \text{ (i)} \quad (1+x)^{\frac{1}{3}}$$

$$\begin{aligned} &= 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \dots \\ &= 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2 + \dots \\ &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \dots \end{aligned}$$

$$(ii) (a) (8 + 16x)^{\frac{1}{3}}$$

$$= [8(1 + 2x)]^{\frac{1}{3}}$$

$$= 8^{\frac{1}{3}}(1 + 2x)^{\frac{1}{3}}$$

$$= 2(1 + 2x)^{\frac{1}{3}}$$

$$= 2\left(1 + \frac{1}{3}(2x) - \frac{1}{9}(2x)^2 + \dots\right)$$

$$= 2\left(1 + \frac{2}{3}x - \frac{4}{9}x^2 + \dots\right)$$

$$= 2 + \frac{4}{3}x - \frac{8}{9}x^2 + \dots$$

$$(b) \text{ Valid for } |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

$$4 (i) (1 + ax)^{-3}$$

$$= 1 + (-3)(ax) + \frac{-3(-3-1)}{2!}(ax)^2 + \dots$$

$$= 1 - 3ax + 6a^2x^2 + \dots$$

$$-3a = 6a^2$$

$$0 = 6a^2 + 3a$$

$$0 = 3a(2a + 1)$$

$$\text{Since } a \neq 0, a = -\frac{1}{2}$$

$$(ii) \left(1 - \frac{1}{2}x\right)^{-3}$$

$$= 1 + (-3)\left(-\frac{1}{2}x\right) + \frac{-3(-3-1)}{2!}\left(-\frac{1}{2}x\right)^2 + \dots$$

$$= 1 + \frac{3}{2}x + \frac{3}{2}x^2 + \dots$$

$$5 (i) \frac{6c^2d}{5e} \div \frac{18d^2}{25e^2} = \frac{6c^2d}{5e} \times \frac{25e^2}{18d^2} = \frac{5c^2e}{3d}$$

$$(ii) \frac{3g+1}{4} \times \frac{g+1}{3g^2+4g+1} = \frac{3g+1}{4} \times \frac{g+1}{(3g+1)(g+1)}$$

$$= \frac{1}{4}$$

$$(iii) \frac{j^4 - k^4}{j^2 + k^2} \times \frac{j+k}{j-k}$$

$$= \frac{(j^2 - k^2)(j^2 + k^2)}{j^2 + k^2} \times \frac{j+k}{j-k}$$

$$= \frac{(j-k)(j+k)(j^2 + k^2)}{j^2 + k^2} \times \frac{j+k}{j-k}$$

$$= (j+k)^2$$

$$6 (i) \frac{5n}{3} - \frac{n+1}{4} = \frac{20n}{12} - \frac{3(n+1)}{12} = \frac{17n-3}{12}$$

$$(ii) \frac{2q}{q+1} - \frac{4}{q-1}$$

$$= \frac{2q(q-1) - 4(q+1)}{(q+1)(q-1)}$$

$$= \frac{2q^2 - 2q - 4q - 4}{(q+1)(q-1)}$$

$$= \frac{2q^2 - 6q - 4}{(q+1)(q-1)}$$

$$(iii) \frac{s^2+4}{s^2-9} - \frac{s}{s-3}$$

$$= \frac{s^2+4}{(s-3)(s+3)} - \frac{s}{s-3}$$

$$= \frac{s^2+4-s(s+3)}{(s-3)(s+3)}$$

$$= \frac{s^2+4-s^2-3s}{(s-3)(s+3)}$$

$$= \frac{4-3s}{(s-3)(s+3)}$$

$$7 (i) \frac{2x+14}{(x-1)(x+3)} \equiv \frac{A}{x-1} + \frac{B}{x+3}$$

$$2x+14 \equiv A(x+3) + B(x-1)$$

$$x=1 \Rightarrow 16 = 4A \Rightarrow A=4$$

$$x=-3 \Rightarrow 8 = -4B \Rightarrow B=-2$$

$$\text{So } \frac{2x+14}{(x-1)(x+3)} \equiv \frac{4}{x-1} - \frac{2}{x+3}$$

$$(ii) \frac{16-3x}{x^2+x-6} = \frac{16-3x}{(x+3)(x-2)}$$

$$\frac{16-3x}{(x+3)(x-2)} \equiv \frac{A}{x+3} + \frac{B}{x-2}$$

$$16-3x \equiv A(x-2) + B(x+3)$$

$$x=2 \Rightarrow 10 = 5A \Rightarrow A=2$$

$$x=-3 \Rightarrow 25 = -5B \Rightarrow B=-5$$

$$\text{So } \frac{16-3x}{(x+3)(x-2)} \equiv \frac{2}{x+3} - \frac{5}{x-2}$$

$$(iii) \frac{x-10}{2x^2-5x-3} = \frac{x-10}{(2x+1)(x-3)}$$

$$\frac{x-10}{(2x+1)(x-3)} \equiv \frac{A}{2x+1} + \frac{B}{x-3}$$

$$x-10 \equiv A(x-3) + B(2x+1)$$

$$x=3 \Rightarrow -7 = 7B \Rightarrow B=-1$$

$$x=-\frac{1}{2} \Rightarrow -\frac{21}{2} = -\frac{7}{2}A \Rightarrow A=3$$

$$\text{So } \frac{x-10}{(2x+1)(x-3)} \equiv \frac{3}{2x+1} - \frac{1}{x-3}$$

$$(iv) \frac{5x^2 + 20x - 32}{x^3 - 16x} = \frac{5x^2 + 20x - 32}{x(x^2 - 16)} = \frac{5x^2 + 20x - 32}{x(x-4)(x+4)}$$

$$\frac{5x^2 + 20x - 32}{x(x-4)(x+4)} \equiv \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+4}$$

$$5x^2 + 20x - 32 \equiv A(x-4)(x+4) + Bx(x+4) + Cx(x-4)$$

$$x = 0 \Rightarrow -32 = -16A \Rightarrow A = 2$$

$$x = 4 \Rightarrow 128 = 32B \Rightarrow B = 4$$

$$x = -4 \Rightarrow -32 = 32C \Rightarrow C = -1$$

$$\text{So } \frac{5x^2 + 20x - 32}{x(x-4)(x+4)} \equiv \frac{2}{x} + \frac{4}{x-4} - \frac{1}{x+4}$$

$$(v) \frac{3x^2 + 2x - 20}{x^2 - 4} = 3 + \frac{2x - 8}{x^2 - 4}$$

$$\frac{2x - 8}{x^2 - 4} = \frac{2x - 8}{(x-2)(x+2)} \equiv \frac{A}{x-2} + \frac{B}{x+2}$$

$$2x - 8 \equiv A(x+2) + B(x-2)$$

$$x = 2 \Rightarrow -4 = 4A \Rightarrow A = -1$$

$$x = -2 \Rightarrow -12 = -4B \Rightarrow B = 3$$

$$\text{So } \frac{2x - 8}{(x-2)(x+2)} \equiv -\frac{1}{x-2} + \frac{3}{x+2}$$

$$\text{and } \frac{3x^2 + 2x - 20}{x^2 - 4} = 3 + \frac{3}{x+2} - \frac{1}{x-2}$$

$$8 \quad (i) \frac{3x^2 + 2x - 3}{(x-1)(x+1)^2} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$3x^2 + 2x - 3 \equiv A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$x = 1 \Rightarrow 2 = 4A \Rightarrow A = \frac{1}{2}$$

$$x = -1 \Rightarrow -2 = -2C \Rightarrow C = 1$$

$$x^2 \text{ terms} \Rightarrow 3 = A + B \Rightarrow B = \frac{5}{2}$$

$$\text{So } \frac{3x^2 + 2x - 3}{(x-1)(x+1)^2} \equiv \frac{\frac{1}{2}}{x-1} + \frac{\frac{5}{2}}{x+1} + \frac{1}{(x+1)^2}$$

$$= \frac{1}{2(x-1)} + \frac{5}{2(x+1)} + \frac{1}{(x+1)^2}$$

$$(ii) \frac{x-1}{(x+1)^3} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x-1 \equiv A(x+1)^2 + B(x+1) + C$$

$$x = -1 \Rightarrow -2 = C$$

$$x^2 \text{ terms} \Rightarrow 0 = A$$

$$x = 0 \Rightarrow -1 = A + B + C \Rightarrow B = 1$$

$$\text{So } \frac{x-1}{(x+1)^3} \equiv \frac{0}{x+1} + \frac{1}{(x+1)^2} + \frac{-2}{(x+1)^3}$$

$$= \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}$$

$$(iii) \frac{-2x+4}{(x^2+1)(x-1)^2} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$-2x+4 \equiv (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$x = 1 \Rightarrow 2 = 2D \Rightarrow D = 1$$

$$(Ax+B)(x-1)^2 \equiv Ax^3 + (B-2A)x^2 + (A-2B)x + B$$

$$C(x^2+1)(x-1) \equiv C(x^3 - x^2 + x - 1)$$

$$x^3 \text{ terms} \Rightarrow 0 = A + C$$

$$x = 0 \Rightarrow 4 = B - C + D \Rightarrow B - C = 3$$

$$x^2 \text{ terms} \Rightarrow 0 = B - 2A - C$$

$$\text{Since } B - C = 3, 3 - 2A = 0 \Rightarrow A = \frac{3}{2}$$

$$A + C = 0 \Rightarrow C = -\frac{3}{2}$$

$$B - C = 3 \Rightarrow B = \frac{3}{2}$$

$$\text{So } \frac{-2x+4}{(x^2+1)(x-1)^2} \equiv \frac{\frac{3}{2}x + \frac{3}{2}}{x^2+1} + \frac{-\frac{3}{2}}{x-1} + \frac{1}{(x-1)^2}$$

$$= \frac{3x+3}{2(x^2+1)} - \frac{3}{2(x-1)} + \frac{1}{(x-1)^2}$$

$$(iv) \frac{2x^2+3x+1}{(x-1)(x^2+2)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

$$2x^2+3x+1 \equiv A(x^2+2) + (Bx+C)(x-1)$$

$$x = 1 \Rightarrow 6 = 3A \Rightarrow A = 2$$

$$x^2 \text{ terms} \Rightarrow 2 = A + B \Rightarrow B = 0$$

$$x = 0 \Rightarrow 1 = 2A - C \Rightarrow C = 3$$

$$\text{So } \frac{2x^2+3x+1}{(x-1)(x^2+2)} \equiv \frac{2}{x-1} + \frac{3}{x^2+2}$$

$$9 \quad \frac{2x^2-3x+3}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$$2x^2-3x+3 \equiv A(1+x^2) + (Bx+C)(1+x)$$

$$x = -1 \Rightarrow 8 = 2A \Rightarrow A = 4$$

$$x^2 \text{ terms} \Rightarrow 2 = A + B \Rightarrow B = -2$$

$$x = 0 \Rightarrow 3 = A + C \Rightarrow C = -1$$

$$\text{So } \frac{2x^2-3x+3}{(1+x)(1+x^2)} \equiv \frac{4}{1+x} + \frac{-2x-1}{1+x^2}$$

$$= \frac{4}{1+x} - \frac{2x+1}{1+x^2}$$

$$\frac{4}{1+x} = 4(1+x)^{-1}$$

$$= 4(1-x+x^2-\dots)$$

$$= 4-4x+4x^2-\dots$$

$$\frac{2x+1}{1+x^2} = (2x+1)(1+x^2)^{-1}$$

$$= (2x+1)(1-x^2+x^4-\dots)$$

$$= 2x-2x^3+2x^5+1-x^2+x^4+\dots$$

Valid for $|x^2| < 1 \Rightarrow |x| < 1$

$$\begin{aligned}\frac{4}{2+x^2} &= 4(2+x^2)^{-1} \\ &= 4(2+x^2)^{-1} \\ &= 4\left(2^{-1}\left(1+\frac{1}{2}x^2\right)^{-1}\right) \\ &= 4\left(\frac{1}{2}\left(1+(-1)\left(\frac{1}{2}x^2\right)+\frac{-1(-1-1)}{2!}\left(\frac{1}{2}x^2\right)^2\right)\right) \\ &= 2\left(1-\frac{1}{2}x^2+\frac{1}{4}x^4+\dots\right) \\ &= 2-x^2+\frac{1}{2}x^4+\dots\end{aligned}$$

So $\frac{2-4x-3x^2}{(2-x)(2+x^2)} = \frac{4}{2+x^2} - \frac{3}{2-x}$

$$\begin{aligned}&= \left(2-x^2+\frac{1}{2}x^4+\dots\right) \\ &\quad - \left(\frac{3}{2}+\frac{3}{4}x+\frac{3}{8}x^2+\frac{3}{16}x^3+\dots\right) \\ &= \frac{1}{2}-\frac{3}{4}x-\frac{11}{8}x^2-\frac{3}{16}x^3+\dots\end{aligned}$$

So $p(x) = (2x - 1)(x^2 + 2)$

$$\begin{aligned} \text{(iii)} \quad \frac{8x-13}{(2x-1)(x^2+2)} &\equiv \frac{A}{2x-1} + \frac{Bx+C}{x^2+2} \\ 8x-13 &\equiv A(x^2+2) + (Bx+C)(2x-1) \\ x = \frac{1}{2} &\Rightarrow -9 \equiv \frac{9}{4}A \Rightarrow A = -4 \\ x = 0 &\Rightarrow -13 = 2A - C \Rightarrow C = 5 \\ x^2 \text{ term} &\Rightarrow 0 = A + 2B \Rightarrow B = 2 \\ \text{So } \frac{8x-13}{p(x)} &\equiv \frac{-4}{2x-1} + \frac{2x+5}{x^2+2} \end{aligned}$$

$$\begin{aligned} 3 \quad \frac{1}{\sqrt{1-2x}} &= (1-2x)^{-\frac{1}{2}} \\ &= 1 + \frac{-\frac{1}{2}(-2x)}{1!} + \frac{\frac{-\frac{1}{2}(-2x)}{2!}(-2x)}{2!} + \frac{\frac{-\frac{1}{2}(-2x)}{3!}(-2x)^2}{3!} + \dots \\ &= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \dots \\ \text{Valid for } |2x| < 1 &\Rightarrow |x| < \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 4 \quad f(x) &= \frac{5x^2-7x+4}{(3x+2)(x^2+5)} \\ \text{(i)} \quad \frac{5x^2-7x+4}{(3x+2)(x^2+5)} &= \frac{A}{3x+2} + \frac{Bx+C}{x^2+5} \\ 5x^2-7x+4 &\equiv A(x^2+5) + (Bx+C)(3x+2) \\ x = -\frac{2}{3} &\Rightarrow \frac{98}{9} = \frac{49}{9}A \Rightarrow A = 2 \\ x^2 \text{ terms} &\Rightarrow 5 = A + 3B \Rightarrow B = 1 \\ x = 0 &\Rightarrow 4 = 5A + 2C \Rightarrow C = -3 \\ \text{So } \frac{5x^2-7x+4}{(3x+2)(x^2+5)} &\equiv \frac{2}{3x+2} + \frac{x-3}{x^2+5} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad 2(3x+2)^{-1} &= 2 \left[2 \left(1 + \frac{3}{2}x \right) \right]^{-1} \\ &= 2 \left(2^{-1} \left(1 + \frac{3}{2}x \right)^{-1} \right) \\ &= 2 \left(\frac{1}{2} \left(1 + \frac{3}{2}x \right)^{-1} \right) \\ &= \left(1 + \frac{3}{2}x \right)^{-1} \\ &= \left(1 - \frac{3}{2}x + \frac{9}{4}x^2 - \dots \right) \end{aligned}$$

$$\begin{aligned} \frac{x-3}{x^2+5} &= (x-3)(x^2+5)^{-1} \\ &= (x-3) \left(5^{-1} \left(1 + \frac{1}{5}x^2 \right)^{-1} \right) \\ &= (x-3) \left(\frac{1}{5} \left(1 + (-1) \left(\frac{1}{5}x^2 \right) + \frac{-1(-1-1)}{2!} \left(\frac{1}{5}x^2 \right)^2 \right) \right) \\ &= (x-3) \left(\frac{1}{5} \left(1 - \frac{1}{5}x^2 + \frac{1}{25}x^4 + \dots \right) \right) \\ &= (x-3) \left(\frac{1}{5} - \frac{1}{25}x^2 + \frac{1}{125}x^4 + \dots \right) \\ &= -\frac{3}{5} + \frac{1}{5}x + \frac{3}{25}x^2 + \dots \end{aligned}$$

$$\begin{aligned} \text{So } \frac{5x^2-7x+4}{(3x+2)(x^2+5)} &\equiv \frac{2}{3x+2} + \frac{x-3}{x^2+5} \\ &= \left(1 - \frac{3}{2}x + \frac{9}{4}x^2 - \dots \right) \\ &\quad + \left(-\frac{3}{5} + \frac{1}{5}x + \frac{3}{25}x^2 + \dots \right) \\ &= \frac{2}{5} - \frac{13}{10}x + \frac{237}{100}x^2 + \dots \end{aligned}$$

Stretch and challenge (Page 13)

$$\begin{aligned} 1 \quad \text{(i)} \quad \sqrt{4-3x} &= (4-3x)^{\frac{1}{2}} \\ &= (4-3x)^{\frac{1}{2}} \\ &= \left[4 \left(1 - \frac{3}{4}x \right) \right]^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} \left(1 - \frac{3}{4}x \right)^{\frac{1}{2}} \\ &= 2 \left(1 + \frac{1}{2} \left(-\frac{3}{4}x \right) + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} \left(-\frac{3}{4}x \right)^2 + \dots \right) \\ &= 2 \left(1 - \frac{3}{8}x - \frac{9}{128}x^2 - \dots \right) \\ &= 2 - \frac{3}{4}x - \frac{9}{64}x^2 - \dots \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad (1+ax)(4-3x)^{\frac{1}{2}} &= (1+ax) \left(2 - \frac{3}{4}x - \frac{9}{64}x^2 - \dots \right) \\ x^2 \text{ terms are } &-\frac{9}{64}x^2 + ax \times -\frac{3}{4}x \\ &= -\frac{9}{64}x^2 + -\frac{3a}{4}x^2 \\ &= \left(-\frac{9}{64} - \frac{3a}{4} \right) x^2 \\ &= \left(\frac{-9-48a}{64} \right) x^2 \\ \text{So } \frac{-9-48a}{64} &= \frac{111}{64} \\ -9-48a &= 111 \\ -48a &= 120 \\ a &= -\frac{5}{2} \end{aligned}$$

$$2 \frac{(1+3x)^2}{(1+kx)^2}$$

$$\begin{aligned}
 &= (1+3x)^2(1+kx)^{-2} \\
 &= (1+6x+9x^2)(1+kx)^{-2} \\
 &= (1+6x+9x^2)\left(1+(-2)(kx)+\frac{-2(-2-1)}{2!}(kx)^2\right) \\
 &= (1+6x+9x^2)(1-2kx+3k^2x^2-\dots) \\
 &\quad x^2 \text{ terms are } 3k^2x^2+6x \times -2kx+9x^2 \\
 &= (3k^2-12k+9)x^2 \\
 \text{So } 3k^2-12k+9 &= 105 \\
 k^2-4k-32 &= 0 \\
 (k+4)(k-8) &= 0 \\
 k &= -4 \text{ or } 8
 \end{aligned}$$

3 You want to find n such that

$$\begin{aligned}
 \frac{n(n-1)(n-2)}{3!} &< \frac{n(n-1)(n-2)(n-3)}{4!} \\
 24n(n-1)(n-2) &< 6n(n-1)(n-2)(n-3) \\
 24 &< 6(n-3) \\
 24 &< 6n-18 \\
 42 &< 6n \\
 n &> 7
 \end{aligned}$$

$$\begin{aligned}
 4 \text{ (i)} \quad (1+ax)^b &= 1+bax+\frac{b(b-1)}{2!}(ax)^2+\dots \\
 &= 1+abx+\frac{a^2b(b-1)}{2}x^2+\dots
 \end{aligned}$$

$$ab = -1 \Rightarrow a = -\frac{1}{b}$$

$$\frac{a^2b(b-1)}{2} = 2 \Rightarrow \frac{\left(-\frac{1}{b}\right)^2 b(b-1)}{2} = 2$$

$$\frac{\frac{1}{b^2}b(b-1)}{2} = 2$$

$$\frac{1}{b}(b-1) = 4$$

$$b-1 = 4b$$

$$-1 = 3b$$

$$b = -\frac{1}{3}$$

$$a = -\frac{1}{-\frac{1}{3}} = 3$$

$$\text{(ii) Valid for } |3x| < 1 \Rightarrow |x| < \frac{1}{3}$$

2 Further calculus

2.1 Differentiating $\tan^{-1}x$ (Page 14)

$$1 \quad y = \tan^{-1}x$$

$$\tan y = x$$

$$\frac{\sin y}{\cos y} = x$$

$$\frac{dx}{dy} = \frac{\cos y \times \cos y - (-\sin y)(\sin y)}{\cos^2 y}$$

$$= \frac{\cos^2 y + \sin^2 y}{\cos^2 y}$$

$$= \frac{\cos^2 y}{\cos^2 y} + \frac{\sin^2 y}{\cos^2 y}$$

$$= 1 + \tan^2 y$$

$$= 1 + x^2$$

$$\text{So } \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$2 \text{ (i)} \quad y = 2 \tan^{-1} 8x$$

$$\frac{dy}{dx} = \frac{2}{1+(8x)^2} \times 8 = \frac{16}{1+64x^2}$$

$$\text{(ii)} \quad y = \frac{1}{2} \tan^{-1}(x^3)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{1+(x^3)^2} \times 3x^2 = \frac{3x^2}{2+2x^6}$$

$$\text{(iii)} \quad y = \tan^{-1}(\sin 2x)$$

$$\frac{dy}{dx} = \frac{2 \cos 2x}{1 + \sin^2 2x}$$

$$\text{(iv)} \quad y = \ln\left(\frac{\cos x}{e^{\tan^{-1}x}}\right)^2$$

$$y = 2 \ln\left(\frac{\cos x}{e^{\tan^{-1}x}}\right)$$

$$y = 2 \left[\ln(\cos x) - \ln(e^{\tan^{-1}x}) \right]$$

$$y = 2 \left[\ln(\cos x) - \tan^{-1}x \right]$$

$$\frac{dy}{dx} = 2 \left[\frac{-\sin x}{\cos x} - \frac{1}{1+x^2} \right]$$

$$= -2 \tan x - \frac{2}{1+x^2}$$

$$3 \quad y = \tan^{-1}(e^x)$$

$$\frac{dy}{dx} = \frac{e^x}{1+e^{2x}}$$

$$\text{When } x = 0, \frac{dy}{dx} = \frac{e^0}{1+e^{2 \times 0}} = \frac{1}{2}$$

$$\text{When } x = 0, y = \tan^{-1}(e^0) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$y = mx + c \Rightarrow \frac{\pi}{4} = \frac{1}{2} \times 0 + c \Rightarrow c = \frac{\pi}{4}$$

$$\text{So } y = \frac{1}{2}x + \frac{\pi}{4}$$

$$4y = 2x + \pi$$

$$2x - 4y + \pi = 0$$

2.2 Integration by substitution (Page 15)

$$1 \quad (i) \quad u = 2x + 1 \Rightarrow du = 2dx$$

$$\begin{aligned} \int (2x+1) dx &= \int u \frac{du}{2} \\ &= \frac{1}{2} \int u du \\ &= \frac{1}{2} \times \frac{u^2}{2} + c \\ &= \frac{1}{4} u^2 + c \\ &= \frac{1}{4} (2x+1)^2 + c \end{aligned}$$

$$(ii) \quad u = 1 - x^2 \Rightarrow du = -2x dx \Rightarrow dx = -\frac{du}{2x}$$

$$\begin{aligned} \int x\sqrt{1-x^2} dx &= \int x\sqrt{u} \left(-\frac{du}{2x}\right) \\ &= -\frac{1}{2} \int \sqrt{u} du \\ &= -\frac{1}{2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] + c \\ &= -\frac{1}{2} \left[\frac{2u^{\frac{3}{2}}}{3} \right] + c \\ &= -\frac{1}{3} u^{\frac{3}{2}} + c \\ &= -\frac{1}{3} (1-x^2)^{\frac{3}{2}} + c \\ &= -\frac{1}{3} \sqrt{(1-x^2)^3} + c \end{aligned}$$

$$(iii) \quad u = x + 4 \Rightarrow du = dx \text{ and } x = u - 4$$

$$\begin{aligned} \int 2x\sqrt{x+4} dx &= \int 2(u-4)\sqrt{u} du \\ &= 2 \int (u^{\frac{3}{2}} - 4u^{\frac{1}{2}}) du \\ &= 2 \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4u^{\frac{3}{2}}}{\frac{3}{2}} \right] + c \end{aligned}$$

$$\begin{aligned} &= 2 \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{8u^{\frac{3}{2}}}{3} \right] + c \\ &= \frac{4u^{\frac{5}{2}}}{5} - \frac{16u^{\frac{3}{2}}}{3} + c \\ &= \frac{4(x+4)^{\frac{5}{2}}}{5} - \frac{16(x+4)^{\frac{3}{2}}}{3} + c \\ &= \frac{4\sqrt{(x+4)^5}}{5} - \frac{16\sqrt{(x+4)^3}}{3} + c \end{aligned}$$

$$(iv) \quad u = x - 1 \Rightarrow du = dx$$

$$\text{Also } x = u + 1$$

$$\begin{aligned} \int x^2 \sqrt{x-1} dx &= \int (u+1)^2 \sqrt{u} du \\ &= \int (u^2 + 2u + 1) \sqrt{u} du \\ &= \int (u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) du \\ &= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2u^{\frac{7}{2}}}{7} + \frac{4u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + c \\ &= \frac{2(x-1)^{\frac{7}{2}}}{7} + \frac{4(x-1)^{\frac{5}{2}}}{5} + \frac{2(x-1)^{\frac{3}{2}}}{3} + c \\ &= \frac{2\sqrt{(x-1)^7}}{7} + \frac{4\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + c \end{aligned}$$

$$2 \quad (i) \quad u = x^4 + 1 \Rightarrow du = 4x^3 dx \Rightarrow dx = \frac{du}{4x^3}$$

$$x = 1 \Rightarrow u = 2, x = 0 \Rightarrow u = 1$$

$$\begin{aligned} \int_0^1 x^3 \sqrt{x^4+1} dx &= \int_{u=1}^{u=2} x^3 \sqrt{u} \frac{du}{4x^3} \\ &= \frac{1}{4} \int_1^2 \sqrt{u} du \\ &= \frac{1}{4} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^2 \\ &= \frac{1}{4} \left[\frac{2u^{\frac{3}{2}}}{3} \right]_1^2 \\ &= \frac{1}{4} \left[\left(\frac{2 \times 2^{\frac{3}{2}}}{3} \right) - \left(\frac{2 \times 1^{\frac{3}{2}}}{3} \right) \right] \\ &= 0.305 \quad (3 \text{ s.f.}) \end{aligned}$$

(ii) $u = 2x^2 - x + 1 \Rightarrow du = (4x - 1) dx$

$x = 3 \Rightarrow u = 16, x = 0 \Rightarrow u = 1$

$$\int_0^3 (4x - 1) \sqrt{2x^2 - x + 1} dx$$

$$= \int_{u=1}^{u=16} (4x - 1) \sqrt{u} \frac{du}{4x - 1}$$

$$= \int_1^{16} \sqrt{u} du$$

$$= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^{16}$$

$$= \left[\frac{2\sqrt{u^3}}{3} \right]_1^{16}$$

$$= \left(\frac{2\sqrt{16^3}}{3} \right) - \left(\frac{2\sqrt{1^3}}{3} \right)$$

$$= 42$$

(iii) $u = x^2 + 2x \Rightarrow du = (2x + 2) dx$

$x = \infty \Rightarrow u = \infty, x = 1 \Rightarrow u = 3$

$$\int_1^\infty \frac{x+1}{(x^2+2x)^5} dx$$

$$= \int_{u=3}^{u=\infty} \frac{x+1}{(u)^5} \frac{du}{2x+2}$$

$$= \frac{1}{2} \int_3^\infty \frac{1}{u^5} du$$

$$= \frac{1}{2} \int_3^\infty u^{-5} du$$

$$= \frac{1}{2} \left[\frac{u^{-4}}{-4} \right]_3^\infty$$

$$= \frac{1}{2} \left[-\frac{1}{4u^4} \right]_3^\infty$$

$$= \frac{1}{2} \left[\left(-\frac{1}{4 \times \infty^4} \right) - \left(-\frac{1}{4 \times 3^4} \right) \right]$$

$$= \frac{1}{2} \left[\frac{1}{324} \right]$$

$$= \frac{1}{648} \approx 0.00154$$

(iv) $u = x^2 - 1 \Rightarrow du = 2x dx$

$x = \sqrt{2} \Rightarrow u = 1, x = 1 \Rightarrow u = 0$

$$\int_1^{\sqrt{2}} x^3 (x^2 - 1)^6 dx$$

$$= \int_{u=0}^{u=1} x^3 (u)^6 \frac{du}{2x}$$

$$= \frac{1}{2} \int_0^1 x^2 (u)^6 du$$

$$= \frac{1}{2} \int_0^1 (u+1) u^6 du$$

$$= \frac{1}{2} \int_0^1 (u^7 + u^6) du$$

$$= \frac{1}{2} \left[\frac{u^8}{8} + \frac{u^7}{7} \right]_0^1$$

$$= \frac{1}{2} \left[\left(\frac{1^8}{8} + \frac{1^7}{7} \right) - 0 \right]$$

$$= \frac{15}{112} \approx 0.134$$

3 $u = 1 + \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx \Rightarrow dx = -x^2 du$

$x = 2 \Rightarrow u = \frac{3}{2}, x = 1 \Rightarrow u = 2$

$$\int_1^2 \frac{\left(1 + \frac{1}{x}\right)^3}{x^2} dx$$

$$= \int_{u=2}^{u=\frac{3}{2}} \frac{(u)^3}{x^2} (-x^2 du)$$

$$= -\int_2^{\frac{3}{2}} u^3 du$$

$$= -\left[\frac{u^4}{4} \right]_2^{\frac{3}{2}}$$

$$= -\left[\frac{\left(\frac{3}{2}\right)^4}{4} - \left(\frac{2^4}{4}\right) \right]$$

$$= -\left[\frac{81}{64} - \frac{16}{4} \right]$$

$$= \frac{175}{64} \approx 2.73$$

4 $u = \sqrt{x+2} \Rightarrow u^2 = x+2 \Rightarrow x = u^2 - 2$

$dx = 2u du$

$x = 7 \Rightarrow u = 3, x = -1 \Rightarrow u = 1$

$$\int_{-1}^7 \frac{x^2}{\sqrt{x+2}} dx$$

$$= \int_{u=1}^{u=3} \frac{(u^2 - 2)^2}{u} 2u du$$

$$= 2 \int_1^3 (u^4 - 4u^2 + 4) du$$

$$= 2 \left[\frac{u^5}{5} - \frac{4u^3}{3} + 4u \right]_1^3$$

$$= 2 \left[\left(\frac{3^5}{5} - \frac{4 \times 3^3}{3} + 4 \times 3 \right) - \left(\frac{1^5}{5} - \frac{4 \times 1^3}{3} + 4 \times 1 \right) \right]$$

$$= 2 \left[\left(\frac{123}{5} \right) - \left(\frac{43}{15} \right) \right]$$

$$= \frac{652}{15}$$

2.3 Integrals involving exponentials and natural logarithms (Page 16)

1 (i) $u = x^2 + 3 \Rightarrow du = 2x dx$

$$\begin{aligned} & \int 8x e^{x^2+3} dx \\ &= \int 8x e^u \frac{du}{2x} \\ &= 4 \int e^u du \\ &= 4e^u + c \\ &= 4e^{x^2+3} + c \end{aligned}$$

(ii) $u = e^x - 1 \Rightarrow du = e^x dx$

$$\begin{aligned} & \int \frac{e^x}{e^x - 1} dx \\ &= \int \frac{e^x du}{u e^x} \\ &= \int \frac{1}{u} du \\ &= \ln|u| + c \\ &= \ln|e^x - 1| + c \end{aligned}$$

(iii) $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx \Rightarrow dx = -x^2 du$

$$\begin{aligned} & \int \frac{e^{\frac{1}{x}}}{x^2} dx \\ &= \int \frac{e^u}{x^2} (-x^2 du) \\ &= -\int e^u du \\ &= -e^u + c \\ &= -e^{\frac{1}{x}} + c \end{aligned}$$

(iv) $u = 1 - \cos x \Rightarrow du = \sin x dx$

$$\begin{aligned} & \int e^{1-\cos x} \sin x dx \\ &= \int e^u \sin x \frac{du}{\sin x} \\ &= \int e^u du \\ &= e^u + c \\ &= e^{1-\cos x} + c \end{aligned}$$

2 $u = 2x + 1 \Rightarrow du = 2dx$

$x = 1 \Rightarrow u = 3, x = 0 \Rightarrow u = 1$

$$\begin{aligned} & \int_0^1 \frac{x}{2x+1} dx \\ &= \int_{u=1}^{u=3} \frac{u-1}{u} \frac{du}{2} \\ &= \frac{1}{4} \int_1^3 \frac{u-1}{u} du \\ &= \frac{1}{4} \int_1^3 \left(1 - \frac{1}{u}\right) du \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4} [u - \ln u]_1^3 \\ &= \frac{1}{4} [(3 - \ln 3) - (1 - \ln 1)] \\ &= \frac{1}{4} [(3 - \ln 3) - 1] \\ &= \frac{1}{4} (2 - \ln 3) \end{aligned}$$

3 $u = 1 + e^{2x} \Rightarrow du = 2e^{2x} dx$

$x = \ln 2 \Rightarrow u = 1 + e^{2 \ln 2} = 1 + e^{\ln 4} = 5$

$x = 0 \Rightarrow u = 1 + e^{2 \times 0} = 1 + 1 = 2$

$$\begin{aligned} & \int_0^{\ln 2} 32e^{2x} (1 + e^{2x})^4 dx \\ &= \int_{u=2}^{u=5} 32e^{2x} (u)^4 \frac{du}{2e^{2x}} \\ &= 16 \int_2^5 u^4 du \\ &= 16 \left[\frac{u^5}{5} \right]_2^5 \\ &= 16 \left[\frac{5^5}{5} - \frac{2^5}{5} \right] \\ &= 9897.6 \end{aligned}$$

4 $u = 2 + \ln t \Rightarrow du = \frac{1}{t} dt \Rightarrow dt = t du$

$t = e \Rightarrow u = 3, t = 1 \Rightarrow u = 2$

$$\begin{aligned} & \int_1^e \frac{1}{t(2 + \ln t)^2} dt \\ &= \int_{u=2}^{u=3} \frac{1}{t(u)^2} t du \\ &= \int_2^3 \frac{1}{u^2} du \\ &= \int_2^3 u^{-2} du \\ &= \left[\frac{u^{-1}}{-1} \right]_2^3 \\ &= \left[-\frac{1}{u} \right]_2^3 \\ &= \left[\left(-\frac{1}{3}\right) - \left(-\frac{1}{2}\right) \right] = \frac{1}{6} \end{aligned}$$

5 $y = \frac{e^{4x}}{1 + e^{4x}}$

(i) P is where $x = 0 \Rightarrow y = \frac{e^0}{1 + e^0} = \frac{1}{2}$

$$\begin{aligned} \text{(ii)} \quad \frac{dy}{dx} &= \frac{4e^{4x}(1 + e^{4x}) - 4e^{4x} \times e^{4x}}{(1 + e^{4x})^2} \\ &= \frac{4e^{4x}}{(1 + e^{4x})^2} \end{aligned}$$

When $x = 0, \frac{dy}{dx} = \frac{4e^0}{(1 + e^0)^2} = \frac{4}{4} = 1$

$$\begin{aligned}
 \text{(iii) Area} &= \int_0^1 \frac{e^{4x}}{1+e^{4x}} dx \\
 &= \frac{1}{4} \int_0^1 \frac{4e^{4x}}{1+e^{4x}} dx \\
 &= \frac{1}{4} \left[\ln|1+e^{4x}| \right]_0^1 \\
 &= \frac{1}{4} \left[\ln(1+e^4) - \ln|1+e^0| \right] \\
 &= \frac{1}{4} \left[\ln(1+e^4) - \ln 2 \right] \\
 &= \frac{1}{4} \ln \frac{(1+e^4)}{2}
 \end{aligned}$$

2.4 Integrals involving trigonometrical functions (Page 19)

1 (i) $u = x^2 \Rightarrow du = 2x dx$

$$\begin{aligned}
 &\int x \sin(x^2) dx \\
 &= \int x \sin(u) \frac{du}{2x} \\
 &= \frac{1}{2} \int \sin(u) du \\
 &= -\frac{1}{2} \cos(u) + c \\
 &= -\frac{1}{2} \cos(x^2) + c
 \end{aligned}$$

(ii) $u = \cos x \Rightarrow du = -\sin x dx$

$$\begin{aligned}
 &\int \sin x e^{\cos x} dx \\
 &= \int \sin x e^u \frac{du}{-\sin x} \\
 &= -\int e^u du \\
 &= -e^u + c \\
 &= -e^{\cos x} + c
 \end{aligned}$$

(iii) $u = \sin x \Rightarrow du = \cos x dx$

$$\begin{aligned}
 &\int \cot x dx \\
 &= \int \frac{\cos x}{\sin x} dx \\
 &= \int \frac{\cos x}{u} \frac{du}{\cos x} \\
 &= \int \frac{1}{u} du \\
 &= \ln|u| + c \\
 &= \ln|\sin x| + c
 \end{aligned}$$

(iv) $u = \cos x \Rightarrow du = -\sin x dx$

$$\begin{aligned}
 &\int \cos^5 x \sin x dx \\
 &= \int u^5 \sin x \frac{du}{-\sin x} \\
 &= -\int u^5 du \\
 &= -\frac{u^6}{6} + c \\
 &= -\frac{\cos^6 x}{6} + c
 \end{aligned}$$

2 $u = 1 - \cos x \Rightarrow du = \sin x dx$

$$\begin{aligned}
 &\int \sin x \cos x (1 - \cos x)^3 dx \\
 &= \int \sin x \cos x (u)^3 \frac{du}{\sin x} \\
 &= \int (1-u)u^3 du \\
 &= \int (u^3 - u^4) du \\
 &= \frac{u^4}{4} - \frac{u^5}{5} + c \\
 &= \frac{(1 - \cos x)^4}{4} - \frac{(1 - \cos x)^5}{5} + c
 \end{aligned}$$

3 $x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta d\theta$

$$x = 1 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$x = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0$$

$$\begin{aligned}
 &\int_0^1 \sqrt{4-x^2} dx \\
 &= \int_{\theta=0}^{\theta=\frac{\pi}{6}} \sqrt{4-(2\sin\theta)^2} 2\cos\theta d\theta \\
 &= \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta} 2\cos\theta d\theta \\
 &= \int_0^{\frac{\pi}{6}} \sqrt{4(1-\sin^2\theta)} 2\cos\theta d\theta \\
 &= \int_0^{\frac{\pi}{6}} \sqrt{4\cos^2\theta} 2\cos\theta d\theta \\
 &= \int_0^{\frac{\pi}{6}} 2\cos\theta \times 2\cos\theta d\theta \\
 &= \int_0^{\frac{\pi}{6}} 4\cos^2\theta d\theta \\
 &= \int_0^{\frac{\pi}{6}} 4\left(\frac{1}{2}(\cos 2\theta + 1)\right) d\theta \\
 &= 2 \int_0^{\frac{\pi}{6}} (\cos 2\theta + 1) d\theta \\
 &= 2 \left[\frac{1}{2} \sin 2\theta + \theta \right]_0^{\frac{\pi}{6}} \\
 &= 2 \left[\left(\frac{1}{2} \sin 2 \times \frac{\pi}{6} + \frac{\pi}{6} \right) - \left(\frac{1}{2} \sin 2 \times 0 + 0 \right) \right] \\
 &= 2 \left[\left(\frac{1}{2} \sin \frac{\pi}{3} + \frac{\pi}{6} \right) - 0 \right] \\
 &= 2 \left[\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6} \right) - 0 \right] \\
 &= \frac{\sqrt{3}}{2} + \frac{\pi}{3}
 \end{aligned}$$

4 (i) $x = \sin^2 \theta \Rightarrow dx = 2 \sin \theta \cos \theta d\theta$

$$x = 1 \Rightarrow \sin^2 \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$x = 0 \Rightarrow \sin^2 \theta = 0 \Rightarrow \theta = 0$$

$$\begin{aligned} \int_0^1 \sqrt{\frac{1-x}{x}} dx &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sqrt{\frac{1-\sin^2 \theta}{\sin^2 \theta}} 2 \sin \theta \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos^2 \theta}{\sin^2 \theta}} 2 \sin \theta \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta} 2 \sin \theta \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} 2 \cos^2 \theta d\theta \end{aligned}$$

(ii) $\int_0^1 \sqrt{\frac{1-x}{x}} dx$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} 2 \cos^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} 2 \left(\frac{1}{2} (\cos 2\theta + 1) \right) d\theta \\ &= \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1) d\theta \\ &= \left[\frac{1}{2} \sin 2\theta + \theta \right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{1}{2} \sin 2 \times \frac{\pi}{2} + \frac{\pi}{2} \right) - \left(\frac{1}{2} \sin 2 \times 0 + 0 \right) \\ &= \left(\frac{1}{2} \sin \pi + \frac{\pi}{2} \right) - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

5 $x = 2 \sec \theta = \frac{2}{\cos \theta} = 2(\cos \theta)^{-1}$

$$\begin{aligned} \frac{dx}{d\theta} &= -2(\cos \theta)^{-2} \times -\sin \theta = \frac{2 \sin \theta}{\cos^2 \theta} \\ \int \frac{1}{x^2 \sqrt{x^2 - 4}} dx &= \int \frac{1}{(2 \sec \theta)^2 \sqrt{(2 \sec \theta)^2 - 4}} \frac{2 \sin \theta}{\cos^2 \theta} d\theta \\ &= \int \frac{1}{4 \sec^2 \theta \sqrt{4 \sec^2 \theta - 4}} \frac{2 \sin \theta}{\cos^2 \theta} d\theta \\ &= \int \frac{1}{4 \sec^2 \theta \sqrt{4(\sec^2 \theta - 1)}} \frac{2 \sin \theta}{\cos^2 \theta} d\theta \\ &= \int \frac{1}{4 \sec^2 \theta \sqrt{4 \tan^2 \theta}} \frac{2 \sin \theta}{\cos^2 \theta} d\theta \\ &= \int \frac{1}{4 \sec^2 \theta \times 2 \tan \theta} \frac{2 \sin \theta}{\cos^2 \theta} d\theta \\ &= \int \frac{1}{\frac{4}{\cos^2 \theta} \times \frac{2 \sin \theta}{\cos \theta}} \frac{2 \sin \theta}{\cos^2 \theta} d\theta \end{aligned}$$

$$= \int \frac{\cos^3 \theta \cdot 2 \sin \theta}{8 \sin \theta \cos^2 \theta} d\theta$$

$$= \frac{1}{4} \int \cos \theta d\theta$$

$$= \frac{1}{4} \sin \theta + c$$

$$x = 2 \sec \theta = \frac{2}{\cos \theta} \Rightarrow \cos \theta = \frac{2}{x} \Rightarrow \theta = \cos^{-1} \frac{2}{x}$$

$$\text{So } \int \frac{1}{x^2 \sqrt{x^2 - 4}} dx = \frac{1}{4} \sin \left(\cos^{-1} \frac{2}{x} \right) + c$$

6 (i) $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2} dy$

$$\begin{aligned} \int \frac{1}{x \sqrt{x^2 - 1}} dx &= \int \frac{1}{\frac{1}{y} \sqrt{\frac{1}{y^2} - 1}} \left(-\frac{1}{y^2} dy \right) \\ &= - \int \frac{1}{y \sqrt{\frac{1}{y^2} - 1}} dy \\ &= - \int \frac{1}{y \sqrt{\frac{1 - y^2}{y^2}}} dy \\ &= - \int \frac{1}{y \frac{\sqrt{1 - y^2}}{y}} dy \\ &= - \int \frac{1}{\sqrt{1 - y^2}} dy \end{aligned}$$

(ii) $\int \frac{1}{x \sqrt{x^2 - 1}} dx = - \int \frac{1}{\sqrt{1 - y^2}} dy$

$$y = \sin \theta \Rightarrow dy = \cos \theta d\theta$$

$$\begin{aligned} \text{So } - \int \frac{1}{\sqrt{1 - y^2}} dy &= - \int \frac{1}{\sqrt{1 - \sin^2 \theta}} \cos \theta d\theta \\ &= - \int \frac{1}{\sqrt{\cos^2 \theta}} \cos \theta d\theta \\ &= - \int d\theta \\ &= -\theta + c \\ &= -\sin^{-1} y + c \\ &= -\sin^{-1} \left(\frac{1}{x} \right) + c \end{aligned}$$

$$7 \text{ (i)} \int \frac{2}{1+x^2} dx = 2 \tan^{-1} x + c$$

$$\begin{aligned} \text{(ii)} \int \frac{1}{9+x^2} dx &= \int \frac{1}{9\left(1+\frac{1}{9}x^2\right)} dx \\ &= \frac{1}{9} \int \frac{1}{\left(1+\left(\frac{x}{3}\right)^2\right)} dx \\ &= \frac{1}{9} \tan^{-1}\left(\frac{x}{3}\right) \times 3 + c \\ &= \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + c \end{aligned}$$

$$\begin{aligned} \text{(iii)} \int \frac{1}{2+3x^2} dx &= \int \frac{1}{2\left(1+\frac{3}{2}x^2\right)} dx \\ &= \frac{1}{2} \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}x\right) \times \frac{\sqrt{2}}{\sqrt{3}} + c \\ &= \frac{\sqrt{2}}{2\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}x\right) + c \\ &= \frac{\sqrt{6}}{6} \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}x\right) + c \end{aligned}$$

2.5 The use of partial fractions in integration (Page 22)

$$\begin{aligned} 1 \text{ (i)} \int \frac{7x+8}{(x+2)(x-1)} dx &= \int \left(\frac{2}{x+2} + \frac{5}{x-1}\right) dx \\ &= 2 \ln|x+2| + 5 \ln|x-1| + c \end{aligned}$$

$$\begin{aligned} \text{(ii)} \int \frac{4x^2-8x}{(x-1)^2(x+1)} dx &= \int \left(\frac{1}{x-1} - \frac{2}{(x-1)^2} + \frac{3}{x+1}\right) dx \\ &= \ln|x-1| + \frac{2}{x-1} + 3 \ln|x+1| + c \end{aligned}$$

$$2 \text{ (i)} \frac{x-5}{x^2-1} = \frac{3}{x+1} - \frac{2}{x-1}$$

$$\begin{aligned} \text{(ii)} \int_2^3 \frac{x-5}{x^2-1} dx &= \int_2^3 \left(\frac{3}{x+1} - \frac{2}{x-1}\right) dx \\ &= \left[3 \ln|x+1| - 2 \ln|x-1|\right]_2^3 \\ &= \left[\ln|x+1|^3 - \ln|x-1|^2\right]_2^3 \\ &= \left[\ln \frac{|x+1|^3}{|x-1|^2}\right]_2^3 \\ &= \left(\ln \frac{|3+1|^3}{|3-1|^2}\right) - \left(\ln \frac{|2+1|^3}{|2-1|^2}\right) \\ &= \ln \frac{4^3}{2^2} - \ln \frac{3^3}{1^2} \\ &= \ln 16 - \ln 27 \\ &= \ln \frac{16}{27} \end{aligned}$$

$$3 \text{ (i)} \frac{2x+1}{(x-3)^2} = \frac{2}{x-3} + \frac{7}{(x-3)^2}$$

$$\begin{aligned} \text{(ii)} \int_4^{10} \frac{2x+1}{(x-3)^2} dx &= \int_4^{10} \left(\frac{2}{x-3} + \frac{7}{(x-3)^2}\right) dx \\ &= \left[2 \ln|x-3| - \frac{7}{x-3}\right]_4^{10} \\ &= \left(2 \ln|10-3| - \frac{7}{10-3}\right) - \left(2 \ln|4-3| - \frac{7}{4-3}\right) \\ &= (2 \ln 7 - 1) - (2 \ln 1 - 7) \\ &= 2 \ln 7 + 6 \end{aligned}$$

$$4 \text{ (i)} u = \sqrt{2-x} \Rightarrow du = -\frac{1}{2\sqrt{2-x}} dx$$

$$dx = -2\sqrt{2-x} du$$

$$u^2 = 2-x \Rightarrow x = 2-u^2$$

$$x = 2 \Rightarrow u = 0, x = 1 \Rightarrow u = 1$$

$$\begin{aligned} I &= \int_1^2 \frac{3}{x + \sqrt{2-x}} dx \\ &= \int_{u=1}^{u=0} \left(\frac{3}{2-u^2+u}\right) (-2\sqrt{2-x} du) \\ &= -\int_1^0 \frac{6u}{2-u^2+u} du \\ &= \int_0^1 \frac{6u}{(2-u)(1+u)} du \end{aligned}$$

$$\begin{aligned} \text{(ii)} \int_0^1 \frac{6u}{(2-u)(1+u)} du &= \int_0^1 \left(\frac{4}{2-u} - \frac{2}{1+u}\right) du \\ &= \left[-4 \ln(2-u) - 2 \ln(1+u)\right]_0^1 \\ &= \left[(-4 \ln(2-1) - 2 \ln(1+1))\right. \\ &\quad \left.- (-4 \ln(2-0) - 2 \ln(1+0))\right] \\ &= (-4 \ln 1 - 2 \ln 2) - (-4 \ln 2 - 2 \ln 1) \\ &= (0 - \ln 4) - (-\ln 16) \\ &= \ln 16 - \ln 4 \\ &= \ln 4 \\ &= 2 \ln 2 \end{aligned}$$

2.6 Integration by parts (Page 24)

$$1 \text{ (i)} u = x \quad v' = \sin x$$

$$u' = 1 \quad v = -\cos x$$

$$\begin{aligned} \int x \sin x dx &= -x \cos x - \int -\cos x dx \\ &= -x \cos x + \sin x + c \end{aligned}$$

$$\text{(ii)} u = 4x \quad v' = e^x$$

$$u' = 4 \quad v = e^x$$

$$\begin{aligned} \int 4x e^x dx &= 4x e^x - \int 4e^x dx \\ &= 4x e^x - 4e^x + c \end{aligned}$$

(iii) $u = \ln x \quad v' = x^2$

$$u' = \frac{1}{x} \quad v = \frac{x^3}{3}$$

$$\begin{aligned} \int x^2 \ln x \, dx &= \frac{x^3}{3} \ln x - \int \frac{1}{x} \times \frac{x^3}{3} \, dx \\ &= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx \\ &= \frac{x^3}{3} \ln x - \frac{x^3}{9} + c \end{aligned}$$

2 (ii) $u = 2x - 1 \quad v' = e^{2x}$

$$u' = 2 \quad v = \frac{1}{2} e^{2x}$$

$$\begin{aligned} \int_1^2 (2x-1)e^{2x} \, dx &= \left[(2x-1) \frac{1}{2} e^{2x} \right]_1^2 - \int_1^2 2 \times \frac{1}{2} e^{2x} \, dx \\ &= \left[(2x-1) \frac{1}{2} e^{2x} \right]_1^2 - \int_1^2 e^{2x} \, dx \\ &= \left[(2x-1) \frac{1}{2} e^{2x} - \frac{1}{2} e^{2x} \right]_1^2 \\ &= \left[x e^{2x} - e^{2x} \right]_1^2 \\ &= \left[e^{2x} (x-1) \right]_1^2 \\ &= e^4 (2-1) - e^2 (1-1) \\ &= e^4 \end{aligned}$$

(iii) $u = x \quad v' = \cos 2x$

$$u' = 1 \quad v = \frac{1}{2} \sin 2x$$

$$\begin{aligned} \int_0^{\frac{\pi}{6}} x \cos 2x \, dx &= \left[\frac{1}{2} x \sin 2x \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{1}{2} \sin 2x \, dx \\ &= \left[\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{6}} \\ &= \left[\left(\frac{1}{2} \times \frac{\pi}{6} \sin \frac{\pi}{3} + \frac{1}{4} \cos \frac{\pi}{3} \right) - \left(\frac{1}{2} \times 0 \sin 0 + \frac{1}{4} \cos 0 \right) \right] \\ &= \left(\frac{1}{2} \times \frac{\pi}{6} \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{1}{2} \right) - \left(0 + \frac{1}{4} \right) \\ &= \left(\frac{\sqrt{3}\pi}{24} + \frac{1}{8} \right) - \left(\frac{1}{4} \right) \\ &= \frac{\sqrt{3}\pi}{24} - \frac{1}{8} \end{aligned}$$

3 (i) $y = x \cos 3x$

P and Q are when $y = 0$

$$x \cos 3x = 0$$

$$\cos 3x = 0$$

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \dots$$

P is $\left(\frac{\pi}{6}, 0\right)$ and Q is $\left(\frac{\pi}{2}, 0\right)$

(ii) $\frac{dy}{dx} = 1 \times \cos 3x + -3 \sin 3x \times x$
 $= \cos 3x - 3x \sin 3x$

$$\begin{aligned} \text{At } x = \frac{\pi}{6}, \frac{dy}{dx} &= \cos 3 \times \frac{\pi}{6} - 3 \times \frac{\pi}{6} \sin 3 \times \frac{\pi}{6} \\ &= \cos \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{2} \\ &= 0 - \frac{\pi}{2} \\ &= -\frac{\pi}{2} \end{aligned}$$

(iii) $u = x \quad v' = \cos 3x$

$$u' = 1 \quad v = \frac{1}{3} \sin 3x$$

$$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{6}} x \cos 3x \, dx \\ &= \left[\frac{1}{3} x \sin 3x \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{1}{3} \sin 3x \, dx \\ &= \left[\frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x \right]_0^{\frac{\pi}{6}} \\ &= \left[\left(\frac{1}{3} \times \frac{\pi}{6} \sin 3 \times \frac{\pi}{6} + \frac{1}{9} \cos 3 \times \frac{\pi}{6} \right) - \left(\frac{1}{3} \times 0 \sin 3 \times 0 + \frac{1}{9} \cos 3 \times 0 \right) \right] \\ &= \left(\frac{\pi}{18} \sin \frac{\pi}{2} + \frac{1}{9} \cos \frac{\pi}{2} \right) - \left(0 + \frac{1}{9} \right) \\ &= \left(\frac{\pi}{18} \right) - \left(0 + \frac{1}{9} \right) \\ &= \frac{\pi}{18} - \frac{1}{9} \\ &= \frac{\pi-2}{18} \end{aligned}$$

4 (i) $\frac{d}{dx} \left(\frac{\ln x}{x^2} \right) = \frac{\frac{1}{x} \times x^2 - 2x \times \ln x}{(x^2)^2}$
 $= \frac{x - 2x \ln x}{x^4}$
 $= \frac{x(1 - 2 \ln x)}{x^4}$

$$(ii) \quad u = \ln x \quad v' = \frac{1}{x^2}$$

$$u' = \frac{1}{x} \quad v = -\frac{1}{x}$$

$$\begin{aligned} & \int \frac{\ln x}{x^2} dx \\ &= \int \frac{1}{x^2} \ln x dx \\ &= -\frac{1}{x} \ln x - \int \left(\frac{1}{x} \times -\frac{1}{x} \right) dx \\ &= -\frac{1}{x} \ln x - \int \left(-\frac{1}{x^2} \right) dx \\ &= -\frac{1}{x} \ln x - \frac{1}{x} + c \\ &= -\frac{1}{x} (\ln x + 1) + c \end{aligned}$$

$$5 \quad u = \ln x \quad v' = x^4$$

$$u' = \frac{1}{x} \quad v = \frac{x^5}{5}$$

$$\begin{aligned} & \int_1^e x^4 \ln x dx \\ &= \left[\frac{x^5}{5} \ln x \right]_1^e - \int_1^e \left(\frac{1}{x} \times \frac{x^5}{5} \right) dx \\ &= \left[\frac{x^5}{5} \ln x \right]_1^e - \int_1^e \left(\frac{x^4}{5} \right) dx \\ &= \left[\frac{x^5}{5} \ln x - \frac{x^5}{25} \right]_1^e \\ &= \left(\frac{e^5}{5} \ln e - \frac{e^5}{25} \right) - \left(\frac{1^5}{5} \ln 1 - \frac{1^5}{25} \right) \\ &= \left(\frac{e^5}{5} - \frac{e^5}{25} \right) - \left(-\frac{1}{25} \right) \\ &= \frac{4e^5 + 1}{25} \end{aligned}$$

$$6 \quad (i) \quad \frac{d}{dx} (e^{\cos x}) = (-\sin x) e^{\cos x}$$

$$(ii) \quad u = \cos x \quad v' = \sin x e^{\cos x}$$

$$u' = -\sin x \quad v = -e^{\cos x}$$

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \cos x \sin x e^{\cos x} dx \\ &= \int_0^{\frac{1}{2}\pi} \cos x \times \sin x e^{\cos x} dx \\ &= \left[-\cos x e^{\cos x} \right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \sin x e^{\cos x} dx \\ &= \left[-\cos x e^{\cos x} + e^{\cos x} \right]_0^{\frac{1}{2}\pi} \\ &= \left[(1 - \cos x) e^{\cos x} \right]_0^{\frac{1}{2}\pi} \\ &= \left(1 - \cos \frac{1}{2}\pi \right) e^{\cos \frac{1}{2}\pi} - (1 - \cos 0) e^{\cos 0} \\ &= e^0 - (1 - 1) e^{\cos 0} \\ &= 1 \end{aligned}$$

$$7 \quad u = e^x \quad v' = \sin x$$

$$u' = e^x \quad v = -\cos x$$

$$\begin{aligned} I &= \int e^x \sin x dx \\ &= -e^x \cos x - \int -e^x \cos x dx \\ &= -e^x \cos x + \int e^x \cos x dx \\ u &= e^x \quad v' = \cos x \\ u' &= e^x \quad v = \sin x \\ I &= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x dx \right) \\ &= -e^x \cos x + e^x \sin x - I + c \\ 2I &= -e^x \cos x + e^x \sin x + c \\ &= e^x (\sin x - \cos x) + c \\ I &= \frac{e^x (\sin x - \cos x)}{2} + c \end{aligned}$$

Further practice (Page 27)

$$1 \quad (i) \quad u = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow dx = \frac{du}{2x}$$

$$\begin{aligned} & \int 2x(x^2 + 1)^4 dx \\ &= \int 2x(u)^4 \frac{du}{2x} \\ &= \int u^4 du \\ &= \frac{u^5}{5} + c \\ &= \frac{(x^2 + 1)^5}{5} + c \end{aligned}$$

$$(ii) \quad u = 3 + 2x^3 \Rightarrow du = 6x^2 dx \Rightarrow dx = \frac{du}{6x^2}$$

$$\begin{aligned} & \int \frac{x^2}{3 + 2x^3} dx \\ &= \int \frac{x^2}{(u) 6x^2} \frac{du}{6x^2} \\ &= \frac{1}{6} \int \frac{1}{u} du \\ &= \frac{1}{6} \ln |u| + c \\ &= \frac{1}{6} \ln |3 + 2x^3| + c \end{aligned}$$

$$(iii) \quad u = 4 - x \Rightarrow du = -dx \Rightarrow dx = -du$$

$$\text{Also } x = 4 - u$$

$$\begin{aligned} & \int \frac{2x}{(4-x)^5} dx \\ &= \int \frac{2(4-u)}{(u)^5} (-du) \\ &= 2 \int \frac{u-4}{u^5} du \\ &= 2 \int \left(\frac{u}{u^5} - \frac{4}{u^5} \right) du \end{aligned}$$

$$\begin{aligned}
&= 2 \int (u^{-4} - 4u^{-5}) du \\
&= 2 \left[\frac{u^{-3}}{-3} - \frac{4u^{-4}}{-4} \right] + c \\
&= 2 \left[-\frac{1}{3u^3} + \frac{1}{u^4} \right] + c \\
&= -\frac{2}{3u^3} + \frac{2}{u^4} + c \\
&= -\frac{2}{3(4-x)^3} + \frac{2}{(4-x)^4} + c
\end{aligned}$$

2 (i) $u = 2 + x \Rightarrow du = dx$, also $x = u - 2$

$x = 2 \Rightarrow u = 4$, $x = 1 \Rightarrow u = 3$

$$\begin{aligned}
&\int_1^2 \frac{x}{(2+x)^3} dx \\
&= \int_{u=3}^{u=4} \frac{u-2}{u^3} du \\
&= \int_3^4 (u^{-2} - 2u^{-3}) du \\
&= \left[\frac{u^{-1}}{-1} - \frac{2u^{-2}}{-2} \right]_3^4 \\
&= \left[-\frac{1}{u} + \frac{1}{u^2} \right]_3^4 \\
&= \left(-\frac{1}{4} + \frac{1}{4^2} \right) - \left(-\frac{1}{3} + \frac{1}{3^2} \right) \\
&= \left(-\frac{3}{16} \right) - \left(-\frac{2}{9} \right) \\
&= \frac{5}{144}
\end{aligned}$$

(ii) $u = x^2 + 1 \Rightarrow du = 2x dx$

$x = \sqrt{3} \Rightarrow u = 4$, $x = 0 \Rightarrow u = 1$

$u = x^2 + 1 \Rightarrow x^2 = u - 1 \Rightarrow x^4 = (u - 1)^2$

$$\begin{aligned}
&\int_0^{\sqrt{3}} x^5 \sqrt{x^2 + 1} dx \\
&= \int_{u=1}^{u=4} x^5 \sqrt{u} \frac{du}{2x} \\
&= \frac{1}{2} \int_{u=1}^{u=4} x^4 \sqrt{u} du \\
&= \frac{1}{2} \int_1^4 (u-1)^2 \sqrt{u} du \\
&= \frac{1}{2} \int_1^4 (u^2 - 2u + 1) \sqrt{u} du \\
&= \frac{1}{2} \int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du \\
&= \frac{1}{2} \left[\frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\
&= \frac{1}{2} \left[\frac{2u^{\frac{7}{2}}}{7} - \frac{4u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} \right]_1^4
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\left(\frac{2 \times 4^{\frac{7}{2}}}{7} - \frac{4 \times 4^{\frac{5}{2}}}{5} + \frac{2 \times 4^{\frac{3}{2}}}{3} \right) \right. \\
&\quad \left. - \left(\frac{2 \times 1^{\frac{7}{2}}}{7} - \frac{4 \times 1^{\frac{5}{2}}}{5} + \frac{2 \times 1^{\frac{3}{2}}}{3} \right) \right] \\
&= \frac{1}{2} \left[\frac{1712}{105} - \frac{16}{105} \right] \\
&= \frac{848}{105} \approx 8.08
\end{aligned}$$

3 (i) $u = x - x^2 \Rightarrow du = (1 - 2x) dx$

$$\begin{aligned}
&\int \frac{1-2x}{x-x^2} dx \\
&= \int \frac{1-2x}{u} \frac{du}{1-2x} \\
&= \int \frac{1}{u} du \\
&= \ln|u| + c \\
&= \ln|x - x^2| + c
\end{aligned}$$

(ii) $u = 1 + e^{3x} \Rightarrow du = 3e^{3x} dx$

$$\begin{aligned}
&\int \frac{e^{3x}}{(1 + e^{3x})^3} dx \\
&= \int \frac{e^{3x}}{(u)^3} \frac{du}{3e^{3x}} \\
&= \frac{1}{3} \int \frac{1}{u^3} du \\
&= \frac{1}{3} \left(\frac{u^{-2}}{-2} \right) + c \\
&= \frac{1}{3} \left(-\frac{1}{2u^2} \right) + c \\
&= -\frac{1}{6u^2} + c \\
&= -\frac{1}{6(1 + e^{3x})^2} + c
\end{aligned}$$

(iii) $u = 1 - 3x^3 \Rightarrow du = -9x^2 dx$

$$\begin{aligned}
&\int \frac{6x^2}{1-3x^3} dx \\
&= \int \frac{6x^2}{u} \frac{du}{-9x^2} \\
&= -\frac{2}{3} \int \frac{1}{u} du \\
&= -\frac{2}{3} \ln|u| + c \\
&= -\frac{2}{3} \ln|1 - 3x^3| + c
\end{aligned}$$

4 (i) $u = 3x - 1 \Rightarrow du = 3dx$

$$x = 1 \Rightarrow u = 2, x = \frac{2}{3} \Rightarrow u = 1$$

$$\begin{aligned} \text{Area} &= \int_{\frac{2}{3}}^1 \frac{x^2}{3x-1} dx \\ &= \int_{u=1}^{u=2} \frac{\left(\frac{u+1}{3}\right)^2}{u} \frac{du}{3} \\ &= \int_1^2 \frac{\left(\frac{u^2 + 2u + 1}{9}\right)}{u} \frac{du}{3} \\ &= \frac{1}{27} \int_1^2 \frac{u^2 + 2u + 1}{u} du \\ &= \frac{1}{27} \int_1^2 \left(u + 2 + \frac{1}{u}\right) du \\ &= \frac{1}{27} \left[\frac{u^2}{2} + 2u + \ln u \right]_1^2 \\ &= \frac{1}{27} \left[\left(\frac{2^2}{2} + 2 \times 2 + \ln 2 \right) - \left(\frac{1^2}{2} + 2 \times 1 + \ln 1 \right) \right] \\ &= \frac{1}{27} \left[(6 + \ln 2) - \left(\frac{5}{2} \right) \right] \\ &= \frac{1}{27} \left(\frac{7}{2} + \ln 2 \right) \approx 0.155 \end{aligned}$$

(ii) $\frac{dy}{dx} = \frac{2x(3x-1) - 3x^2}{(3x-1)^2}$

$$\begin{aligned} &= \frac{3x^2 - 2x}{(3x-1)^2} \\ &= \frac{x(3x-2)}{(3x-1)^2} \end{aligned}$$

Stationary point when $\frac{dy}{dx} = 0$

$$\frac{x(3x-2)}{(3x-1)^2} = 0 \Rightarrow x(3x-2) = 0$$

$$x = 0 \text{ or } \frac{2}{3}$$

$$\text{When } x = 0, y = \frac{(0)^2}{3(0)-1} = 0 \Rightarrow (0, 0)$$

$$\text{When } x = \frac{2}{3}, y = \frac{\left(\frac{2}{3}\right)^2}{3\left(\frac{2}{3}\right)-1} = \frac{4}{9} \Rightarrow \left(\frac{2}{3}, \frac{4}{9}\right)$$

5 (i) $u = 1 + \sin 3x \Rightarrow du = 3 \cos 3x dx$

$$\begin{aligned} &\int \cos 3x (1 + \sin 3x)^3 dx \\ &= \int \cos 3x (u)^3 \frac{du}{3 \cos 3x} \\ &= \frac{1}{3} \int u^3 du \\ &= \frac{1}{3} \left[\frac{u^4}{4} \right] + c \\ &= \frac{u^4}{12} + c \\ &= \frac{(1 + \sin 3x)^4}{12} + c \end{aligned}$$

(ii) $u = \cos 4x \Rightarrow du = -4 \sin 4x dx$

$$\begin{aligned} &\int \tan 4x dx \\ &= \int \frac{\sin 4x}{\cos 4x} dx \\ &= \int \frac{\sin 4x}{u} \frac{du}{-4 \sin 4x} \\ &= -\frac{1}{4} \int \frac{1}{u} du \\ &= -\frac{1}{4} \ln |u| + c \\ &= -\frac{1}{4} \ln |\cos 4x| + c \end{aligned}$$

(iii) $u = \sin 3x \Rightarrow du = 3 \cos 3x dx$

$$\begin{aligned} &\int 2 \sin^3 3x \cos 3x dx \\ &= \int 2u^3 \cos 3x \frac{du}{3 \cos 3x} \\ &= \frac{2}{3} \int u^3 du \\ &= \frac{2}{3} \left(\frac{u^4}{4} \right) + c \\ &= \frac{\sin^4 3x}{3} + c \end{aligned}$$

(iv) $u = 2\theta \Rightarrow du = 2d\theta$

$$\begin{aligned} &\int \frac{1}{\cos^2 2\theta} d\theta \\ &= \int \frac{1}{\cos^2 u} \frac{du}{2} \\ &= \frac{1}{2} \int \frac{1}{\cos^2 u} du \\ &= \frac{1}{2} \int \sec^2 u du \\ &= \frac{1}{2} \tan u + c \\ &= \frac{1}{2} \tan 2\theta + c \end{aligned}$$

6 $x = \cos \theta \Rightarrow dx = -\sin \theta d\theta$

$$x = 1 \Rightarrow \cos \theta = 1 \Rightarrow \theta = 0$$

$$x = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$\begin{aligned} &\int_0^1 \frac{x^2}{\sqrt{1-x^2}} dx \\ &= \int_{\theta=\frac{\pi}{2}}^{\theta=0} \frac{\cos^2 \theta}{\sqrt{1-\cos^2 \theta}} (-\sin \theta d\theta) \\ &= \int_{\frac{\pi}{2}}^0 \frac{\cos^2 \theta}{\sqrt{\sin^2 \theta}} (-\sin \theta d\theta) \\ &= \int_{\frac{\pi}{2}}^0 \frac{\cos^2 \theta}{\sin \theta} (-\sin \theta d\theta) \\ &= -\int_{\frac{\pi}{2}}^0 \cos^2 \theta d\theta \end{aligned}$$

$$\begin{aligned}
&= -\int_{\frac{\pi}{2}}^0 \left(\frac{1}{2}(\cos 2\theta + 1) \right) d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1) d\theta \\
&= \frac{1}{2} \left[\frac{1}{2} \sin 2\theta + \theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2} \left[\left(\frac{1}{2} \sin 2 \times \frac{\pi}{2} + \frac{\pi}{2} \right) - \left(\frac{1}{2} \sin 2 \times 0 + 0 \right) \right] \\
&= \frac{1}{2} \left[\left(\frac{1}{2} \sin \pi + \frac{\pi}{2} \right) - 0 \right] \\
&= \frac{1}{2} \left(0 + \frac{\pi}{2} \right) \\
&= \frac{\pi}{4}
\end{aligned}$$

7 $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$$x = 1 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$x = 0 \Rightarrow \tan \theta = 0 \Rightarrow \theta = 0$$

$$\begin{aligned}
&\int_0^1 \frac{1}{(x^2 + 1)^2} dx \\
&= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{1}{(\tan^2 \theta + 1)^2} \sec^2 \theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{(\sec^2 \theta)^2} \sec^2 \theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 \theta} d\theta \\
&= \int_0^{\frac{\pi}{4}} \cos^2 \theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}(\cos 2\theta + 1) \right) d\theta \\
&= \frac{1}{2} \left[\frac{1}{2} \sin 2\theta + \theta \right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2} \left[\left(\frac{1}{2} \sin 2 \times \frac{\pi}{4} + \frac{\pi}{4} \right) - \left(\frac{1}{2} \sin 2 \times 0 + 0 \right) \right] \\
&= \frac{1}{2} \left[\left(\frac{1}{2} \sin \frac{\pi}{2} + \frac{\pi}{4} \right) - 0 \right] \\
&= \frac{1}{2} \left[\left(\frac{1}{2} + \frac{\pi}{4} \right) - 0 \right] \\
&= \frac{1}{4} + \frac{\pi}{8}
\end{aligned}$$

8 $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$$x = \sqrt{3} \Rightarrow \tan \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

$$x = 1 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$\begin{aligned}
&\int_1^{\sqrt{3}} \frac{1-x^2}{1+x^2} dx \\
&= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} \frac{1-\tan^2 \theta}{1+\tan^2 \theta} \sec^2 \theta d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1-(\sec^2 \theta - 1)}{\sec^2 \theta} \sec^2 \theta d\theta
\end{aligned}$$

$$\begin{aligned}
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (2 - \sec^2 \theta) d\theta \\
&= \left[2\theta - \tan \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
&= \left(2 \times \frac{\pi}{3} - \tan \frac{\pi}{3} \right) - \left(2 \times \frac{\pi}{4} - \tan \frac{\pi}{4} \right) \\
&= \left(\frac{2\pi}{3} - \sqrt{3} \right) - \left(\frac{\pi}{2} - 1 \right) \\
&= \frac{\pi}{6} + 1 - \sqrt{3}
\end{aligned}$$

9 $\int \frac{5x^2 + 12x + 1}{(x+3)(x^2+1)} dx = \int \left(\frac{1}{x+3} + \frac{4x}{x^2+1} \right) dx$
 $= \ln|x+3| + 2\ln(x^2+1) + c$

10 (i) $\frac{2x^3 + 3x^2 + 9x + 12}{x^2 + 4} = 2x + 3 + \frac{x}{x^2 + 4}$
 $A = 2, B = 3, C = 1, D = 0$

(ii) $\int_1^3 \frac{2x^3 + 3x^2 + 9x + 12}{x^2 + 4} dx$
 $= \int_1^3 \left(2x + 3 + \frac{x}{x^2 + 4} \right) dx$
 $= \left[x^2 + 3x + \frac{1}{2} \ln(x^2 + 4) \right]_1^3$
 $= \left[\left(3^2 + 3 \times 3 + \frac{1}{2} \ln(3^2 + 4) \right) - \left(1^2 + 3 \times 1 + \frac{1}{2} \ln(1^2 + 4) \right) \right]$
 $= \left(18 + \frac{1}{2} \ln(13) \right) - \left(4 + \frac{1}{2} \ln(5) \right)$
 $= 14 + \frac{1}{2} (\ln 13 - \ln 5)$
 $= 14 + \frac{1}{2} \ln \frac{13}{5}$
 $= 14 + \ln \sqrt{\frac{13}{5}}$

11 $u = \sin \theta \Rightarrow du = \cos \theta d\theta$

$$\theta = \frac{\pi}{2} \Rightarrow u = 1, \theta = 0 \Rightarrow u = 0$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin^2 \theta - 5 \sin \theta + 6} d\theta \\
&= \int_{u=0}^{u=1} \frac{\cos \theta}{u^2 - 5u + 6} \frac{du}{\cos \theta} \\
&= \int_0^1 \frac{1}{u^2 - 5u + 6} du \\
&= \int_0^1 \frac{1}{(u-3)(u-2)} du \\
&= \int_0^1 \left(\frac{1}{u-3} - \frac{1}{u-2} \right) du \\
&= \left[\ln|u-3| - \ln|u-2| \right]_0^1
\end{aligned}$$

$$\begin{aligned}
&= \left[\ln \left| \frac{u-3}{u-2} \right| \right]_0^1 \\
&= \ln \left| \frac{1-3}{1-2} \right| - \ln \left| \frac{0-3}{0-2} \right| \\
&= \ln 2 - \ln \frac{3}{2} \\
&= \ln \left(\frac{2}{\frac{3}{2}} \right) \\
&= \ln \frac{4}{3}
\end{aligned}$$

12 (i) $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2\sqrt{x} du$

$$\begin{aligned}
&\int \frac{1}{x(1+\sqrt{x})} dx \\
&= \int \frac{1}{u^2(1+u)} 2\sqrt{x} du \\
&= \int \frac{1}{u^2(1+u)} 2u du \\
&= \int \frac{2}{u(1+u)} du
\end{aligned}$$

(ii) $\int_1^9 \frac{1}{x(1+\sqrt{x})} dx$
 $x = 9 \Rightarrow u = 3, x = 1 \Rightarrow u = 1$

$$\begin{aligned}
&\int_1^9 \frac{1}{x(1+\sqrt{x})} dx \\
&= \int_{u=1}^{u=3} \frac{2}{u(1+u)} du \\
&= \int_1^3 \left(\frac{2}{u} - \frac{2}{1+u} \right) du \\
&= [2\ln|u| - 2\ln|1+u|]_1^3 \\
&= \left[\ln \left(\frac{u}{1+u} \right)^2 \right]_1^3 \\
&= \ln \left(\frac{3}{1+3} \right)^2 - \ln \left(\frac{1}{1+1} \right)^2 \\
&= \ln \frac{9}{16} - \ln \frac{1}{4} \\
&= \ln \frac{9}{4}
\end{aligned}$$

13 $\frac{4x^2+4x-17}{2x^2+5x-3} = 2 - \frac{6x+11}{2x^2+5x-3}$

$$\begin{aligned}
&= 2 - \frac{6x+11}{(2x-1)(x+3)} \\
&= 2 - \left(\frac{4}{2x-1} + \frac{1}{x+3} \right) \\
&= 2 - \frac{4}{2x-1} - \frac{1}{x+3}
\end{aligned}$$

$$\begin{aligned}
&\int_1^2 \frac{4x^2+4x-17}{2x^2+5x-3} dx \\
&= \int_1^2 \left(2 - \frac{4}{2x-1} - \frac{1}{x+3} \right) dx \\
&= [2x - 2\ln(2x-1) - \ln(x+3)]_1^2 \\
&= [2x - (\ln(2x-1)^2 + \ln(x+3))]_1^2 \\
&= [2x - (\ln(2x-1)^2(x+3))]_1^2 \\
&= (4 - \ln(9 \times 5)) - (2 - \ln(1 \times 4)) \\
&= 4 - \ln 45 - 2 + \ln 4 \\
&= 2 + \ln \frac{4}{45} \\
&= 2 - \ln \frac{45}{4}
\end{aligned}$$

14 (i) $\frac{d}{dx}(\sec x) = \frac{d}{dx} \left(\frac{1}{\cos x} \right)$

$$\begin{aligned}
&= \frac{0 \times \cos x - (-\sin x) \times 1}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

(ii) $u = x^3 \Rightarrow du = 3x^2 dx$

$$\begin{aligned}
&\int x^2 \sec x^3 \tan x^3 dx \\
&= \int x^2 \sec u \tan u \frac{du}{3x^2} \\
&= \frac{1}{3} \int \sec u \tan u du \\
&= \frac{1}{3} \sec u + c \\
&= \frac{1}{3} \sec x^3 + c
\end{aligned}$$

(iii) $\int_0^{\frac{\pi}{3}} (\sec x + \tan x)^2 dx$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{3}} (\sec^2 x + 2\sec x \tan x + \tan^2 x) dx \\
&= \int_0^{\frac{\pi}{3}} (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1) dx \\
&= \int_0^{\frac{\pi}{3}} (2\sec^2 x + 2\sec x \tan x - 1) dx \\
&= [2\tan x + 2\sec x - x]_0^{\frac{\pi}{3}} \\
&= \left(2\tan \frac{\pi}{3} + 2\sec \frac{\pi}{3} - \frac{\pi}{3} \right) - (2\tan 0 + 2\sec 0 - 0) \\
&= \left(2\sqrt{3} + 2 \times 2 - \frac{\pi}{3} \right) - (2) \\
&= 2\sqrt{3} + 2 - \frac{\pi}{3} \\
&= 2(\sqrt{3} + 1) - \frac{\pi}{3}
\end{aligned}$$

$$\begin{aligned}
 15 \text{ (i)} \quad \frac{dy}{dx} &= 8 \sin x \cos x \times \cos^3 x + 3 \cos^2 x \times -\sin x \times 4 \sin^2 x \\
 &= 8 \sin x \cos^4 x - 12 \cos^2 x \sin^3 x \\
 &= 4 \sin x \cos^2 x (2 \cos^2 x - 3 \sin^2 x)
 \end{aligned}$$

$$\frac{dy}{dx} = 0 \text{ when}$$

$$\sin x = 0 \Rightarrow x = 0 \text{ (not M)}$$

$$\cos^2 x = 0 \Rightarrow x = \frac{\pi}{2} \text{ (not M)}$$

$$2 \cos^2 x - 3 \sin^2 x = 0$$

$$\frac{2 \cos^2 x}{\cos^2 x} - \frac{3 \sin^2 x}{\cos^2 x} = \frac{0}{\cos^2 x}$$

$$2 - 3 \tan^2 x = 0$$

$$\tan^2 x = \frac{2}{3}$$

$$\tan x = \pm \sqrt{\frac{2}{3}}$$

$$x = 0.685 \text{ (3 s.f.)}$$

$$(ii) \text{ Area} = \int_0^{\frac{\pi}{2}} 4 \sin^2 x \cos^3 x \, dx$$

$$u = \sin x \Rightarrow du = \cos x \, dx$$

$$x = \frac{\pi}{2} \Rightarrow u = 1, \quad x = 0 \Rightarrow u = 0$$

$$\int_0^{\frac{\pi}{2}} 4 \sin^2 x \cos^3 x \, dx$$

$$= 4 \int_0^1 u^2 \cos^2 x \frac{du}{\cos x}$$

$$= 4 \int_0^1 u^2 \cos^2 x \, du$$

$$= 4 \int_0^1 u^2 (1 - \sin^2 x) \, du$$

$$= 4 \int_0^1 u^2 (1 - u^2) \, du$$

$$= 4 \int_0^1 (u^2 - u^4) \, du$$

$$= 4 \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1$$

$$= 4 \left[\left(\frac{1^3}{3} - \frac{1^5}{5} \right) - 0 \right]$$

$$= 4 \left(\frac{2}{15} \right)$$

$$= \frac{8}{15}$$

$$\begin{aligned}
 16 \text{ (i)} \quad \frac{dy}{dx} &= 1 \times \sqrt{\ln x} + \frac{1}{2} (\ln x)^{-\frac{1}{2}} \times -\frac{1}{x} \times x \\
 &= \sqrt{\ln x} - \frac{1}{2\sqrt{\ln x}}
 \end{aligned}$$

$$\text{When } x = 2,$$

$$\frac{dy}{dx} = \sqrt{\ln 2} - \frac{1}{2\sqrt{\ln 2}} = \frac{2\ln 2 - 1}{2\sqrt{\ln 2}} = 0.232 \text{ (3 s.f.)}$$

$$y = mx + c$$

$$\Rightarrow 2\sqrt{\ln 2} = \frac{2\ln 2 - 1}{2\sqrt{\ln 2}} \times 2 + c$$

$$c = 2\sqrt{\ln 2} - \frac{4\ln 2 - 2}{2\sqrt{\ln 2}}$$

$$= \frac{4\ln 2 - (4\ln 2 - 2)}{2\sqrt{\ln 2}}$$

$$= \frac{2}{2\sqrt{\ln 2}}$$

$$= \frac{1}{\sqrt{\ln 2}}$$

$$y = \frac{2\ln 2 - 1}{2\sqrt{\ln 2}} x + \frac{1}{\sqrt{\ln 2}}$$

$$(ii) \quad x\sqrt{\ln x} = 0$$

$$\Rightarrow x = 0 \text{ or } \ln x = 0 \Rightarrow x = 1$$

$$\text{Volume} = \int_1^e \pi y^2 \, dx$$

$$= \pi \int_1^e (x\sqrt{\ln x})^2 \, dx$$

$$= \pi \int_1^e x^2 \ln x \, dx$$

$$u = \ln x, \quad v' = x^2$$

$$u' = \frac{1}{x}, \quad v' = \frac{x^3}{3}$$

$$\text{Volume} = \pi \left[\frac{x^3}{3} \ln x \right]_1^e - \pi \int_1^e \frac{1}{x} \times \frac{x^3}{3} \, dx$$

$$= \pi \left[\frac{x^3}{3} \ln x \right]_1^e - \pi \int_1^e \frac{x^2}{3} \, dx$$

$$= \pi \left[\frac{x^3}{3} \ln x - \frac{x^3}{9} \right]_1^e$$

$$= \pi \left[\left(\frac{e^3}{3} \ln e - \frac{e^3}{9} \right) - \left(\frac{1^3}{3} \ln 1 - \frac{1^3}{9} \right) \right]$$

$$= \pi \left[\left(\frac{e^3}{3} - \frac{e^3}{9} \right) - \left(-\frac{1}{9} \right) \right]$$

$$= \pi \left(\frac{2e^3}{9} + \frac{1}{9} \right)$$

$$= \pi \left(\frac{2e^3 + 1}{9} \right) \approx 14.4$$

$$17 \text{ (i)} \quad \int \sin(1 - 4x) \, dx$$

$$= -\cos(1 - 4x) \times -\frac{1}{4} + c$$

$$= \frac{1}{4} \cos(1 - 4x) + c$$

$$(ii) \quad \int \frac{e^{2x}}{e^{2x} + 1} \, dx$$

$$= \frac{1}{2} \int \frac{2e^{2x}}{e^{2x} + 1} \, dx$$

$$= \frac{1}{2} \ln |e^{2x} + 1| + c$$

$$(iii) \int x^2 e^{1-x^3} dx$$

$$u = 1 - x^3 \Rightarrow \frac{du}{dx} = -3x^2$$

$$\begin{aligned} \int x^2 e^{1-x^3} dx &= \int x^2 e^u \frac{du}{-3x^2} \\ &= -\frac{1}{3} \int e^u du \\ &= -\frac{1}{3} e^u + c \\ &= -\frac{1}{3} e^{1-x^3} + c \end{aligned}$$

$$(iv) \int 2x \cos 2x dx$$

$$u = 2x \quad v' = \cos 2x$$

$$u' = 2 \quad v = \frac{1}{2} \sin 2x$$

$$\begin{aligned} \int 2x \cos 2x dx &= 2x \left(\frac{1}{2} \sin 2x \right) - \int \left(2 \times \frac{1}{2} \sin 2x \right) dx \\ &= x \sin 2x - \int \sin 2x dx \\ &= x \sin 2x + \frac{1}{2} \cos 2x + c \end{aligned}$$

$$(v) \int \frac{1}{x^2 + 25} dx$$

$$= \frac{1}{5} \tan^{-1} \left(\frac{x}{5} \right) + c$$

$$(vi) \int \frac{2x}{(4-x)^5} dx$$

$$u = 4 - x \Rightarrow du = -dx$$

$$\begin{aligned} \int \frac{2x}{(4-x)^5} dx &= \int \frac{2(4-u)}{u^5} (-du) \\ &= -2 \int \frac{(4-u)}{u^5} du \\ &= -2 \int (4u^{-5} - u^{-4}) du \\ &= -2 \left(\frac{4u^{-4}}{-4} - \frac{u^{-3}}{-3} \right) + c \\ &= -2 \left(-\frac{1}{u^4} + \frac{1}{3u^3} \right) + c \\ &= \frac{2}{u^4} - \frac{2}{3u^3} + c \\ &= \frac{2}{(4-x)^4} - \frac{2}{3(4-x)^3} + c \end{aligned}$$

$$(vii) \int \frac{x-5}{x^2-1} dx$$

$$\frac{x-5}{(x-1)(x+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1}$$

$$x-5 \equiv A(x+1) + B(x-1)$$

$$x=1 \Rightarrow -4 = 2A \Rightarrow A = -2$$

$$x=-1 \Rightarrow -6 = -2B \Rightarrow B = 3$$

$$\frac{x-5}{(x-1)(x+1)} \equiv \frac{-2}{x-1} + \frac{3}{x+1}$$

$$\begin{aligned} \int \frac{x-5}{x^2-1} dx &= \int \left(\frac{3}{x+1} - \frac{2}{x-1} \right) dx \\ &= 3 \ln|x+1| - 2 \ln|x-1| + c \end{aligned}$$

$$(viii) \int \frac{\ln x}{x^2} dx$$

$$u = \ln x \quad v' = x^{-2}$$

$$u' = \frac{1}{x} \quad v = -x^{-1}$$

$$\begin{aligned} \int \frac{\ln x}{x^2} dx &= -\frac{1}{x} \ln x - \int \left(\frac{1}{x} \times -x^{-1} \right) dx \\ &= -\frac{1}{x} \ln x + \int x^{-2} dx \\ &= -\frac{1}{x} \ln x - \frac{1}{x} + c \end{aligned}$$

Past exam questions (Page 29)

$$\begin{aligned} 1 \quad (i) \quad \frac{2}{(x+1)(x+3)} &\equiv \frac{A}{x+1} + \frac{B}{x+3} \\ 2 &\equiv A(x+3) + B(x+1) \end{aligned}$$

$$x = -1 \Rightarrow 2 \equiv 2A \Rightarrow A = 1$$

$$x = -3 \Rightarrow 2 \equiv -2B \Rightarrow B = -1$$

$$\text{So } \frac{2}{(x+1)(x+3)} = \frac{1}{x+1} - \frac{1}{x+3}$$

$$\begin{aligned} (ii) \quad \left(\frac{2}{(x+1)(x+3)} \right)^2 &= \left(\frac{1}{x+1} - \frac{1}{x+3} \right)^2 \\ &= \frac{1}{(x+1)^2} - \frac{2}{(x+1)(x+3)} + \frac{1}{(x+3)^2} \\ &= \frac{1}{(x+1)^2} - \left(\frac{1}{x+1} - \frac{1}{x+3} \right) + \frac{1}{(x+3)^2} \\ &= \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2} \end{aligned}$$

$$\begin{aligned} (iii) \quad \int_0^1 \frac{4}{(x+1)^2(x+3)^2} dx &= \int_0^1 \left[(x+1)^{-2} - \frac{1}{x+1} + \frac{1}{x+3} + (x+3)^{-2} \right] dx \\ &= \left[-\frac{1}{x+1} - \ln(x+1) + \ln(x+3) + \left(-\frac{1}{x+3} \right) \right]_0^1 \\ &= \left(-\frac{1}{2} - \ln 2 + \ln 4 - \frac{1}{4} \right) - \left(-1 - \ln 1 + \ln 3 - \frac{1}{3} \right) \\ &= \left(-\frac{3}{4} + \ln 2 \right) - \left(-\frac{4}{3} + \ln 3 \right) \\ &= \frac{7}{12} + \ln \frac{2}{3} \\ &= \frac{7}{12} - \ln \frac{3}{2} \end{aligned}$$

2 (i) $y = x^3 \ln x$

$$\frac{dy}{dx} = 3x^2 \ln x + \frac{1}{x} \times x^3 = x^2(3 \ln x + 1)$$

$$\frac{dy}{dx} = 0 \text{ when } 3 \ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{3}$$

$$x = e^{-\frac{1}{3}} \text{ or } \frac{1}{\sqrt[3]{e}}$$

(ii) $y = x^3 \ln x$

$$y = 0 \text{ when } \ln x = 0 \text{ so } x = 1$$

$$u' = x^3 \quad v = \ln x$$

$$u = \frac{x^4}{4} \quad v' = \frac{1}{x}$$

$$\text{So } A = \int_1^2 x^3 \ln x \, dx$$

$$= \left[\frac{x^4}{4} \ln x \right]_1^2 - \int_1^2 \frac{x^4}{4} \times \frac{1}{x} \, dx$$

$$= \left[\frac{x^4}{4} \ln x \right]_1^2 - \int_1^2 \frac{x^3}{4} \, dx$$

$$= \left[\frac{x^4}{4} \ln x - \frac{x^4}{16} \right]_1^2$$

$$= \left(\frac{2^4}{4} \ln 2 - \frac{2^4}{16} \right) - \left(\frac{1^4}{4} \ln 1 - \frac{1^4}{16} \right)$$

$$= (4 \ln 2 - 1) - \left(-\frac{1}{16} \right)$$

$$= 4 \ln 2 - \frac{15}{16}$$

3 (i) $x = 2 \sin \theta \quad 1 = 2 \sin \theta \Rightarrow \theta = \frac{\pi}{6}$
 $0 = 2 \sin \theta \Rightarrow \theta = 0$

$$\frac{dx}{d\theta} = 2 \cos \theta$$

$$dx = 2 \cos \theta \, d\theta$$

$$\text{So } \int_0^1 \frac{x^2}{\sqrt{4-x^2}} \, dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{4 \sin^2 \theta}{\sqrt{4-4 \sin^2 \theta}} 2 \cos \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{4 \sin^2 \theta}{\sqrt{4(1-\sin^2 \theta)}} 2 \cos \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{4 \sin^2 \theta}{\sqrt{4(\cos^2 \theta)}} 2 \cos \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{4 \sin^2 \theta}{2 \cos \theta} 2 \cos \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 4 \sin^2 \theta \, d\theta$$

(ii) $\cos 2\theta = 1 - 2 \sin^2 \theta$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\text{So } I = \int_0^{\frac{\pi}{6}} 4 \left[\frac{1}{2}(1 - \cos 2\theta) \right] d\theta$$

$$= 2 \int_0^{\frac{\pi}{6}} (1 - \cos 2\theta) \, d\theta$$

$$= 2 \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{6}}$$

$$= 2 \left[\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right] - 0$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

4 (i) $A = \int_0^3 x^2 e^{-x} \, dx$

$$u = x^2 \quad v' = e^{-x}$$

$$u' = 2x \quad v = -e^{-x}$$

$$= [-x^2 e^{-x}]_0^3 - \int_0^3 -2x e^{-x} \, dx$$

$$= [-x^2 e^{-x}]_0^3 + \int_0^3 2x e^{-x} \, dx$$

$$u = 2x \quad v' = e^{-x}$$

$$u' = 2 \quad v = -e^{-x}$$

$$= [-x^2 e^{-x}]_0^3 + \left[-2x e^{-x} - \int -2e^{-x} \, dx \right]_0^3$$

$$= [-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}]_0^3$$

$$= [-e^{-x}(x^2 + 2x + 2)]_0^3$$

$$= (-e^{-3}(3^2 + 2 \times 3 + 2)) - (-e^0(0^2 + 2 \times 0 + 2))$$

$$= -e^{-3}(17) + 2$$

$$= 2 - \frac{17}{e^3}$$

(ii) Maximum point when $\frac{dy}{dx} = 0$

$$\Rightarrow 2x e^{-x} + (-e^{-x})x^{-2} = 0$$

$$x e^{-x}(2-x) = 0$$

$$x = 2$$

(iii) Equation of tangent is $y = mx = c$ and $c = 0$

$$x^2 e^{-x} = x e^{-x}(2-x) \times x$$

$$x^2 e^{-x} = 2x^2 e^{-x} - x^3 e^{-x}$$

$$0 = x^2 e^{-x} - x^3 e^{-x}$$

$$0 = x^2 e^{-x}(1-x)$$

$$\text{So } x = 1$$

5 (i) $u = \tan x$

$$u = \tan \frac{\pi}{4} = 1 \quad u = \tan 0 = 0$$

$$du = \sec^2 x \, dx$$

$$dx = \frac{du}{\sec^2 x} \Rightarrow dx = \cos^2 x \, du$$

$$\int_0^{\frac{\pi}{4}} (\tan^{n+2} x + \tan^n x) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x (\tan^2 x + 1) \, dx$$

$$= \int_0^1 u^n (\sec^2 x) \cos^2 x \, du$$

$$= \int_0^1 u^n \, du$$

$$= \left[\frac{u^{n+1}}{n+1} \right]_0^1$$

$$= \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}$$

$$= \frac{1}{n+1}$$

$$(ii) (a) \int_0^{\frac{\pi}{4}} (\sec^4 x - \sec^2 x) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x (\sec^2 x - 1) \, dx$$

$$= \int_0^{\frac{\pi}{4}} (1 + \tan^2 x) \tan^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan^4 x + \tan^2 x) \, dx$$

$$= \frac{1}{3} \quad (n=2)$$

$$(b) \tan^9 x + 5 \tan^7 x + 5 \tan^5 x + \tan^3 x \\ = \tan^9 x + \tan^7 x + 4(\tan^7 x + \tan^5 x) \\ + \tan^5 x + \tan^3 x$$

$$I = \int_0^{\frac{\pi}{4}} (\tan^9 x + \tan^7 x) \, dx$$

$$+ 4 \int_0^{\frac{\pi}{4}} (\tan^7 x + \tan^5 x) \, dx$$

$$+ \int_0^{\frac{\pi}{4}} (\tan^5 x + \tan^3 x) \, dx$$

$$= \frac{1}{8} + 4 \times \frac{1}{6} + \frac{1}{4} = \frac{25}{24}$$

$$6 (i) y = \frac{x^2}{1+x^3}$$

$$\frac{dy}{dx} = \frac{2x(1+x^3) - 3x^2(x^2)}{(1+x^3)^2}$$

$$= \frac{2x - x^4}{(1+x^3)^2}$$

$$\frac{2x - x^4}{(1+x^3)^2} = 0 \Rightarrow 2x - x^4 = 0$$

$$x(2 - x^3) = 0$$

$$\Rightarrow x = 0 \text{ or } \sqrt[3]{2}$$

Since clearly the x -coordinate is > 0 ,

$$x = \sqrt[3]{2}$$

$$(ii) \int_1^p \frac{x^2}{1+x^3} \, dx = 1$$

$$\frac{1}{3} \int_1^p \frac{3x^2}{1+x^3} \, dx = 1$$

$$\left[\frac{1}{3} \ln |1+x^3| \right]_1^p = 1$$

$$\frac{1}{3} \ln |1+p^3| - \frac{1}{3} \ln |1+1^3| = 1$$

$$\frac{1}{3} \ln |1+p^3| - \frac{1}{3} \ln 2 = 1$$

$$\frac{1}{3} \ln \left| \frac{1+p^3}{2} \right| = 1$$

$$\ln \left| \frac{1+p^3}{2} \right| = 3$$

$$\frac{1+p^3}{2} = e^3$$

$$1+p^3 = 2e^3$$

$$p = \sqrt[3]{2e^3 - 1} = 3.40 \text{ (3 s.f.)}$$

$$7 \int_0^{\frac{1}{2}\pi} x^2 \sin 2x \, dx$$

$$u = x^2 \quad v' = \sin 2x$$

$$u' = 2x \quad v = -\frac{1}{2} \cos 2x$$

$$\int x^2 \sin x \, dx$$

$$= x^2 \left(-\frac{1}{2} \cos 2x \right) - \int 2x \left(-\frac{1}{2} \cos 2x \right) \, dx$$

$$= -\frac{1}{2} x^2 \cos 2x + \int x \cos 2x \, dx$$

$$u = x \quad v' = \cos 2x$$

$$u' = 1 \quad v = \frac{1}{2} \sin 2x$$

$$\text{So } \int x^2 \sin x \, dx$$

$$= -\frac{1}{2} x^2 \cos 2x + \left[\frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx \right]$$

$$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x$$

$$\text{So } \int_0^{\frac{1}{2}\pi} x^2 \sin x \, dx$$

$$= \left[-\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\frac{1}{2}\pi}$$

$$= \left(-\frac{1}{2} \left(\frac{1}{2} \pi \right)^2 (-1) + \frac{1}{4} (-1) \right) - \left(0 + \frac{1}{4} (1) \right)$$

$$= \left(\frac{\pi^2}{8} - \frac{1}{4} \right) - \left(\frac{1}{4} \right)$$

$$= \frac{\pi^2}{8} - \frac{1}{2}$$

$$8 \quad I = \int_1^4 \frac{\sqrt{x}-1}{2(x+\sqrt{x})} dx$$

$$(i) \quad u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x} du = 2u du$$

$$x = 4 \Rightarrow u = 2, x = 1 \Rightarrow u = 1$$

$$I = \int_1^2 \frac{u-1}{2(u^2+u)} 2u du$$

$$= \int_1^2 \frac{u-1}{2u(u+1)} 2u du$$

$$= \int_1^2 \frac{u-1}{u+1} du$$

$$(ii) \quad I = \int_1^2 \frac{u-1}{u+1} du$$

Using division,

$$I = \int_1^2 \left(1 - \frac{2}{u+1}\right) du$$

$$= \left[u - 2\ln(u+1) \right]_1^2$$

$$= (2 - 2\ln 3) - (1 - 2\ln 2)$$

$$= 1 - 2\ln 3 + 2\ln 2$$

$$= 1 - \ln 9 + \ln 4$$

$$= 1 + \ln 4 - \ln 9$$

$$= 1 + \ln \frac{4}{9}$$

$$9 \quad y = (\ln x)^2$$

$$(i) \quad \frac{dy}{dx} = 2\ln x \times \frac{1}{x}$$

$$\text{When } x = e, \frac{dy}{dx} = \frac{2}{e}$$

$$\text{When } x = e, y = 1$$

$$\text{Gradient of normal at P} = -\frac{e}{2}$$

$$y = mx + c \Rightarrow 1 = -\frac{e}{2} \times e + c$$

$$c = 1 + \frac{e^2}{2}$$

$$\text{So Q is when } 0 = -\frac{e}{2}x + 1 + \frac{e^2}{2}$$

$$0 = -ex + 2 + e^2$$

$$x = \frac{2+e^2}{e} \approx 3.45 \text{ (3 s.f.)}$$

$$(ii) \quad \int \ln x dx = \int 1 \times \ln x dx$$

$$u = \ln x \quad v' = 1$$

$$u' = \frac{1}{x} \quad v = x$$

$$\int \ln x dx = x \ln x - \int \frac{1}{x} \times x dx$$

$$= x \ln x - \int 1 dx$$

$$= x \ln x - x + c$$

(iii) Area = area of triangle + area under curve

$$= \frac{1}{2} \left(e + \frac{2}{e} - e \right) \times 1 + \int_1^e (\ln x)^2 dx$$

$$= \frac{1}{e} + \int_1^e (\ln x)^2 dx$$

$$\text{Let } e^u = x \Rightarrow \frac{dx}{du} = e^u$$

$$\int_1^e (\ln x)^2 dx = \int_0^1 u^2 e^u du$$

Integrate by parts (twice)

$$u = u^2 \quad v' = e^u$$

$$u' = 2u \quad v = e^u$$

$$\int_0^1 u^2 e^u du = \left[u^2 e^u \right]_0^1 - \int_0^1 2u e^u du$$

$$u = 2u \quad v' = e^u$$

$$u' = 2 \quad v = e^u$$

$$\int_0^1 u^2 e^u du = \left[u^2 e^u \right]_0^1 - \left[2u e^u - \int_0^1 2e^u du \right]$$

$$= \left[u^2 e^u \right]_0^1 - \left[2u e^u - 2e^u \right]_0^1$$

$$= \left[u^2 e^u - 2u e^u + 2e^u \right]_0^1$$

$$= \left[e^u (u^2 - 2u + 2) \right]_0^1$$

$$= e - 2$$

$$\text{Area} = \frac{1}{e} + e - 2$$

Stretch and challenge (Page 31)

$$1 \quad (i) \quad u = 1 + x \Rightarrow du = dx$$

$$x = 1 \Rightarrow u = 2, x = 0 \Rightarrow u = 1$$

$$\int_0^1 \frac{x^3}{1+x} dx$$

$$= \int_{u=1}^{u=2} \frac{(u-1)^3}{u} du$$

$$= \int_1^2 \frac{u^3 - 3u^2 + 3u - 1}{u} du$$

$$= \int_1^2 \left(u^2 - 3u + 3 - \frac{1}{u} \right) du$$

$$= \left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln|u| \right]_1^2$$

$$= \left[\left(\frac{2^3}{3} - \frac{3 \times 2^2}{2} + 3 \times 2 - \ln|2| \right) - \left(\frac{1^3}{3} - \frac{3 \times 1^2}{2} + 3 \times 1 - \ln|1| \right) \right]$$

$$= \left(\frac{8}{3} - 6 + 6 - \ln 2 \right) - \left(\frac{1}{3} - \frac{3}{2} + 3 - 0 \right)$$

$$= \left(\frac{8}{3} - \ln 2 \right) - \left(\frac{11}{6} \right)$$

$$= \frac{5}{6} - \ln 2$$

$$(ii) (a) \frac{dy}{dx} = 2x \ln(1+x) + \frac{x^2}{1+x}$$

$$(b) \text{Area} = \int_0^1 x^2 \ln(1+x) dx$$

$$u = \ln(1+x) \quad v' = x^2$$

$$u' = \frac{1}{1+x} \quad v = \frac{x^3}{3}$$

$$\begin{aligned} & \int_0^1 x^2 \ln(1+x) dx \\ &= \left[\frac{x^3}{3} \ln(1+x) \right]_0^1 - \int_0^1 \frac{1}{1+x} \times \frac{x^3}{3} dx \\ &= \left[\frac{x^3}{3} \ln(1+x) \right]_0^1 - \frac{1}{3} \int_0^1 \frac{x^3}{1+x} dx \\ &= \left[\frac{x^3}{3} \ln(1+x) \right]_0^1 - \frac{1}{3} \left(\frac{5}{6} - \ln 2 \right) \\ &= \left(\frac{1}{3} \ln 2 - 0 \right) - \frac{1}{3} \left(\frac{5}{6} - \ln 2 \right) \\ &= \frac{2}{3} \ln 2 - \frac{5}{18} \approx 0.184 \end{aligned}$$

$$2 \quad (i) \quad u = x^n \quad v' = \cos x$$

$$u' = nx^{n-1} \quad v = \sin x$$

$$\begin{aligned} I_n &= \int_0^{\frac{1}{2}\pi} x^n \cos x dx \\ &= \left[x^n \sin x \right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} nx^{n-1} \sin x dx \\ &\quad \begin{array}{ll} u = nx^{n-1} & v' = \sin x \\ u' = n(n-1)x^{n-2} & v = -\cos x \end{array} \\ &= \left[x^n \sin x \right]_0^{\frac{1}{2}\pi} - \left(- \int_0^{\frac{1}{2}\pi} n(n-1)x^{n-2} (-\cos x) dx \right) \\ &= \left[x^n \sin x + nx^{n-1} \cos x \right]_0^{\frac{1}{2}\pi} \\ &\quad - n(n-1) \int_0^{\frac{1}{2}\pi} x^{n-2} \cos x dx \\ &= \left[\left(\frac{\pi}{2} \right)^n \sin \frac{\pi}{2} + n \left(\frac{\pi}{2} \right)^{n-1} \cos \frac{\pi}{2} \right] - n(n-1) I_{n-2} \\ &= \left[0^n \sin 0 + n 0^{n-1} \cos 0 \right] \\ &= \left(\frac{\pi}{2} \right)^n - n(n-1) I_{n-2} \end{aligned}$$

OR

$$u = x^n \quad v' = \cos x$$

$$u' = nx^{n-1} \quad v = \sin x$$

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} x^n \cos x dx &= \left[x^n \sin x \right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} nx^{n-1} \sin x dx \\ &= \left[x^n \sin x \right]_0^{\frac{1}{2}\pi} - n \left[\left(x^{n-1} \times -\cos x \right) \right. \\ &\quad \left. - \int_0^{\frac{1}{2}\pi} (n-1)x^{n-2} \times -\cos x dx \right]_0^{\frac{1}{2}\pi} \end{aligned}$$

$$= \left[x^n \sin x + nx^{n-1} \cos x \right]_0^{\frac{1}{2}\pi} - n(n-1) \int_0^{\frac{1}{2}\pi} nx^{n-2} \cos x dx$$

$$= \left(\frac{1}{2}\pi \right)^n + 0 - n(n-1) \int_0^{\frac{1}{2}\pi} x^{n-2} \cos x dx$$

$$= \left(\frac{1}{2}\pi \right)^n - n(n-1) I_{n-2}$$

$$(ii) \quad I_4 = \left(\frac{1}{2}\pi \right)^4 - 4 \times 3 \times I_2$$

$$I_2 = \left(\frac{1}{2}\pi \right)^2 - 2 \times 1 \times I_0$$

$$I_0 = \int_0^{\frac{1}{2}\pi} \cos x dx = \left[\sin x \right]_0^{\frac{1}{2}\pi} = \sin \frac{1}{2}\pi - \sin 0 = 1$$

$$I_2 = \left(\frac{1}{2}\pi \right)^2 - 2 \times 1 \times 1 = \frac{1}{4}\pi^2 - 2$$

$$\begin{aligned} I_4 &= \left(\frac{1}{2}\pi \right)^4 - 4 \times 3 \times \left(\frac{1}{4}\pi^2 - 2 \right) \\ &= \frac{1}{16}\pi^4 - 3\pi^2 + 24 \end{aligned}$$

OR

$$\begin{aligned} I_4 &= \left(\frac{1}{2}\pi \right)^4 - 4(4-1)I_{4-2} \\ &= \frac{1}{16}\pi^4 - 12I_2 \\ &= \frac{1}{16}\pi^4 - 12 \left(\left(\frac{1}{2}\pi \right)^2 - 2(2-1)I_0 \right) \\ &= \frac{1}{16}\pi^4 - 12 \left(\frac{1}{4}\pi^2 - 2I_0 \right) \\ &= \frac{1}{16}\pi^4 - 12 \left(\frac{1}{4}\pi^2 - 2 \left(\left(\frac{1}{2}\pi \right)^0 - 0 \right) \right) \\ &= \frac{1}{16}\pi^4 - 12 \left(\frac{1}{4}\pi^2 - 2(1) \right) \\ &= \frac{1}{16}\pi^4 - 3\pi^2 + 24 \end{aligned}$$

$$3 \quad (i) \quad \text{Consider } I = \int_0^a f(a-x) dx.$$

$$\text{Let } u = a - x, \quad \text{then } \frac{du}{dx} = -1,$$

and when $x = 0, u = a$; when $x = a, u = 0$, so

$$\begin{aligned} I &= \int_a^0 -f(u) du \\ &= \int_0^a f(u) du = \int_0^a f(x) dx \end{aligned}$$

OR

Let $F(x)$ be an antiderivative of f

$$\int_0^a f(x) dx = [F(x)]_0^a = F(a) - F(0)$$

$$\int_0^a f(a-x) dx = [-F(a-x)]_0^a = -F(0) + F(a)$$

and hence result.

(ii) Using the result in 3 (i) with $a = \frac{\pi}{2}$,

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^n(\frac{\pi}{2} - x)}{\sin^n(\frac{\pi}{2} - x) + \cos^n(\frac{\pi}{2} - x)} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x} dx \end{aligned}$$

since $\sin(\frac{\pi}{2} - x) = \cos x$ and so

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \\ &= \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x} dx \right) \\ &= \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} dx \right) = \frac{1}{2} \left(\frac{\pi}{2} \right) = \frac{\pi}{4} \end{aligned}$$

4 (i) $\frac{dy}{dx} = \frac{2x(1+x^2) - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$

When $\frac{dy}{dx} = \frac{1}{2}$

$$\frac{2x}{(1+x^2)^2} = \frac{1}{2}$$

$$x^4 + 2x^2 - 4x + 1 = 0$$

But $x = 1$ is a solution, so by division or otherwise

$$(x-1)(x^3 + x^2 + 3x - 1) = 0$$

and any other solutions are from $x^3 + x^2 + 3x - 1 = 0$

Let $g(x) = x^3 + x^2 + 3x - 1$, then

$$g\left(\frac{1}{4}\right) = \frac{1+4+48-64}{64} = -\frac{11}{64} < 0$$

$$g\left(\frac{1}{2}\right) = \frac{1+2+12-8}{8} = \frac{7}{8} > 0$$

Hence there is a root in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$$g(x) = 0 \text{ for some } \frac{1}{4} < x < \frac{1}{2}.$$

OR

$$f(x) = x^4 + 2x^2 - 4x + 1$$

$$f\left(\frac{1}{4}\right) = \frac{33}{256} = 0.1289 > 0$$

$$f\left(\frac{1}{2}\right) = -\frac{7}{16} = -0.4375 < 0$$

Hence there is a root in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$.

OR

$$\frac{dy}{dx} = \frac{2x}{(1+x^2)^2}$$

$$\text{When } x = \frac{1}{4}, \frac{dy}{dx} = \frac{\frac{1}{2}}{\left(1 + \frac{1}{16}\right)^2} = \frac{128}{289} < \frac{1}{2}$$

$$\text{and when } x = \frac{1}{2}, \frac{dy}{dx} = \frac{1}{\left(1 + \frac{1}{4}\right)^2} = \frac{16}{25} > \frac{1}{2}$$

Hence there is a value of $\frac{dy}{dx} = \frac{1}{2}$ in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$$\frac{dy}{dx} = \frac{1}{2} \text{ for some } \frac{1}{4} < x < \frac{1}{2}$$

(ii) First translate the function 1 unit in the negative

$$x \text{ direction to get } y = \frac{x^2}{1+x^2}.$$

The volume of revolution is then

$$V = \int_0^{\frac{1}{2}} \pi x^2 dy$$

$$\text{and } y = \frac{x^2}{1+x^2}, \text{ so rearrange to get } x^2 = \frac{y}{1-y}$$

$$\text{and } V = \pi \int_0^{\frac{1}{2}} \left(\frac{1}{1-y} - 1 \right) dy$$

$$V = \pi \left[-\ln|1-y| - y \right]_0^{\frac{1}{2}}$$

$$= \pi \left(-\ln\left(\frac{1}{2}\right) - \frac{1}{2} \right)$$

$$= \pi \left(\ln(2) - \frac{1}{2} \right) \text{ units}^3$$

OR to integrate using substitution,

$$V = \int_0^{\frac{1}{2}} \pi \frac{y}{1-y} dy \quad \text{Let } u = 1-y$$

$$du = -dy$$

$$y = 1-u$$

$$V = \int_1^{\frac{1}{2}} \pi \frac{1-u}{u} (-du)$$

$$= \int_{\frac{1}{2}}^1 \pi \left(\frac{1}{u} - 1 \right) du$$

$$= \pi \left[\ln|u| - u \right]_{\frac{1}{2}}^1$$

$$= \pi \left[(0-1) - \left(\ln\left(\frac{1}{2}\right) - \frac{1}{2} \right) \right]$$

$$= \pi \left(\ln 2 - \frac{1}{2} \right) \text{ units}^3$$

3 Differential equations

3.1 Forming differential equations from rates of change (Page 32)

- 1 (i) $\frac{dy}{dt} = k$
 (ii) $\frac{dB}{dt} = kB$
 (iii) $\frac{dh}{dt} = k\sqrt[3]{t}$
 (iv) $\frac{dV}{dt} = k\sqrt{V}$
 (v) $\frac{dr}{dt} = \frac{k}{\sqrt{r}}$
 (vi) $\frac{dP}{dt} = k(N - P)$
 (vii) $\frac{dV}{dt} = kS$
 (viii) $\frac{dA}{dt} = kr^2$

3.2 Solving differential equations (Page 33)

- 1 (i) $\frac{dy}{dx} = 1 - 2x$
 $\int dy = \int (1 - 2x) dx$
 $y = x - x^2 + c$
 (ii) $\frac{dy}{dx} = \frac{4x}{y}$
 $\int y dy = \int 4x dx$
 $\frac{y^2}{2} = 2x^2 + c_1$
 $y^2 = 4x^2 + c_2$
 $y = \pm \sqrt{4x^2 + c_2}$
 (iii) $\frac{dA}{dt} = 0.01A$
 $\int \frac{1}{A} dA = \int 0.01 dt$
 $\ln|A| = 0.01t + c$
 $A = e^{0.01t+c}$
 $A = e^{0.01t} \times e^c$
 $A = Ke^{0.01t}$
 (iv) $\frac{dy}{dx} = y - y \sin x$
 $\frac{dy}{dx} = y(1 - \sin x)$
 $\int \frac{1}{y} dy = \int (1 - \sin x) dx$
 $\ln|y| = x + \cos x + c$

$$y = e^{x+\cos x+c}$$

$$y = e^{x+\cos x} \times e^c$$

$$y = Ae^{x+\cos x}$$

- 2 (i) $\frac{dy}{dx} = e^{x+y}$
 $\frac{dy}{dx} = e^x \times e^y$
 $\int e^{-y} dy = \int e^x dx$
 $-e^{-y} = e^x + c$
 $x = 0, y = 2 \Rightarrow$
 $-e^{-2} = e^0 + c \Rightarrow c = -e^{-2} - 1$
 $-e^{-y} = e^x - e^{-2} - 1$
 $e^{-y} = -e^x + e^{-2} + 1$
 $-y = \ln(-e^x + e^{-2} + 1)$
 $y = -\ln(-e^x + e^{-2} + 1)$

- (ii) $(1+x^2)\frac{dy}{dx} = 2xy$
 $\int \frac{1}{y} dy = \int \frac{2x}{(1+x^2)} dx$
 $\ln|y| = \ln(1+x^2) + c$
 $x = 0, y = 2 \Rightarrow$
 $\ln 2 = \ln(1+0^2) + c \Rightarrow c = \ln 2$
 $\ln|y| = \ln(1+x^2) + \ln 2$
 $\ln|y| = \ln(2(1+x^2))$
 $e^{\ln|y|} = e^{\ln(2(1+x^2))}$
 $y = 2(1+x^2)$

- 3 (i) $\frac{\sec^2 y}{\cos^2 2x} \frac{dy}{dx} = 2$
 $\int \sec^2 y dy = \int 2 \cos^2 2x dx$
 $\int \sec^2 y dy = \int (\cos 4x + 1) dx$
 $\tan y = \frac{1}{4} \sin 4x + x + c$
 (ii) $\tan \frac{1}{4}\pi = \frac{1}{4} \sin 0 + 0 + c \Rightarrow c = 1$
 $\tan y = \frac{1}{4} \sin 4x + x + 1$
 $y = \tan^{-1} \left(\frac{1}{4} \sin 4x + x + 1 \right)$
 $x = \frac{\pi}{6}$
 So $y = \tan^{-1} \left(\frac{1}{4} \sin \frac{4\pi}{6} + \frac{\pi}{6} + 1 \right)$
 $= 1.05 \text{ (3.s.f.)}$

4 (i) $\frac{dh}{dt} = \frac{6-h}{20}$
 $\int \frac{20}{6-h} dh = \int dt$
 $-20 \ln|6-h| = t + c_1$
 $\ln|6-h| = -\frac{1}{20}t + c_2$
 $t = 0, h = 1 \Rightarrow$
 $\ln|6-1| = c_2 \Rightarrow c_2 = \ln 5$
 $\ln|6-h| = -\frac{1}{20}t + \ln 5$
 $\ln|6-h| - \ln 5 = -\frac{1}{20}t$
 $\ln\left|\frac{6-h}{5}\right| = -\frac{1}{20}t$
 $t = -20 \ln\left|\frac{6-h}{5}\right|$

(ii) $t = -20 \ln\left|\frac{6-2}{5}\right| = 4.46$ years

(iii) $10 = -20 \ln\left|\frac{6-h}{5}\right|$
 $-\frac{1}{2} = \ln\left|\frac{6-h}{5}\right|$
 $e^{-\frac{1}{2}} = \left|\frac{6-h}{5}\right|$
 $5e^{-\frac{1}{2}} = |6-h|$
 $h = 6 - 5e^{-\frac{1}{2}} = 2.97$ m

(iv) $t = -20 \ln\left|\frac{6-h}{5}\right|$
 $e^{-\frac{t}{20}} = \frac{6-h}{5}$
 $h = 6 - 5e^{-\frac{t}{20}}$
As $t \rightarrow \infty, h \rightarrow 6$
Maximum possible height is 6 m.

5 $\frac{d\theta}{dt} = k(160 - \theta)$
 $\int \frac{1}{160 - \theta} d\theta = \int k dt$
 $\ln|160 - \theta| = kt + c$
 $t = 0, \theta = 20 \Rightarrow \ln 140 = c$
 $\ln|160 - \theta| = kt + \ln 140$
 $t = 5, \theta = 65 \Rightarrow \ln 95 = 5k + \ln 140$
 $k = \frac{\ln 95 - \ln 140}{5}$
 $= -0.0776$
 $\ln|160 - \theta| = -0.0776t + \ln 140$
 $160 - \theta = e^{-0.0776t + \ln 140}$
 $160 - \theta = e^{-0.0776t} \times e^{\ln 140}$
 $160 - \theta = 140e^{-0.0776t}$
 $\theta = 160 - 140e^{-0.0776t}$
 $t = 10 \Rightarrow \theta = 160 - 140e^{-0.0776 \times 10} = 95.5^\circ \text{C}$

6 (i) $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2$
 $\int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx$
 $-\frac{1}{y} = -\frac{1}{x} + c$
 $-\frac{1}{y} = \frac{-1+cx}{x}$
 $-y = \frac{x}{-1+cx}$
 $y = \frac{x}{1-cx}$

(ii) $1 = \frac{2}{1-2c} \Rightarrow 1-2c = 2 \Rightarrow c = -\frac{1}{2}$
 $y = \frac{x}{1+\frac{1}{2}x} = \frac{2x}{2+x}$
 $y = \frac{2 \times 8}{2+8} = -\frac{16}{10} = -\frac{8}{5}$

7 (i) $\frac{dx}{dt} = k\sqrt{x}$
(ii) $\frac{dy}{dt} = \frac{10\,000}{\sqrt{y}}$
 $\int \sqrt{y} dy = \int 10\,000 dt$

$$\frac{y^{\frac{3}{2}}}{\frac{3}{2}} = 10\,000t + c$$

$$\frac{2\sqrt{y^3}}{3} = 10\,000t + c$$

$$t = 0, y = 900 \Rightarrow c = \frac{2\sqrt{900^3}}{3} = 18\,000$$

$$\frac{2\sqrt{y^3}}{3} = 10\,000t + 18\,000$$

$$\sqrt{y^3} = 15\,000t + 27\,000$$

$$y = \sqrt[3]{(15\,000t + 27\,000)^2}$$

After 10 minutes,
 $y = \sqrt[3]{(15\,000 \times 10 + 27\,000)^2} = 3152.45 \approx 3150$

8 $\frac{dv}{dt} = 10e^{-\frac{1}{2}t}$
(ii) $v = \int 10e^{-\frac{1}{2}t} dt$
 $= -20e^{-\frac{1}{2}t} + c$
When $t = 0, v = 0 \Rightarrow 0 = -20 + c \Rightarrow c = 20$
So $v = 20 - 20e^{-\frac{1}{2}t}$

(ii) As $t \rightarrow \infty, e^{-\frac{1}{2}t} \rightarrow 0 \Rightarrow v \rightarrow 20$

So long term speed is 20 m s^{-1}

(iii) $\frac{1}{(w-4)(w+5)} = \frac{1}{9(w-4)} - \frac{1}{9(w+5)}$

$$\begin{aligned}
 \text{(iv)} \quad \frac{dw}{dt} &= -\frac{1}{2}(w-4)(w+5) \\
 \Rightarrow \int \frac{dw}{(w-4)(w+5)} &= \int -\frac{1}{2} dt \\
 \Rightarrow \int \left(\frac{1}{9(w-4)} - \frac{1}{9(w+5)} \right) dw &= \int -\frac{1}{2} dt \\
 \Rightarrow \frac{1}{9} \ln(w-4) - \frac{1}{9} \ln(w+5) &= -\frac{1}{2}t + c \\
 \Rightarrow \frac{1}{9} \ln \frac{w-4}{w+5} &= -\frac{1}{2}t + c \\
 \text{When } t=0, w=10 \Rightarrow c &= \frac{1}{9} \ln \frac{6}{15} = \frac{1}{9} \ln \frac{2}{5} \\
 \Rightarrow \ln \frac{w-4}{w+5} &= -\frac{9}{2}t + \ln \frac{2}{5} \\
 \Rightarrow \frac{w-4}{w+5} &= e^{-\frac{9}{2}t + \ln \frac{2}{5}} = \frac{2}{5} e^{-\frac{9}{2}t} = 0.4e^{-4.5t}
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad \text{As } t \rightarrow \infty, e^{-4.5t} &\rightarrow 0 \Rightarrow w-4 \rightarrow 0 \\
 \text{So long term speed is } &4 \text{ m s}^{-1}.
 \end{aligned}$$

9 θ : temperature,

t : time in hours from 4 p.m.

$$\begin{aligned}
 \frac{d\theta}{dt} &= k(\theta-6) \\
 \int \frac{1}{(\theta-6)} d\theta &= \int k dt \\
 \ln|\theta-6| &= kt + c \\
 t=0, \theta=16 \Rightarrow \ln|16-6| &= c \Rightarrow c = \ln 10 \\
 t=1, \theta=8 \Rightarrow \ln|8-6| &= k + \ln 10 \Rightarrow k = \ln \frac{1}{5} \\
 \ln|\theta-6| &= \left(\ln \frac{1}{5} \right) t + \ln 10 \\
 |\theta-6| &= \left(e^{\ln \frac{1}{5}} \right)^t \times e^{\ln 10} \\
 \theta &= 10 \left(\frac{1}{5} \right)^t + 6 \\
 37 &= 10 \left(\frac{1}{5} \right)^t + 6 \\
 31 &= 10 \left(\frac{1}{5} \right)^t \\
 3.1 &= \left(\frac{1}{5} \right)^t \\
 \ln 3.1 &= t \ln \left(\frac{1}{5} \right) \\
 t &= \frac{\ln 3.1}{\ln \left(\frac{1}{5} \right)} = -0.703
 \end{aligned}$$

i.e. 42 minutes ago = 3.18 p.m.

$$\begin{aligned}
 10 \quad \frac{dV}{dt} &= k\sqrt{V} \\
 \int \frac{1}{\sqrt{V}} dV &= \int k dt \\
 \int V^{-\frac{1}{2}} dV &= \int k dt
 \end{aligned}$$

$$\begin{aligned}
 \frac{V^{\frac{1}{2}}}{\frac{1}{2}} &= kt + c \\
 2\sqrt{V} &= kt + c \\
 t=0, V=900 \Rightarrow 2\sqrt{900} &= kt + c \Rightarrow c=60 \\
 t=1, V=841 \Rightarrow 2\sqrt{841} &= k+60 \Rightarrow k=-2 \\
 2\sqrt{V} &= -2t + 60 \\
 2\sqrt{0} &= -2t + 60 \\
 2t &= 60 \\
 t &= 30
 \end{aligned}$$

The tank would empty in 30 days.

11 y : amount of substance A in grams

$$\begin{aligned}
 \frac{dy}{dt} &= ky^2 \\
 \int \frac{1}{y^2} dy &= \int k dt \\
 \int y^{-2} dy &= \int k dt \\
 \frac{y^{-1}}{-1} &= kt + c \\
 -\frac{1}{y} &= kt + c \\
 y &= -\frac{1}{kt + c} \\
 t=0, y=60 \Rightarrow 60 &= -\frac{1}{c} \Rightarrow c = -\frac{1}{60} \\
 t=1, y=10 \Rightarrow 10 &= -\frac{1}{k - \frac{1}{60}} \Rightarrow k = -\frac{1}{12} \\
 y &= -\frac{1}{-\frac{1}{12}t - \frac{1}{60}} = \frac{60}{5t+1} \\
 \text{When } t=2, y &= \frac{60}{5 \times 2 + 1} = 5.45 \text{ g}
 \end{aligned}$$

Further practice (Page 39)

$$\begin{aligned}
 1 \quad \text{(i)} \quad \frac{dx}{dt} &= xe^{2t} \\
 \int \frac{1}{x} dx &= \int e^{2t} dt \\
 \ln|x| &= \frac{1}{2}e^{2t} + c \\
 e^{\ln|x|} &= e^{\frac{1}{2}e^{2t} + c} \\
 x &= Ae^{\frac{1}{2}e^{2t}} \\
 \text{(ii)} \quad \frac{dy}{dt} &= \frac{\cos^2 y}{e^t} \\
 \int \frac{1}{\cos^2 y} dy &= \int \frac{1}{e^t} dt \\
 \int \sec^2 y dy &= \int e^{-t} dt \\
 \tan y &= -e^{-t} + c \\
 y &= \tan^{-1}(-e^{-t} + c)
 \end{aligned}$$

2 (i) $(x^2 - 5x + 6) \div (x - 1) = (x - 4) + \frac{2}{x - 1}$

(ii) (a) $\left(\frac{x-1}{x^2-5x+6}\right) \frac{dy}{dx} = y - 5$

$$\int \frac{1}{y-5} dy = \int \frac{x^2-5x+6}{x-1} dx$$

$$\int \frac{1}{y-5} dy = \int \left(x - 4 + \frac{2}{x-1}\right) dx$$

$$\ln|y-5| = \frac{x^2}{2} - 4x + 2\ln|x-1| + c$$

$$y-5 = e^{\frac{x^2}{2} - 4x + 2\ln|x-1| + c}$$

$$y-5 = e^{\frac{x^2}{2} - 4x + \ln(x-1)^2 + c}$$

$$y-5 = e^{\frac{x^2}{2} - 4x} \times e^{\ln(x-1)^2} \times e^c$$

$$y = A(x-1)^2 e^{\frac{x^2}{2} - 4x} + 5$$

(b) $7 = A(8-1)^2 e^{\frac{8^2}{2} - 4 \times 8} + 5$

$$7 = 49Ae^0 + 5$$

$$A = \frac{2}{49}$$

$$y = \frac{2}{49}(x-1)^2 e^{\frac{x^2}{2} - 4x} + 5$$

$$y = \frac{2}{49}(6-1)^2 e^{\frac{6^2}{2} - 4 \times 6} + 5$$

$$= 5.00$$

3 (i) $\frac{dx}{dt} = -4(x-8)^{\frac{1}{3}}$

$$\int \frac{1}{(x-8)^{\frac{1}{3}}} dx = \int -4 dt$$

$$\int (x-8)^{-\frac{1}{3}} dx = \int -4 dt$$

$$\frac{(x-8)^{\frac{2}{3}}}{\frac{2}{3}} = -4t + c$$

$$\frac{3(x-8)^{\frac{2}{3}}}{2} = -4t + c$$

$$t=0, x=72 \Rightarrow \frac{3(72-8)^{\frac{2}{3}}}{2} = c \Rightarrow c = 24$$

$$\frac{3(x-8)^{\frac{2}{3}}}{2} = -4t + 24$$

$$4t = 24 - \frac{3(x-8)^{\frac{2}{3}}}{2}$$

$$t = 6 - \frac{3(x-8)^{\frac{2}{3}}}{8}$$

When $x = 35$,

$$t = 6 - \frac{3(35-8)^{\frac{2}{3}}}{8} = \frac{21}{8} = 2.625 \text{ mins}$$

(ii) $t = 6 - \frac{3(x-8)^{\frac{2}{3}}}{8}$

When $x = 0$,

$$t = 6 - \frac{3(0-8)^{\frac{2}{3}}}{8} = 4.5 \text{ mins}$$

4 (i) $\frac{1}{(2x+1)(x+1)} = \frac{2}{2x+1} - \frac{1}{x+1}$

(ii) $\frac{dy}{dx} = \frac{y}{(2x+1)(x+1)}$

$$\int \frac{1}{y} dy = \int \frac{1}{(2x+1)(x+1)} dx$$

$$\int \frac{1}{y} dy = \int \left(\frac{2}{2x+1} - \frac{1}{x+1}\right) dx$$

$$\ln|y| = \ln|2x+1| - \ln|x+1| + c$$

$$\ln|y| = \ln\left|\frac{2x+1}{x+1}\right| + c$$

$$x=0, y=2 \Rightarrow \ln 2 = \ln 1 + c \Rightarrow c = \ln 2$$

$$\ln|y| = \ln\left|\frac{2x+1}{x+1}\right| + \ln 2$$

$$e^{\ln|y|} = e^{\ln\left|\frac{2x+1}{x+1}\right| + \ln 2}$$

$$y = e^{\ln\left|\frac{2x+1}{x+1}\right|} \times e^{\ln 2}$$

$$y = 2\left(\frac{2x+1}{x+1}\right) = \frac{4x+2}{x+1}$$

5 (i) $\frac{3}{(y-2)(y+1)} = \frac{A}{y-2} + \frac{B}{y+1}$

$$= \frac{A(y+1) + B(y-2)}{(y-2)(y+1)}$$

$$3 = A(y+1) + B(y-2)$$

$$y = 2 \Rightarrow 3 = 3A \Rightarrow A = 1$$

$$y = -1 \Rightarrow 3 = -3B \Rightarrow B = -1$$

$$\frac{3}{(y-2)(y+1)} = \frac{1}{y-2} - \frac{1}{y+1}$$

(ii) $\frac{dy}{dx} = x^2(y-2)(y+1)$

$$\Rightarrow \int \frac{3}{(y-2)(y+1)} dy = \int 3x^2 dx$$

$$\Rightarrow \int \left(\frac{1}{y-2} - \frac{1}{y+1}\right) dy = \int 3x^2 dx$$

$$\Rightarrow \ln(y-2) - \ln(y+1) = x^3 + c$$

$$\Rightarrow \ln\left(\frac{y-2}{y+1}\right) = x^3 + c$$

$$\Rightarrow \frac{y-2}{y+1} = e^{x^3+c}$$

$$= e^{x^3} \times e^c = Ae^{x^3}$$

6 (i) $y = \operatorname{cosec} x = \frac{1}{\sin x}$

$$\frac{dy}{dx} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cot x$$

$$(ii) \frac{dx}{dt} = -\sin x \tan x \cot t$$

$$\int \frac{1}{-\sin x \tan x} dx = \int \cot t dt$$

$$\int -\operatorname{cosec} x \cot x dx = \int \frac{\cos t}{\sin t} dt$$

$$\operatorname{cosec} x = \ln|\sin t| + c$$

$$\frac{1}{\sin x} = \ln|\sin t| + c$$

$$\sin x = \frac{1}{\ln|\sin t| + c}$$

$$x = \frac{1}{6}\pi, t = \frac{1}{2}\pi \Rightarrow \sin \frac{1}{6}\pi = \frac{1}{\ln|\sin \frac{1}{2}\pi| + c}$$

$$\frac{1}{2} = \frac{1}{\ln 1 + c} \Rightarrow c = 2$$

$$\sin x = \frac{1}{\ln|\sin t| + 2}$$

$$x = \sin^{-1}\left(\frac{1}{\ln|\sin t| + 2}\right)$$

$$7 (i) \frac{1}{(3-x)(6-x)} = \frac{1}{3(3-x)} - \frac{1}{3(6-x)}$$

$$(ii) (a) \frac{dx}{dt} = k(3-x)(6-x)$$

$$\int \frac{1}{(3-x)(6-x)} dx = \int k dt$$

$$\int \left(\frac{1}{3(3-x)} - \frac{1}{3(6-x)} \right) dx = \int k dt$$

$$-\frac{1}{3} \ln(3-x) + \frac{1}{3} \ln(6-x) = kt + c$$

$$\frac{1}{3} (\ln(6-x) - \ln(3-x)) = kt + c$$

$$\frac{1}{3} \ln \frac{(6-x)}{(3-x)} = kt + c$$

$$t = 0, x = 0 \Rightarrow \frac{1}{3} \ln \frac{6}{3} = k \times 0 + c \Rightarrow c = \frac{1}{3} \ln 2$$

$$t = 1, x = 1 \Rightarrow \frac{1}{3} \ln \frac{5}{2} = k + \frac{1}{3} \ln 2$$

$$\Rightarrow k = \frac{1}{3} \ln \frac{5}{2} - \frac{1}{3} \ln 2$$

$$\Rightarrow k = \frac{1}{3} \left(\ln \frac{5}{2} - \ln 2 \right)$$

$$\Rightarrow k = \frac{1}{3} \ln \frac{5}{4}$$

$$(b) \frac{1}{3} \ln \frac{(6-x)}{(3-x)} = \left(\frac{1}{3} \ln \frac{5}{4} \right) t + \frac{1}{3} \ln 2$$

$$\frac{1}{3} \ln \frac{(6-x)}{(3-x)} = \left(\frac{1}{3} \ln \frac{5}{4} \right) \times 2 + \frac{1}{3} \ln 2$$

$$\ln \frac{(6-x)}{(3-x)} = 2 \ln \frac{5}{4} + \ln 2$$

$$\frac{6-x}{3-x} = e^{\left(2 \ln \frac{5}{4} \right) + \ln 2}$$

$$\frac{6-x}{3-x} = e^{\left(2 \ln \frac{5}{4} \right)} \times e^{\ln 2}$$

$$\frac{6-x}{3-x} = 2e^{\ln \left(\frac{5}{4} \right)^2}$$

$$\frac{6-x}{3-x} = 2 \times \frac{25}{16}$$

$$\frac{6-x}{3-x} = \frac{25}{8}$$

$$8(6-x) = 25(3-x)$$

$$48 - 8x = 75 - 25x$$

$$17x = 27$$

$$x = \frac{27}{17} \approx 1.59$$

$$8 e^{2y} \frac{dy}{dx} + \tan x = 0$$

$$e^{2y} \frac{dy}{dx} = -\tan x$$

$$\int e^{2y} dy = -\int \frac{\sin x}{\cos x} dx$$

$$\frac{1}{2} e^{2y} = \ln|\cos x| + c$$

$$x = 0, y = 0 \Rightarrow \frac{1}{2} e^0 = \ln|\cos 0| + c \Rightarrow c = \frac{1}{2}$$

$$\frac{1}{2} e^{2y} = \ln|\cos x| + \frac{1}{2}$$

$$e^{2y} = 2 \ln|\cos x| + 1$$

$$2y = \ln(\ln(\cos^2 x) + 1)$$

$$y = \frac{1}{2} \ln(\ln(\cos^2 x) + 1)$$

$$9 (i) \frac{dA}{dt} = k\sqrt{3A-2}$$

$$(ii) \int \frac{1}{\sqrt{3A-2}} dA = \int k dt$$

$$\int (3A-2)^{-\frac{1}{2}} dA = \int k dt$$

$$\frac{(3A-2)^{\frac{1}{2}}}{\frac{1}{2}} \times \frac{1}{3} = kt + c$$

$$\frac{2\sqrt{3A-2}}{3} = kt + c$$

$$t = 0, A = 6 \Rightarrow \frac{2\sqrt{3 \times 6 - 2}}{3} = k \times 0 + c \Rightarrow c = \frac{8}{3}$$

$$t = 4, A = 17 \Rightarrow \frac{2\sqrt{3 \times 17 - 2}}{3} = 4k + \frac{8}{3} \Rightarrow k = \frac{1}{2}$$

$$\frac{2\sqrt{3A-2}}{3} = \frac{1}{2} t + \frac{8}{3}$$

$$\frac{2\sqrt{3A-2}}{3} = \frac{1}{2} \times 8 + \frac{8}{3}$$

$$\frac{2\sqrt{3A-2}}{3} = \frac{20}{3}$$

$$2\sqrt{3A-2} = 20$$

$$\sqrt{3A-2} = 10$$

$$3A-2 = 100$$

$$A = 34 \text{ m}^2$$

$$10 \quad xy^2 \frac{dy}{dx} = y^3 - 1$$

$$\int \frac{y^2}{y^3 - 1} dy = \int \frac{1}{x} dx$$

$$\frac{1}{3} \ln|y^3 - 1| = \ln|x| + c$$

$$x = 2, y = 0 \Rightarrow \frac{1}{3} \ln|0^3 - 1| = \ln|2| + c \Rightarrow c = -\ln 2$$

$$\frac{1}{3} \ln|y^3 - 1| = \ln|x| - \ln 2$$

$$\ln|y^3 - 1| = 3(\ln|x| - \ln 2)$$

$$\ln|y^3 - 1| = 3 \ln \frac{x}{2}$$

$$\ln|y^3 - 1| = \ln \left(\frac{x}{2} \right)^3$$

$$y^3 = \frac{x^3}{8} + 1$$

Past exam questions (Page 40)

$$1 \quad (i) \quad \frac{dy}{dx} = kxy$$

$$x = 1, y = 2 \Rightarrow 4 = k \times 1 \times 2 \Rightarrow k = 2$$

$$\Rightarrow \frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln y = x^2 + C$$

$$\ln 2 = 1^2 + C$$

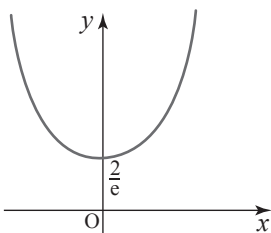
$$1C = \ln 2 - 1$$

$$\ln y = x^2 + \ln 2 - 1$$

$$y = e^{(x^2 + \ln 2 - 1)}$$

$$y = e^{x^2} \times e^{\ln 2} \times e^{-1}$$

$$(ii) \quad \frac{dy}{dx} = 2 \times (-1) \times 2 = -4$$



$$2 \quad (i) \quad \frac{dN}{dt} = 1.2e^{-0.02t} N^{0.5}$$

$$\int \frac{1}{N^{0.5}} dN = \int 1.2e^{-0.02t} dt$$

$$\int N^{-\frac{1}{2}} dN = 1.2 \int e^{-0.02t} dt$$

$$\frac{N^{\frac{1}{2}}}{\frac{1}{2}} = 1.2 \times \frac{1}{-0.02} e^{-0.02t} + C$$

$$2\sqrt{N} = -60e^{-0.02t} + C$$

$$t = 0, N = 100 \Rightarrow 2\sqrt{100} = -60e^{-0.20 \times 0} + C$$

$$20 = -60 + C \Rightarrow C = 80$$

$$\Rightarrow 2\sqrt{N} = -60e^{-0.02t} + 80$$

$$\sqrt{N} = 40 - 30e^{-0.02t}$$

$$N = (40 - 30e^{-0.02t})^2$$

$$(ii) \quad \text{As } t \rightarrow \infty, e^{-0.02t} \rightarrow 0$$

$$\text{So } N = 40^2 = 1600$$

$$3 \quad (i) \quad \frac{dx}{dt} = \frac{xe^{-t}}{k + e^{-t}}$$

$$\int \frac{1}{x} dx = \int \frac{e^{-t}}{k + e^{-t}} dt$$

$$\ln x = -\ln(k + e^{-t}) + c$$

$$\ln 10 = -\ln(k + e^0) + c$$

$$\ln 10 = -\ln(k + 1) + c$$

$$c = \ln 10 + \ln(k + 1) = \ln 10(k + 1)$$

$$\Rightarrow \ln x = -\ln(k + e^{-t}) + \ln 10(k + 1)$$

OR

$$\ln x - \ln 10 = -\ln(k + e^{-t}) + \ln(k + 1)$$

$$(ii) \quad \ln 20 - \ln 10 = -\ln(k + e^{-1}) + \ln(k + 1)$$

$$\ln 2 = \ln \left(\frac{k+1}{k+e^{-1}} \right)$$

$$\frac{k+1}{k+e^{-1}} = 2$$

$$k+1 = 2(k+e^{-1})$$

$$k+1 = 2k+2e^{-1}$$

$$k = 1 - 2e^{-1} = 1 - \frac{2}{e}$$

$$(iii) \quad \text{As } t \rightarrow \infty, e^{-t} \rightarrow 0$$

$$\text{So } \ln x - \ln 10 = -\ln(k + 0) + \ln(k + 1)$$

$$\ln x = -\ln k + \ln(k + 1) + \ln 10$$

$$\ln x = -\ln \left(1 - \frac{2}{e} \right) + \ln \left(1 - \frac{2}{e} + 1 \right) + \ln 10$$

$$\ln x = 3.8679...$$

$$x = 47.844...$$

So x will never reach 48.

$$4 \quad (i) \quad (3 + \cos 2\theta) \frac{dx}{d\theta} = x \sin 2\theta$$

$$\int \frac{1}{x} dx = \int \frac{\sin 2\theta}{3 + \cos 2\theta} d\theta$$

$$\ln x = -\frac{1}{2} \ln(3 + \cos 2\theta) + c$$

$$\ln 3 = -\frac{1}{2} \ln \left(3 + \cos 2 \times \frac{\pi}{4} \right) + c$$

$$\ln 3 = -\frac{1}{2} \ln(3) + c$$

$$c = \frac{3}{2} \ln 3$$

$$\text{so } \ln x = -\frac{1}{2} \ln(3 + \cos 2\theta) + \frac{3}{2} \ln 3$$

$$\begin{aligned} x &= e^{-\frac{1}{2} \ln(3 + \cos 2\theta) + \frac{3}{2} \ln 3} \\ &= e^{-\frac{1}{2} \ln(3 + \cos 2\theta)} \times e^{\frac{3}{2} \ln 3} \\ &= e^{\ln(3 + \cos 2\theta)^{-\frac{1}{2}}} \times e^{\ln 3^{\frac{3}{2}}} \\ &= (3 + \cos 2\theta)^{-\frac{1}{2}} \times 3^{\frac{3}{2}} \\ &= \frac{\sqrt{3^3}}{\sqrt{3 + \cos 2\theta}} \\ &= \sqrt{\frac{27}{3 + \cos 2\theta}} \end{aligned}$$

(ii) Least value is when $\cos 2\theta = 1$

$$x = \sqrt{\frac{27}{3+1}} = \frac{3\sqrt{3}}{2} \approx 2.60 \text{ (3 s.f.)}$$

5 (i) Since $V = 2h$,

$$\frac{dV}{dt} = 2 \frac{dh}{dt}$$

$$\begin{aligned} \frac{dV}{dt} &= \text{volume in} - \text{volume out} \\ &= 1 - 0.2\sqrt{h} \end{aligned}$$

$$1 - 0.2\sqrt{h} = 2 \frac{dh}{dt}$$

$$5 - \sqrt{h} = 10 \frac{dh}{dt}$$

$$\int dt = \int \frac{10}{5 - \sqrt{h}} dh$$

$$t = \int \frac{10}{5 - \sqrt{h}} dh$$

To find t when $h = 4$,

$$T = \int_0^4 \frac{10}{5 - \sqrt{h}} dh$$

$$\text{(ii) } u = 5 - \sqrt{h} \Rightarrow \frac{du}{dh} = -\frac{1}{2\sqrt{h}}$$

$$dh = -2(5 - u) du$$

$$dh = (-10 + 2u) du$$

$$h = 4 \Rightarrow u = 3, h = 0 \Rightarrow u = 5$$

$$T = \int_{u=5}^{u=3} \frac{10}{u} (-10 + 2u) du$$

$$= -20 \int_{u=5}^{u=3} \frac{5-u}{u} du$$

$$= 20 \int_u^5 \frac{5-u}{u} du$$

$$= 20 [5 \ln u - u]_3^5$$

$$= 20 [(5 \ln 5 - 5) - (5 \ln 3 - 3)]$$

$$= 11.1 \text{ (3 s.f.)}$$

6 (i) x : amount of A

y : amount of B

$$\frac{dy}{dt} = -0.2 \times \frac{10}{(1+t)^2} \times y$$

$$\int \frac{1}{y} dy = \int \frac{-2}{(1+t)^2} dt$$

$$\int \frac{1}{y} dy = -2 \int (1+t)^{-2} dt$$

$$\ln y = \frac{2}{1+t} + c$$

When $t = 0$, $y = 100$

$$\ln 100 = \frac{2}{1+0} + c$$

$$c = \ln 100 - 2$$

$$c = \ln 100 - \ln e^2 = \ln \left(\frac{100}{e^2} \right)$$

$$\ln y = \frac{2}{1+t} + \ln \left(\frac{100}{e^2} \right)$$

$$y = e^{\frac{2}{1+t} + \ln \left(\frac{100}{e^2} \right)}$$

$$= e^{\frac{2}{1+t}} \times e^{\ln \left(\frac{100}{e^2} \right)}$$

$$= \frac{100}{e^2} e^{\frac{2}{1+t}}$$

$$= 100e^{-2} e^{\frac{2}{1+t}}$$

$$= 100e^{\frac{2}{1+t} - 2}$$

$$= 100e^{\frac{2}{1+t} - \frac{2(1+t)}{1+t}}$$

$$= 100e^{-\frac{2t}{1+t}}$$

(ii) As $t \rightarrow \infty$, $\frac{-2t}{1+t} \rightarrow -2$

$$y = 100e^{-2} = \frac{100}{e^2}$$

The mass of B approaches $\frac{100}{e^2}$.

$$\text{As } t \rightarrow \infty, \frac{10}{(1+t)^2} \rightarrow 0$$

The mass of A approaches 0.

Stretch and challenge (Page 41)

1 (i) $x = a(1+kt)^{-1}$

$$\frac{dx}{dt} = -ka(1+kt)^{-2}$$

$$= -ka \left(\frac{x}{a} \right)^2$$

$$= -\frac{kx^2}{a}$$

(ii) When $t=0$, $x=a \Rightarrow a=2.5$ When $t=1$, $x=1.6$

$$\Rightarrow 1.6 = \frac{2.5}{1+k}$$

$$1+k=1.5625$$

$$k=0.5625$$

(iii) In the long term, $x \rightarrow 0$

$$(iv) \frac{1}{2y-y^2} = \frac{1}{y(2-y)} = \frac{A}{y} + \frac{B}{2-y}$$

$$\Rightarrow 1 = A(2-y) + By$$

$$y=0 \Rightarrow 2A=1 \Rightarrow A=\frac{1}{2}$$

$$y=2 \Rightarrow 1=2B \Rightarrow B=\frac{1}{2}$$

$$\text{So } \frac{1}{2y-y^2} = \frac{1}{2y} + \frac{1}{2(2-y)}$$

$$(v) \int \frac{1}{2y-y^2} dy = \int dt$$

$$\Rightarrow \int \left(\frac{1}{2y} + \frac{1}{2(2-y)} \right) dy = \int dt$$

$$\Rightarrow \frac{1}{2} \ln y - \frac{1}{2} \ln(2-y) = t + c$$

$$\text{When } t=0, y=1 \Rightarrow 0-0=0+c \Rightarrow c=0$$

$$\Rightarrow \ln y - \ln(2-y) = 2t$$

$$\Rightarrow \ln \frac{y}{2-y} = 2t$$

$$\frac{y}{2-y} = e^{2t}$$

$$\Rightarrow y = 2e^{2t} - ye^{2t}$$

$$\Rightarrow y + ye^{2t} = 2e^{2t}$$

$$\Rightarrow y(1+e^{2t}) = 2e^{2t}$$

$$\Rightarrow y = \frac{2e^{2t}}{1+e^{2t}} = \frac{2}{1+e^{-2t}}$$

(vi) As $t \rightarrow \infty$, $e^{-2t} \rightarrow 0 \Rightarrow y \rightarrow 2$

So the long-term population is 2000.

$$2 \text{ (i) } P = \frac{700}{5+2e^{-\frac{t}{2}}} = 100$$

(ii) As $t \rightarrow \infty$, $e^{-\frac{t}{2}} \rightarrow 0$

$$\text{So } P \rightarrow \frac{700}{5} = 140$$

$$(iii) P = \frac{700}{5+2e^{-\frac{t}{2}}} \Rightarrow 5+2e^{-\frac{t}{2}} = \frac{700}{P}$$

$$e^{-\frac{t}{2}} = \frac{350}{P} - \frac{5}{2}$$

Differentiating with respect to t ,

$$-\frac{1}{2}e^{-\frac{t}{2}} = -\frac{350}{P^2} \frac{dP}{dt}$$

$$\frac{1}{2} \left(\frac{350}{P} - \frac{5}{2} \right) = \frac{350}{P^2} \frac{dP}{dt}$$

$$\frac{P^2}{700} \left(\frac{350}{P} - \frac{5}{2} \right) = \frac{dP}{dt}$$

$$\frac{dP}{dt} = \frac{P}{2} - \frac{P^2}{280}$$

$$\frac{dP}{dt} = \frac{P}{2} \left(1 - \frac{P}{140} \right)$$

(iv) When $t=0$, $P=100 \Rightarrow$

$$\frac{dP}{dt} = \frac{100}{2} \left(1 - \frac{100}{140} \right) = \frac{100}{7} = 14.3$$

$$3 \quad \frac{dy}{dx} = y^{m+1}$$

$$\int y^{-m-1} dy = \int dx$$

$$\frac{y^{-m}}{-m} = x + c \quad (\text{as } m \neq 0)$$

$$y^m = \frac{1}{m(k-x)}$$

$$y = \frac{1}{\sqrt[m]{m(k-x)}}$$

Using the chain rule,

$$\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n) \frac{dy}{dx} = ny^{n-1} y^{m+1} = ny^{n+m},$$

as required.

$$4 \text{ (i) } \frac{d^2x}{dt^2} = 0$$

Integrating with respect to t , $\frac{dx}{dt} = C$ But when $t=0$, $v_x = V \cos \alpha$, so $\frac{dx}{dt} = V \cos \alpha$

$$\frac{d^2y}{dt^2} = -g,$$

Integrating with respect to t , $\frac{dy}{dt} = -gt + K$ But when $t=0$, $v_y = V \sin \alpha$, so $K = V \sin \alpha$ and $\frac{dy}{dt} = -gt + V \sin \alpha$

Integrating again,

$$\frac{dx}{dt} = V \cos \alpha \text{ so } x = Vt \cos \alpha + M$$

But $x=0$ when $t=0$, so $M=0$ and $x = Vt \cos \alpha$

$$\frac{dy}{dt} = -gt + V \sin \alpha \text{ so } y = \frac{-gt^2}{2} + Vt \sin \alpha + N$$

But $y=0$ when $t=0$, so $N=0$ and $y = \frac{-gt^2}{2} + Vt \sin \alpha$ From $x = Vt \cos \alpha$ we get $t = \frac{x}{V \cos \alpha}$ and substituting into $y = \frac{-gt^2}{2} + Vt \sin \alpha$ gives

$$y = \frac{-g \left(\frac{x}{V \cos \alpha} \right)^2}{2} + V \left(\frac{x}{V \cos \alpha} \right) \sin \alpha \text{ so}$$

$$y = x \tan \alpha - \frac{gx^2}{2V^2} \sec^2 \alpha \text{ or}$$

$$y = x \tan \alpha - \frac{gx^2}{2V^2} (1 + \tan^2 \alpha)$$

- (iii) (a) If the centre passes through the point (kh, h) , then, from the above

$$h = kh \tan \alpha - \frac{g(kh)^2}{2V^2} (1 + \tan^2 \alpha)$$

which is a quadratic in $\tan \alpha$ so, rearranging,

$$g(kh)^2 \tan^2 \alpha - 2V^2 kh \tan \alpha + 2V^2 h + g(kh)^2 = 0$$

and, since $h \neq 0$

$$ghk^2 \tan^2 \alpha - 2V^2 k \tan \alpha + 2V^2 + ghk^2 = 0$$

This has two distinct real solutions if and only if $b^2 - 4ac > 0$.

$$(2V^2 k)^2 - 4ghk^2(2V^2 + ghk^2) > 0$$

$$4V^4 k^2 - 4ghk^2(2V^2 + ghk^2) > 0$$

$$V^4 - 2ghV^2 - g^2 h^2 k^2 > 0 \quad (k^2 > 0)$$

The LHS of this is a quadratic in V^2 , and has the form $(V^2 - a)(V^2 - b)$ where a, b arise from

$$V^2 = \frac{2gh \pm \sqrt{(2gh)^2 + 4g^2 h^2 k^2}}{2}$$

$$V^2 = \frac{gh \pm gh\sqrt{1+k^2}}{1}$$

$$V^2 = gh(1 \pm \sqrt{1+k^2})$$

$$\text{But } 1 - \sqrt{1+k^2} < 0$$

$$\text{So } 4V^2 - 8ghV^2 - 4g^2 h^2 k^2 > 0 \text{ when}$$

$$V^2 > gh(1 + \sqrt{1+k^2})$$

OR

$$y = \frac{-gt^2}{2} + Vt \sin \alpha \text{ and } x = Vt \cos \alpha$$

Since it passes through the point (kh, h) ,

$$h = \frac{-gt^2}{2} + Vt \sin \alpha \text{ and } kh = Vt \cos \alpha$$

$$Vt \sin \alpha = h + \frac{gt^2}{2}$$

$$V^2 t^2 \sin^2 \alpha = \left(h + \frac{gt^2}{2}\right)^2$$

$$V^2 t^2 \left(1 - \left(\frac{kh}{Vt}\right)^2\right) = \left(h + \frac{gt^2}{2}\right)^2$$

$$4(V^2 t^2 - k^2 h^2) = 4h^2 + 4ght^2 + g^2 t^4$$

$$g^2 t^4 + 4(gh - V^2)t^2 + 4h^2(1 + k^2) = 0$$

To get two different values of t we need

$$b^2 - 4ac > 0$$

$$16(gh - V^2)^2 - 16g^2 h^2(1 + k^2) > 0$$

$$(gh - V^2)^2 - g^2 h^2(1 + k^2) > 0$$

$$gh - V^2 < -gh\sqrt{1+k^2} \text{ or } gh - V^2 > gh\sqrt{1+k^2}$$

$$\text{So } V^2 > gh(1 + \sqrt{1+k^2})$$

(Since $V^2 < gh(1 + \sqrt{1+k^2})$ is not possible as

$$1 - \sqrt{1+k^2} < 0).$$

(b) Use

$$g(kh)^2 \tan^2 \alpha - 2V^2 kh \tan \alpha + 2V^2 h + g(kh)^2 = 0$$

Let $\tan \alpha_1, \tan \alpha_2$ be the two roots, then by the sum and the product of the roots:

$$\tan \alpha_1 + \tan \alpha_2 = \frac{2V^2 kh}{gk^2 h^2} = \frac{2V^2}{gkh}$$

and

$$\tan \alpha_1 \times \tan \alpha_2 = \frac{2V^2 h + gk^2 h^2}{gk^2 h^2} = \frac{2V^2 + gk^2 h}{gk^2 h}$$

$$\text{and, since } \tan(\alpha_1 + \alpha_2) = \frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \times \tan \alpha_2},$$

$$\begin{aligned} \tan(\alpha_1 + \alpha_2) &= \frac{\frac{2V^2}{gkh}}{1 - \frac{2V^2 + gk^2 h}{gk^2 h}} \\ &= \frac{2kV^2}{gk^2 h - 2V^2 - gk^2 h} \\ &= -k \end{aligned}$$

$$\text{and } \alpha_1 + \alpha_2 = \tan^{-1}(-k).$$

- 5 Let y be the number of tigers at time t .

$$\frac{dy}{dt} = ky \ln \frac{180}{y}$$

$$\frac{dy}{dt} = ky(\ln 180 - \ln y)$$

$$\int \frac{1}{y(\ln 180 - \ln y)} dy = \int k dt$$

$$u = \ln y \Rightarrow du = \frac{1}{y} dy \Rightarrow dy = y du$$

$$\int \frac{1}{y(\ln 180 - u)} y du = \int k dt$$

$$\int \frac{1}{(\ln 180 - u)} du = \int k dt$$

$$-\ln(\ln 180 - u) = kt + c_1$$

$$\ln(\ln 180 - \ln y) = -kt + c_2$$

$$\ln 180 - \ln y = e^{-kt + c_2}$$

$$\ln 180 - \ln y = A e^{-kt}$$

$$t = 0, y = 10 \Rightarrow A = \ln 18$$

$$\ln 180 - \ln y = \ln 18 e^{-kt}$$

$$t = 2, y = 30 \Rightarrow \ln 180 - \ln 30 = \ln 18 e^{-2k}$$

$$\ln 6 = \ln 18 e^{-2k}$$

$$k = 0.239$$

$$\ln 180 - \ln y = \ln 18 e^{-0.239t}$$

$$\ln y = \ln 180 - \ln 18 e^{-0.239t}$$

$$y = e^{\ln 180} \times e^{-\ln 18 e^{-0.239t}}$$

$$y = 180 e^{-\ln 18 e^{-0.239t}}$$

When $t = 8$,

$$y = 180 e^{-\ln 18 e^{-0.239 \times 8}} \approx 117$$

- 6 A : amount of alcohol (litres) in the tank at time t .

$$\frac{dA}{dt} = \text{amount in} - \text{amount out}$$

$$= 2 - \frac{A}{40} \times 4$$

$$= 2 - \frac{A}{10}$$

$$= \frac{20 - A}{10}$$

$$\int \frac{10}{20 - A} dA = \int dt$$

$$-10 \ln|20 - A| = t + c_1$$

$$\ln|20 - A| = -\frac{1}{10}t + c_2$$

$$20 - A = e^{-\frac{1}{10}t + c_2}$$

$$20 - A = K e^{-\frac{1}{10}t}$$

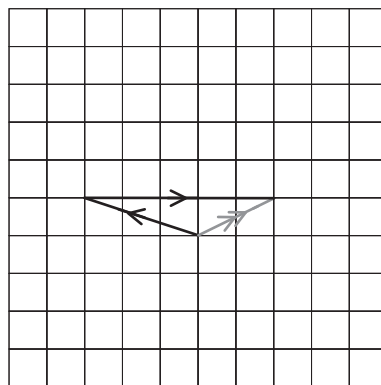
$$A = 20 - K e^{-\frac{1}{10}t}$$

$$t = 0, A = 8 \Rightarrow 8 = 20 - K e^0 \Rightarrow K = 12$$

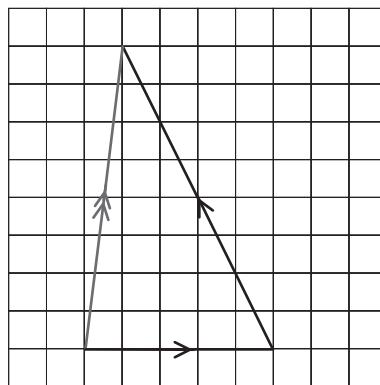
$$A = 20 - 12e^{-\frac{1}{10}t}$$

$$A = 20 - 12e^{-\frac{1}{10} \times 15} = 17.3 \text{ litres}$$

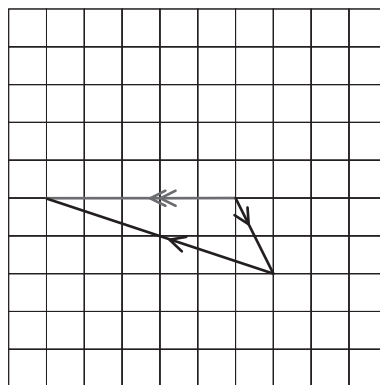
(iii) (a) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



(b) $\mathbf{b} - 2\mathbf{c} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$



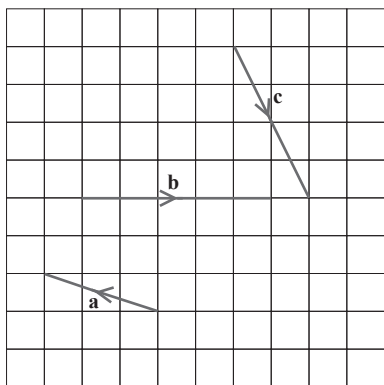
(c) $\frac{1}{2}\mathbf{c} + 2\mathbf{a} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$



4 Vectors

4.1 Vectors in two dimensions (Page 44)

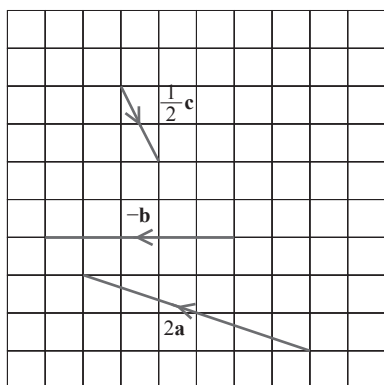
- 1 (i)



(ii) (a) $2\mathbf{a} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$

(b) $-\mathbf{b} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

(c) $-\frac{1}{2}\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$



- 2 (i) $4\mathbf{i} - 5\mathbf{j}$

(ii) $-\mathbf{i} + 2\mathbf{j}$

(iii) $\mathbf{g} = -6\mathbf{j}$

$$\mathbf{h} = 4\mathbf{i} - 5\mathbf{j}$$

$$\mathbf{g} + \mathbf{h} = -6\mathbf{j} + 4\mathbf{i} - 5\mathbf{j} = 4\mathbf{i} - 11\mathbf{j}$$

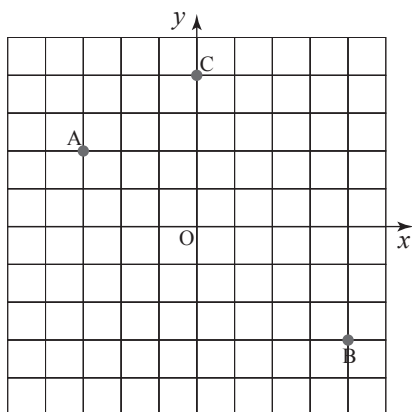
- 3 (i) $l = \sqrt{(-3)^2 + 2^2} = \sqrt{13} = 3.61 \text{ (3 s.f.)}$

(ii) $l = \sqrt{2^2 + (-1)^2} = \sqrt{5} = 2.24 \text{ (3 s.f.)}$

(iii) $\vec{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

$$|\vec{AB}| = \sqrt{(-6)^2 + 1^2} = \sqrt{37} = 6.08 \text{ (3 s.f.)}$$

4



$$(i) \vec{OA} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$(ii) \vec{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$$

$$(iii) \vec{CA} = \mathbf{a} - \mathbf{c} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$(iv) \vec{AD} = 3 \begin{pmatrix} -4 \\ 7 \end{pmatrix} = \begin{pmatrix} -12 \\ 21 \end{pmatrix}$$

$$\begin{aligned} \vec{OD} &= \vec{OA} + \vec{AD} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -12 \\ 21 \end{pmatrix} \\ &= \begin{pmatrix} -15 \\ 23 \end{pmatrix} \end{aligned}$$

So D is $(-15, 23)$.

$$5 (i) \vec{OB} = 2\mathbf{m} + \frac{1}{2}\mathbf{n}$$

$$(ii) \vec{OE} = \frac{5}{2}\mathbf{m} + \frac{5}{2}\mathbf{n}$$

$$(iii) \vec{BD} = 2\mathbf{m} + \frac{3}{2}\mathbf{n}$$

$$(iv) \vec{EC} = \frac{1}{2}\mathbf{m} - \frac{3}{2}\mathbf{n}$$

$$6 (i) l = \sqrt{(-4)^2 + 3^2} = 5$$

$$\text{Unit vector is } \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} = \begin{pmatrix} -0.8 \\ 0.6 \end{pmatrix}$$

$$(ii) \vec{EF} = \mathbf{f} - \mathbf{e} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$|\vec{EF}| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

$$\text{Unit vector is } \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{17}} \\ -\frac{1}{\sqrt{17}} \end{pmatrix}$$

$$\text{or } \frac{1}{\sqrt{17}}(4\mathbf{i} - \mathbf{j})$$

4.2 Vectors in three dimensions (Page 47)

$$1 (i) \begin{pmatrix} 2 \\ 8 \\ -18 \end{pmatrix} \quad (ii) \begin{pmatrix} 3 \\ -8 \\ 13 \end{pmatrix}$$

$$\begin{aligned} 2 \quad \vec{AB} &= \mathbf{b} - \mathbf{a} \\ &= (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) - (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \\ &= -\mathbf{i} + 5\mathbf{j} - 4\mathbf{k} \end{aligned}$$

$$\begin{aligned} 3 (i) \quad \vec{OD} &= 3\mathbf{i} + 4\mathbf{k} \\ (ii) \quad \vec{OF} &= 5\mathbf{i} + 6\mathbf{j} + 4\mathbf{k} \\ (iii) \quad \vec{CF} &= 5\mathbf{i} + 4\mathbf{k} \\ (iv) \quad \vec{BF} &= -3\mathbf{i} + 4\mathbf{k} \\ (v) \quad \vec{GA} &= 5\mathbf{i} - 6\mathbf{j} - 4\mathbf{k} \\ (vi) \quad \vec{DF} &= 2\mathbf{i} + 6\mathbf{j} \end{aligned}$$

$$\begin{aligned} 4 \quad \vec{CD} &= \mathbf{d} - \mathbf{c} = \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 8 \end{pmatrix} \\ |\vec{CD}| &= \sqrt{(-5)^2 + 2^2 + 8^2} = \sqrt{93} \\ &= 9.64 \text{ (3 s.f.)} \end{aligned}$$

4.3 Vector calculations (Page 48)

$$1 (i) \vec{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix}$$

$$(ii) \vec{QR} = \mathbf{r} - \mathbf{q} = \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 12 \end{pmatrix}$$

$$(iii) \text{ Since } \vec{QR} = -2\vec{PQ}$$

- they are parallel
- $\vec{QR}$ is twice the length
- $\vec{QR}$ goes in the opposite direction.

$$(iv) \vec{RP} = \mathbf{p} - \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix}$$

$$\vec{RS} = 2\vec{RP} = 2 \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -12 \end{pmatrix}$$

$$\vec{OS} = \vec{OR} + \vec{RS} = \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \\ -12 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

So S is $(1, 2, -3)$.

$$2 \quad \vec{EF} = \mathbf{f} - \mathbf{e} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$|\vec{EF}| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\vec{EG}| = 6|\vec{EF}|$$

$$\vec{EG} = 6 \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \\ -12 \end{pmatrix}$$

$$\vec{OG} = \vec{OE} + \vec{EG}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ -12 \\ -12 \end{pmatrix} = \begin{pmatrix} 8 \\ -13 \\ -6 \end{pmatrix}$$

So G is (8, -13, -6).

$$3 \quad l = \sqrt{(-1)^2 + (-2)^2 + 2^2} = 3$$

Unit vector is

$$\frac{1}{3}(-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = -\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$4 \quad \vec{AB} = \mathbf{l} - \mathbf{k} = \begin{pmatrix} 5 \\ -1 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$|\vec{AB}| = \sqrt{3^2 + 0^2 + 4^2} = 5$$

$$\vec{AM} = 3\vec{AB} = 3 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 12 \end{pmatrix}$$

$$\vec{OM} = \vec{OA} + \vec{AM} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + \begin{pmatrix} 9 \\ 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ 18 \end{pmatrix}$$

$$|\vec{OM}| = \sqrt{11^2 + (-1)^2 + 18^2} = \sqrt{446}$$

$$\text{The unit vector is } \frac{1}{\sqrt{446}} \begin{pmatrix} 11 \\ -1 \\ 18 \end{pmatrix} = \begin{pmatrix} 0.521 \\ -0.047 \\ 0.852 \end{pmatrix}$$

4.4 The angle between two vectors (Page 49)

$$1 \quad \text{(i)} \quad \begin{pmatrix} 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 3 \times -1 + -4 \times 2 = -11$$

$$l_1 = \sqrt{3^2 + (-4)^2} = 5$$

$$l_2 = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

$$-11 = 5 \times \sqrt{5} \times \cos \theta$$

$$\cos \theta = \frac{-11}{5 \times \sqrt{5}}$$

$$\theta = 169.7^\circ (1 \text{ d.p.})$$

$$\text{(ii)} \quad \begin{pmatrix} 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \end{pmatrix} = 3 \times 1 + 7 \times -6 = -39$$

$$l_1 = \sqrt{3^2 + 7^2} = \sqrt{58}$$

$$l_2 = \sqrt{1^2 + (-6)^2} = \sqrt{37}$$

$$-39 = \sqrt{58} \times \sqrt{37} \times \cos \theta$$

$$\cos \theta = \frac{-39}{\sqrt{58} \times \sqrt{37}}$$

$$\theta = 147.2^\circ (1 \text{ d.p.})$$

$$2 \quad \mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix} = 2 \times 0 + -3 \times 5 + -1 \times -2 = -13$$

$$l_1 = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{14}$$

$$l_2 = \sqrt{0^2 + 5^2 + (-2)^2} = \sqrt{29}$$

$$-13 = \sqrt{14} \times \sqrt{29} \times \cos \theta$$

$$\cos \theta = \frac{-13}{\sqrt{14} \times \sqrt{29}}$$

$$\theta = 130.2^\circ (1 \text{ d.p.})$$

$$3 \quad \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 2 \times 3 + 3 \times -2 + -5 \times -1 = 5$$

$$l_1 = \sqrt{2^2 + 3^2 + (-5)^2} = \sqrt{38}$$

$$l_2 = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$$

$$5 = \sqrt{38} \times \sqrt{14} \times \cos \theta$$

$$\cos \theta = \frac{5}{\sqrt{38} \times \sqrt{14}}$$

$$\theta = 77.5^\circ (1 \text{ d.p.})$$

$$4 \quad \mathbf{g} \cdot \mathbf{h} = 0$$

$$\begin{pmatrix} 8 \\ c \\ -2 \end{pmatrix} \cdot \begin{pmatrix} c \\ -3 \\ -1 \end{pmatrix} = 0$$

$$8c + -3c + 2 = 0$$

$$5c + 2 = 0$$

$$c = -\frac{2}{5}$$

$$5 \quad \text{(i)} \quad \vec{OM} = 14\mathbf{i} + 20\mathbf{k}$$

$$\vec{CM} = 30\mathbf{i} - 20\mathbf{k}$$

$$\vec{MB}' = -14\mathbf{i} + 16\mathbf{j} + 20\mathbf{k}$$

$$\vec{CB}' = 12\mathbf{i} + 16\mathbf{j} + 40\mathbf{k}$$

(ii) Angle between $\vec{OM}$ and $\vec{C'M}$:

$$\begin{pmatrix} 14 \\ 0 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ 0 \\ -20 \end{pmatrix} = 14 \times 30 + 0 \times 0 + 20 \times -20$$

$$= 20$$

$$l_1 = \sqrt{14^2 + 0^2 + 20^2} = \sqrt{596}$$

$$l_2 = \sqrt{30^2 + 0^2 + (-20)^2} = \sqrt{1300}$$

$$20 = \sqrt{596} \times \sqrt{1300} \times \cos \theta$$

$$\cos \theta = \frac{20}{\sqrt{596} \times \sqrt{1300}}$$

$$\theta = 88.7^\circ \text{ (1 d.p.)}$$

Angle between $\vec{MB'}$ and $\vec{CB'}$:

$$\begin{pmatrix} -14 \\ 16 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 16 \\ 40 \end{pmatrix} = -14 \times 12 + 16 \times 16 + 20 \times 40$$

$$= 888$$

$$l_1 = \sqrt{(-14)^2 + 16^2 + 20^2} = \sqrt{852}$$

$$l_2 = \sqrt{12^2 + 16^2 + 40^2} = \sqrt{2000}$$

$$888 = \sqrt{852} \times \sqrt{2000} \times \cos \theta$$

$$\cos \theta = \frac{888}{\sqrt{852} \times \sqrt{2000}}$$

$$\theta = 47.1^\circ \text{ (1 d.p.)}$$

4.5 The vector equation of a line (Page 51)

1 (i) $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + t(-2\mathbf{i} + 5\mathbf{j})$

$$= (3 - 2t)\mathbf{i} + (-1 + 5t)\mathbf{j}$$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

(ii) $\vec{AB} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

$$\mathbf{r} = -2\mathbf{i} + \mathbf{j} + t(2\mathbf{i} + 7\mathbf{j})$$

$$= (-2 + 2t)\mathbf{i} + (1 + 7t)\mathbf{j}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

(iii) $\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 4\mathbf{k})$

$$= \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

(iv) $\vec{AB} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 9 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -9 \\ 4 \end{pmatrix}$

$$\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(\mathbf{i} - 9\mathbf{j} + 4\mathbf{k})$$

$$= (2 + \lambda)\mathbf{i} - 9\lambda\mathbf{j} + (-1 + 4\lambda)\mathbf{k}$$

$$= \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -9 \\ 4 \end{pmatrix}$$

2 (i) We need to find λ such that

$$2 - 3\lambda = -7$$

$$-1 - 2\lambda = -7$$

$$1 + \lambda = 4$$

When $\lambda = 3$, all equations are satisfied, hence the point lies on the line.

(ii) We need to find μ such that

$$1 + \mu = 0$$

$$-2\mu = 2$$

$$-6 + \mu = -8$$

If $\mu = -1$, the first two equations are satisfied, but the last equation is not. Hence the point does *not* lie on the line.

3 $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + t(\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

$$= \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$3 + t = a \quad \text{①}$$

$$-4 + 3t = b \quad \text{②}$$

$$2 - t = 0 \quad \text{③}$$

From ③, $t = 2$

Hence $a = 5$, $b = 2$

4.6 The intersection of two lines (Page 52)

1 (i) Equating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components,

$$4 + 2\lambda = 3 - 5\mu \quad \text{①}$$

$$-1 - 2\lambda = -3 + 2\mu \quad \text{②}$$

$$1 + 3\lambda = 7 \quad \text{③}$$

From ③, $\lambda = 2$

From ②, $-1 - 2 \times 2 = -3 + 2\mu \Rightarrow \mu = -1$

Substituting into ①,

$$4 + 2 \times 2 = 3 - 5 \times -1$$

$$8 = 8$$

So the lines intersect.

Position vector of point of intersection is

$$\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ 7 \end{pmatrix}$$

Point is $(8, -5, 7)$.

(ii) Equating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components,

$$1 - 2\lambda = -1 + 4\mu \quad \text{①}$$

$$3\lambda = 1 - 5\mu \quad \text{②}$$

$$2 + \lambda = -3 + \mu \quad \text{③}$$

Solving ① and ② simultaneously gives

$$\lambda = -3, \mu = 2$$

Substituting into ③,

$$2 + -3 = -3 + 2$$

$$-1 = -1$$

So the lines intersect.

Position vector of point of intersection is

$$\begin{pmatrix} 1 - 2 \times -3 \\ 3 \times -3 \\ 2 + -3 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ -1 \end{pmatrix}$$

Point is (7, -9, -1).

(iii) Equating **i**, **j** and **k** components.

$$-4 - s = 3 + 11t \quad (1)$$

$$4 = 1 - 3t \quad (2)$$

$$-5 + 3s = 8 + t \quad (3)$$

From (2), $t = -1$

From (1), $-4 - s = 3 + 11 \times -1 \Rightarrow s = 4$

Substituting into (3),

$$-5 + 3 \times 4 = 8 + (-1)$$

$$7 = 7$$

So the lines intersect.

Position vector of point of intersection is

$$\mathbf{r} = (-4 - 4)\mathbf{i} + 4\mathbf{j} + (-5 + 3 \times 4)\mathbf{k}$$

$$\mathbf{r} = -8\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$$

$$\text{or } (-8, 4, 7)$$

2 (i) $(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} - 5\mathbf{k})$

$$= 3 \times 2 + -4 \times -1 + 2 \times -5$$

$$= 0$$

(ii) Equating **i**, **j** and **k** components,

$$5 + 3s = 2 + 2t \quad (1)$$

$$-2 - 4s = -2 - t \quad (2)$$

$$-2 + 2s = 7 - 5t \quad (3)$$

Solving (1) and (2) simultaneously gives

$$s = \frac{3}{5}, t = \frac{12}{5}$$

Substituting into (3),

$$-2 + 2 \times \frac{3}{5} = 7 - 5 \times \frac{12}{5}$$

$$-\frac{4}{5} \neq -5$$

Hence there is no point of intersection.

The lines are not parallel so the lines are skew.

3 Equating **i**, **j** and **k** components,

$$1 - \lambda = \mu \quad (1)$$

$$2 + 2\lambda = 6 \quad (2)$$

$$-1 + 3\lambda = 3 - 2\mu \quad (3)$$

From (2), $\lambda = 2$

From (1), $1 - 2 = \mu \Rightarrow \mu = -1$

Substituting into (3),

$$-1 + 3 \times 2 = 3 - 2 \times -1$$

$$5 = 5$$

So the lines intersect.

Position vector of point of intersection is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$$

Point is (-1, 6, 5).

$$4 \text{ (i) } \overrightarrow{\text{PQ}} = \begin{pmatrix} 13 \\ 9 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -5 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Equation is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{\text{RS}} = \begin{pmatrix} 10 \\ k \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ k-2 \\ 3 \end{pmatrix}$$

Equation is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ k-2 \\ 3 \end{pmatrix}$$

Equating **i**, **j** and **k** components,

$$3 + 2\lambda = 1 + 9\mu \quad (1)$$

$$4 + \lambda = 2 + \mu(k-2) \quad (2)$$

$$7 - \lambda = 3 + 3\mu \quad (3)$$

Solving (1) and (3) simultaneously gives

$$\lambda = 2, \mu = \frac{2}{3}$$

Substituting into (2),

$$4 + 2 = 2 + \frac{2}{3}(k-2)$$

$$4 = \frac{2}{3}(k-2)$$

$$k = 8$$

$$(ii) \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix}$$

A is (7, 6, 5).

$$5 \text{ (i) } \begin{pmatrix} 2 \\ a \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} = 0$$

$$4 + 2a - 6 = 0$$

$$a = 1$$

(ii) Equating **i**, **j** and **k** components,

$$2t = 3 + 2s \quad (1)$$

$$1 + at = 2s \quad (2)$$

$$1 + t = -1 - 6s \quad (3)$$

Solving ① and ③ simultaneously gives

$$t = 1, s = -\frac{1}{2}$$

From ②,

$$1 + at = 2s \Rightarrow 1 + a = 2 \times -\frac{1}{2} \Rightarrow a = -2$$

6 (ii) Equating **i**, **j** and **k** components,

$$-3 + \lambda = 7 + a\mu \quad \text{①}$$

$$2 + 2\lambda = 3 + b\mu \quad \text{②}$$

$$3 + \lambda = 3 - 2\mu \quad \text{③}$$

① - ③ gives

$$-6 = 4 + (a + 2)\mu$$

$$-10 = (a + 2)\mu$$

$$\mu = -\frac{10}{a + 2}$$

② - 2 × ③ gives

$$-4 = -3 + (b + 4)\mu$$

$$-1 = (b + 4)\mu$$

$$\mu = -\frac{1}{b + 4}$$

$$\text{So } -\frac{10}{a + 2} = -\frac{1}{b + 4}$$

$$10(b + 4) = a + 2$$

$$10b + 40 = a + 2$$

$$a - 10b = 38$$

$$\text{(ii)} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ -2 \end{pmatrix} = 0$$

$$a + 2b - 2 = 0 \Rightarrow a + 2b = 2$$

Solving simultaneously,

$$a = 8, b = -3$$

$$\text{(iii)} \mu = -\frac{10}{a + 2} = -\frac{10}{8 + 2} = -1$$

Position vector of the point is

$$\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} - (8\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$$

$$= -\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$$

$$2 \text{ (ii)} \begin{pmatrix} 3 \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow 12 + c = 0 \Rightarrow c = -12$$

$$\begin{pmatrix} 3 \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 0 \Rightarrow 12 + 3b + 2c = 0$$

$$\Rightarrow \begin{aligned} 12 + 3b + 2(-12) &= 0 \\ -12 + 3b &= 0 \\ b &= 4 \end{aligned}$$

$$\text{(ii)} \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 18$$

$$\sqrt{4^2 + 0^2 + 1^2} = \sqrt{17}$$

$$\sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$$

$$18 = \sqrt{17}\sqrt{29}\cos\theta$$

$$\cos\theta = \frac{18}{\sqrt{17}\sqrt{29}}$$

$$\theta = 35.8^\circ = 36^\circ \text{ (nearest degree)}$$

3 A(1, 3, -2), B(12, -2, 4) and C(5, -1, a)

$$\vec{AC} = \begin{pmatrix} 5 \\ -1 \\ a \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ a + 2 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} 5 \\ -1 \\ a \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ a - 4 \end{pmatrix}$$

$$\vec{AC} \cdot \vec{BC} = 0 \Rightarrow \begin{pmatrix} 4 \\ -4 \\ a + 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ a - 4 \end{pmatrix} = 0$$

$$-12 - 4 + (a + 2)(a - 4) = 0$$

$$-12 - 4 + a^2 - 2a - 8 = 0$$

$$a^2 - 2a - 24 = 0$$

$$(a + 4)(a - 6) = 0$$

$$a = -4 \text{ or } 6$$

$$4 \quad \vec{OB} = \mathbf{i} + \mathbf{j}, \vec{OD} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} = 1$$

$$|\vec{OB}| = \sqrt{2}, |\vec{OD}| = \sqrt{0.5^2 + 0.5^2 + 1^2} = \frac{\sqrt{6}}{2}$$

$$1 = \sqrt{2} \frac{\sqrt{6}}{2} \cos\theta$$

$$\theta = 54.7^\circ$$

4.7 The angle between two lines (Page 56)

$$1 \quad \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -7$$

$$\sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}$$

$$-7 = \sqrt{14}\sqrt{5}\cos\theta$$

$$\cos\theta = \frac{-7}{\sqrt{14}\sqrt{5}}$$

$$\theta = 146.8^\circ$$

Acute angle is $180^\circ - 146.8^\circ = 33.2^\circ$

4.8 The perpendicular distance from a point to a line (Page 57)

- 1 (i) Let M be the point on the line that gives the minimum distance.

$$\overrightarrow{AM} = \begin{pmatrix} 1-3t \\ -1+4t \\ 1+2t \end{pmatrix} - \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} = \begin{pmatrix} 1-3t \\ 6+4t \\ 4+2t \end{pmatrix}$$

$$\begin{pmatrix} 1-3t \\ 6+4t \\ 4+2t \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = 0$$

$$-3 + 9t + 24 + 16t + 8 + 4t = 0$$

$$29t + 29 = 0$$

$$t = -1$$

$$M \text{ is } \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}$$

$$|\overrightarrow{AM}| = \sqrt{(4-0)^2 + (-5-(-7))^2 + (-1-(-3))^2} = \sqrt{24}$$

- (ii) Let M be the point on the line that gives the minimum distance.

$$BM = \begin{pmatrix} 4+2\lambda \\ -2-\lambda \\ 1-2\lambda \end{pmatrix} - \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix} = \begin{pmatrix} 5+2\lambda \\ 4-\lambda \\ -6-2\lambda \end{pmatrix}$$

$$\begin{pmatrix} 5+2\lambda \\ 4-\lambda \\ -6-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$10 + 4\lambda - 4 + \lambda + 12 + 4\lambda = 0$$

$$18 + 9\lambda = 0$$

$$\lambda = -2$$

$$M \text{ is } \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + (-2) \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

$$|\overrightarrow{BM}| = \sqrt{(-1-0)^2 + (-6-0)^2 + (7-5)^2} = \sqrt{41}$$

- 2 (i) $\overrightarrow{PQ} = (4\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) - (5\mathbf{i} + 2\mathbf{j} - 9\mathbf{k})$
 $= -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$$\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ -9 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

(ii) $\overrightarrow{OT}$ is $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0 \Rightarrow \text{lines are perpendicular.}$$

- (iii) Equating i, j and k components,

$$5 - t = s \quad (1)$$

$$2 + 2t = 2s \quad (2)$$

$$-9 + 3t = -s \quad (3)$$

Solving (1) and (3) simultaneously gives

$$t = 2, s = 3$$

Position vector of point of intersection is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$$

(iv) $\sqrt{3^2 + 6^2 + 3^2} = \sqrt{54}$

- 3 (i) Equation of plane's path is

$$\mathbf{r} = \begin{pmatrix} 1.2 \\ 0.8 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

Let M be the point closest to O.

$$\overrightarrow{OM} \cdot \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1.2-2t \\ 0.8-t \\ t \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$-2.4 + 4t - 0.8 + t + t = 0$$

$$-3.2 + 6t = 0$$

$$t = 0.533 \text{ mins (32 s)}$$

$$M \text{ is } \begin{pmatrix} 1.2 - 2 \times 0.533 \\ 0.8 - 0.533 \\ 0.533 \end{pmatrix} = \begin{pmatrix} 0.133 \\ 0.267 \\ 0.533 \end{pmatrix}$$

$$|\overrightarrow{OM}| = \sqrt{0.133^2 + 0.267^2 + 0.533^2} = 0.611 \text{ km}$$

- (ii) Height is 0.533 km or 533 m.

Time is 0.533 min or 32 seconds.

4 (i) Rabbit: $\mathbf{r} = \begin{pmatrix} 5 \\ 20 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 5 \\ 20 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 30 \\ 0 \end{pmatrix}$$

$$5 + 2t = 25 \Rightarrow t = 10$$

It would take the rabbit 10 seconds.

$$(ii) \text{ Eagle: } \mathbf{r} = \begin{pmatrix} -10 \\ 20 \\ 50 \end{pmatrix} + t \begin{pmatrix} 5 \\ 1 \\ -10 \end{pmatrix}$$

Eagle reaches the ground when

$$50 - 10t = 0 \Rightarrow t = 5$$

$$\text{Eagle is then at } \begin{pmatrix} -10 \\ 20 \\ 50 \end{pmatrix} + 5 \begin{pmatrix} 5 \\ 1 \\ -10 \end{pmatrix} = \begin{pmatrix} 15 \\ 25 \\ 0 \end{pmatrix}$$

$$\text{At } t = 5 \text{ the rabbit is at } \begin{pmatrix} 5 \\ 20 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 25 \\ 0 \end{pmatrix}$$

So the eagle catches the rabbit.

Further practice (Page 60)

$$1 \quad (i) \quad \overrightarrow{OC} = 2\overrightarrow{OA} = 2 \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$$

$$\overrightarrow{OD} = 3\overrightarrow{OB} = 3 \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -15 \end{pmatrix}$$

$$\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = \begin{pmatrix} 12 \\ 0 \\ -15 \end{pmatrix} - \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -21 \end{pmatrix}$$

$$(ii) \quad \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -8 \end{pmatrix}$$

$$\overrightarrow{AE} = \frac{1}{2} \overrightarrow{AB} = \frac{1}{2} \begin{pmatrix} 2 \\ 2 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$$

$$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{EC} = \mathbf{c} - \mathbf{e} = \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$$

$$2 \quad \overrightarrow{GH} = \mathbf{h} - \mathbf{g} = \begin{pmatrix} 9 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -12 \end{pmatrix}$$

$$|\overrightarrow{GH}| = \sqrt{5^2 + 0^2 + (-12)^2} = \sqrt{169} = 13$$

$$\text{Unit vector is } \frac{1}{13} \begin{pmatrix} 5 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ 0 \\ -\frac{12}{13} \end{pmatrix} \text{ or } \frac{1}{13}(5\mathbf{i} - 12\mathbf{k})$$

$$3 \quad (i) \quad \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix} = 1 \times 6 + 3 \times 0 + -2 \times 5 = -4$$

$$l_1 = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

$$l_2 = \sqrt{6^2 + 0^2 + 5^2} = \sqrt{61}$$

$$-4 = \sqrt{14} \times \sqrt{61} \times \cos \theta$$

$$\cos \theta = \frac{-4}{\sqrt{14} \times \sqrt{61}}$$

$$\theta = 97.9^\circ (1 \text{ d.p.})$$

$$(ii) \quad \overrightarrow{CD} = \mathbf{d} - \mathbf{c} = \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix} = 1 \times 5 + 3 \times -3 + -2 \times 7 = -18$$

$$l_1 = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

$$l_2 = \sqrt{5^2 + (-3)^2 + 7^2} = \sqrt{83}$$

$$-18 = \sqrt{14} \times \sqrt{83} \times \cos \theta$$

$$\cos \theta = \frac{-18}{\sqrt{14} \times \sqrt{83}}$$

$$\theta = 121.9^\circ (1 \text{ d.p.})$$

(iii) If perpendicular, then $\overrightarrow{CD} \cdot \overrightarrow{OF} = 0$

$$\overrightarrow{CD} \cdot \overrightarrow{OF} = \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} a \\ 3 \\ 2a \end{pmatrix} = 5a + -9 + 14a$$

$$5a + -9 + 14a = 0$$

$$19a - 9 = 0$$

$$a = \frac{9}{19}$$

4 First find the vector $\overrightarrow{PQ}$:

$$\overrightarrow{PQ} = \begin{pmatrix} 10 \\ 1 \\ 12 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 8 \end{pmatrix} \text{ so}$$

$$\overrightarrow{PR} = \frac{1}{2} \overrightarrow{PQ} = \frac{1}{2} \begin{pmatrix} 8 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$$

$$\text{Now } \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix}$$

$$|\overrightarrow{OR}| = \sqrt{6^2 + 0^2 + 8^2} = \sqrt{100} = 10$$

So the unit vector in the direction of $\overrightarrow{OR}$ is

$$\frac{1}{10} \begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0 \\ 0.8 \end{pmatrix}$$

- 5 When two vectors are perpendicular then

$$\mathbf{a} \cdot \mathbf{b} = 0 \text{ so } 2p - 6 - 3p = 0$$

$$-6 - p = 0$$

$$p = -6$$

$$6 \text{ (i) } \overrightarrow{CD} = \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \\ 5 \end{pmatrix}$$

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(-5\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$$

$$= (2 - 5\lambda)\mathbf{i} + (3 - 7\lambda)\mathbf{j} + 5\lambda\mathbf{k}$$

$$= \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -7 \\ 5 \end{pmatrix}$$

$$\text{(ii) } \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(8\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$= (2 + 8\lambda)\mathbf{i} + (3 - 2\lambda)\mathbf{j} + (-2 + \lambda)\mathbf{k}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix}$$

- 7 (i) Equating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components,

$$1 + 2s = 5 - t \quad \text{①}$$

$$-1 + 4s = 2 + 3t \quad \text{②}$$

$$3 + s = 1 + 2t \quad \text{③}$$

Solving ① and ② simultaneously gives

$$s = \frac{3}{2}, t = 1$$

Substituting into ③,

$$3 + \frac{3}{2} = 1 + 2 \times 1$$

$$\frac{9}{2} \neq 3$$

So the lines do not intersect.

Since the direction vectors are not multiples, the vectors are skew.

- (ii) Equating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components,

$$1 + 4\lambda = 1 - 8\mu \quad \text{①}$$

$$8 - \lambda = 2\mu \quad \text{②}$$

$$2\lambda = 12 - 4\mu \quad \text{③}$$

Solving ① and ② simultaneously gives

$$33 = 1 \text{ so no point of intersection.}$$

The direction vectors are multiples of each other so the lines are parallel.

- (iii) Equating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components,

$$-5 + 3t = -1 + 2s \quad \text{①}$$

$$3 = 4 - s \quad \text{②}$$

$$4 - t = 2 \quad \text{③}$$

$$\text{From ②, } s = 1$$

$$\text{From ③, } t = 2$$

Substituting into ①,

$$-5 + 3 \times 2 = -1 + 2 \times 1$$

$$1 = 1$$

So the lines intersect.

$$\mathbf{r} = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Point of intersection is (1, 3, 2).

$$8 \text{ (i) } \overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$$

$$\overrightarrow{PQ} \text{ is } \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + s \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$$

$$l \text{ is } \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

To intersect,

$$1 + 6s = 1 + t \quad \text{①}$$

$$2 + 3s = 3 - 2t \quad \text{②}$$

$$4 + s = 5 + 2t \quad \text{③}$$

$$6 + 4s = 8 \quad \text{②} + \text{③}$$

$$s = \frac{1}{2}$$

$$t = \frac{-1}{4}$$

Substituting into ①,

$$\Rightarrow 1 + 6 \times \frac{1}{2} = 1 + \left(-\frac{1}{4}\right)$$

$$4 \neq \frac{3}{4}$$

So the lines do not intersect.

$$\text{(ii) } \overrightarrow{AQ} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1+t \\ 3-2t \\ 5+2t \end{pmatrix} = \begin{pmatrix} -t \\ -1+2t \\ -1-2t \end{pmatrix}$$

$$\overrightarrow{AQ} \cdot \overrightarrow{QP} = 0$$

$$\begin{pmatrix} -t \\ -1+2t \\ -1-2t \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$-6t - 3 + 6t - 1 - 2t = 0$$

$$-4 - 2t = 0$$

$$t = -2$$

$$\Rightarrow (1 + (-2), 3 + -2 \times -2, 5 + -2 \times 2)$$

$$A \text{ is } (-1, 7, 1).$$

9 To intersect,

$$2 + 3\lambda = 11 - \mu \quad (1)$$

$$k + 2\lambda = -4 + \mu \quad (2)$$

$$1 + 2\lambda = -1 + 2\mu \quad (3)$$

$$5 + 8\lambda = 21 \quad 2 \times (1) + (3)$$

$$\lambda = 2$$

$$\mu = 3$$

$$k + 2 \times 2 = -4 + 3$$

$$k = -5$$

Point is $(11 + 3 \times -1, -4 + 3 \times 1, -1 + 3 \times 2)$

$$= (8, -1, 5)$$

10 (i) $-6\mathbf{i} + 8\mathbf{j} - 2\mathbf{k} = -2(3\mathbf{i} + c\mathbf{j} + \mathbf{k})$

$$8 = -2c \Rightarrow c = -4$$

(ii) Equating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components,

$$3 + t = 2 + 3u \quad (1)$$

$$-8 + 3t = 1 + cu \quad (2)$$

$$2t = 3 + u \quad (3)$$

Solving (1) and (3) simultaneously gives

$$t = 2, u = 1$$

Substituting into (2),

$$-8 + 3 \times 2 = 1 + c \times 1$$

$$-2 = 1 + c$$

$$c = -3$$

$$11 \text{ (i)} \quad \overrightarrow{AB} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Equation is

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(ii) The position vector for any point on AB is

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 + 2\lambda \\ 1 - \lambda \\ 3 + \lambda \end{pmatrix}$$

$$\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$$

$$\begin{pmatrix} 5 + 2\lambda \\ 1 - \lambda \\ 3 + \lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$10 + 4\lambda - 1 + \lambda + 3 + \lambda = 0$$

$$12 + 6\lambda = 0$$

$$\lambda = -2$$

$$\overrightarrow{OP} = \begin{pmatrix} 5 + 2 \times -2 \\ 1 - (-2) \\ 3 + (-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

So P is $(1, 3, 1)$.

12 Second line has direction

$$\begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} = -6 + -5 + -6 = -17$$

$$-17 = \sqrt{38} \times \sqrt{14} \times \cos \theta$$

$$\theta = 42.5^\circ$$

13 Let M be the point on the line that gives the minimum distance.

$$\overrightarrow{CM} = \begin{pmatrix} 5s \\ 1 - s \\ 3 - 2s \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 + 5s \\ -s \\ 5 - 2s \end{pmatrix}$$

$$\begin{pmatrix} -1 + 5s \\ -s \\ 5 - 2s \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$-5 + 25s + s - 10 + 4s = 0$$

$$-15 + 30s = 0$$

$$s = \frac{1}{2}$$

$$M \text{ is } \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 0.5 \\ 2 \end{pmatrix}$$

$$|\overrightarrow{CM}| = \sqrt{(2.5 - 1)^2 + (0.5 - 1)^2 + (2 - -2)^2} = 4.30$$

$$14 \text{ (i)} \quad P \text{ is } \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 10$$

$$10 = 5 \times \sqrt{6} \times \cos \theta$$

$$\cos \theta = \frac{10}{5 \times \sqrt{6}}$$

$$\theta = 35.3^\circ$$

$$(ii) \quad Q \text{ is } \begin{pmatrix} 3 + t \\ 1 - t \\ 1 + 2t \end{pmatrix}$$

$$\begin{pmatrix} 3 + t \\ 1 - t \\ 1 + 2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$3 + t - 1 + t + 2 + 4t = 0$$

$$4 + 6t = 0$$

$$t = -\frac{2}{3}$$

$$Q \text{ is } \begin{pmatrix} 3 - \frac{2}{3} \\ 1 - \frac{2}{3} \\ 1 + 2\left(-\frac{2}{3}\right) \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -1 \end{pmatrix}$$

$$(iii) |\vec{OQ}| = \sqrt{\left(\frac{7}{3}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = \frac{5\sqrt{3}}{3}$$

Past exam questions (Page 62)

$$1 \text{ (i)} \vec{OB} = \vec{OA} + \vec{AB} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

$$|\vec{OB}| = \sqrt{4^2 + 2^2 + 4^2} = 6$$

$$\text{Unit vector is } \frac{1}{6} \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \text{ or } \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$(ii) \vec{AC} = \vec{AO} + \vec{OC} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$

$$|\vec{AC}| = \sqrt{2^2 + (-4)^2 + (-2)^2} = \sqrt{24}$$

$$\vec{AC} \cdot \vec{OB} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} = 8 - 8 - 8 = -8$$

Angle between diagonals:

$$-8 = 6 \times \sqrt{24} \times \cos \theta$$

$$\cos \theta = \frac{-8}{6 \times \sqrt{24}}$$

$$\theta = 105.8^\circ$$

Acute angle is $180^\circ - 105.8^\circ = 74.2^\circ$

$$(iii) |\vec{OA}| = \sqrt{1^2 + 3^2 + 3^2} = \sqrt{19}$$

$$|\vec{OC}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$$

$$\begin{aligned} \text{Perimeter} &= 2|\vec{OA}| + 2|\vec{OC}| \\ &= 2\sqrt{19} + 2\sqrt{11} = 15.4 \end{aligned}$$

$$2 \text{ (i)} \vec{OA} \cdot \vec{OB} = 2 \times -3 + 1 \times 2 + -3 \times -4 = 8$$

$$|\vec{OA}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$$

$$|\vec{OB}| = \sqrt{(-3)^2 + 2^2 + (-4)^2} = \sqrt{29}$$

$$8 = \sqrt{14} \times \sqrt{29} \times \cos \theta$$

$$\cos \theta = \frac{8}{\sqrt{14} \times \sqrt{29}}$$

$$\theta = 66.6^\circ$$

$$\begin{aligned} (ii) \vec{BC} &= \vec{BO} + \vec{OC} = \vec{BO} + p\vec{OA} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + p \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 3+2p \\ -2+p \\ 4-3p \end{pmatrix} \text{ or } (3+2p)\mathbf{i} + (-2+p)\mathbf{j} + (4-3p)\mathbf{k} \end{aligned}$$

$$(iii) \begin{pmatrix} 3+2p \\ -2+p \\ 4-3p \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 2(3+2p) + (-2+p) - 3(4-3p)$$

$$2(3+2p) + (-2+p) - 3(4-3p) = 0$$

$$6 + 4p - 2 + p - 12 + 9p = 0$$

$$-8 + 14p = 0$$

$$p = \frac{8}{14} = \frac{4}{7}$$

$$3 \text{ (i)} \vec{PQ} = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

$$\vec{RQ} = -3\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$$

$$(ii) \vec{PQ} \cdot \vec{RQ} = 3 \times -3 + 6 \times 8 + -3 \times 3 = 30$$

$$|\vec{PQ}| = \sqrt{3^2 + 6^2 + (-3)^2} = \sqrt{54}$$

$$|\vec{RQ}| = \sqrt{(-3)^2 + 8^2 + 3^2} = \sqrt{82}$$

$$30 = \sqrt{54} \times \sqrt{82} \times \cos \theta$$

$$\cos \theta = \frac{30}{\sqrt{54} \times \sqrt{82}}$$

$$\theta = 63.2^\circ$$

4 Lines are

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$$

(i) At intersection,

$$3 - \lambda = 4 + \mu a \quad (1)$$

$$-2 + 2\lambda = 4 + \mu b \quad (2)$$

$$1 + \lambda = 2 - \mu \quad (3)$$

Adding (1) + (3),

$$4 = 6 + (a-1)\mu \Rightarrow \mu = -\frac{2}{a-1}$$

$$(2) - 2 \times (3)$$

$$4 = (b+2)\mu \Rightarrow \mu = -\frac{4}{b+2}$$

$$-\frac{2}{a-1} = -\frac{4}{b+2}$$

$$2b + 4 = 4a - 4$$

$$8 = 4a - 2b$$

$$2a - b = 4$$

$$(ii) \text{ Perpendicular lines } \Rightarrow \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ -1 \end{pmatrix} = 0$$

$$-a + 2b - 1 = 0 \Rightarrow -a + 2b = 1$$

Solving simultaneously, with $2a - b = 4$,

$$a = 3, b = 2$$

(iii) Point of intersection is

$$\mu = -\frac{2}{3-1} = -1$$

$$\mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Point is $(1, 2, 3)$ or $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

5 (i) Perpendicular means

$$\begin{pmatrix} p-6 \\ 2p-6 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4-2p \\ p \\ 2 \end{pmatrix} = 0$$

$$(p-6)(4-2p) + (2p-6)p + 2 = 0$$

$$4p - 2p^2 - 24 + 12p + 2p^2 - 6p + 2 = 0$$

$$10p - 22 = 0$$

$$p = 2.2$$

(ii) If OAB is a straight line, one vector is a scalar multiple of the other.

Looking at the x -coordinates,

$$\overrightarrow{OB} = 2\overrightarrow{OA}$$

$$\Rightarrow 4 - 2p = 2(p - 6)$$

$$4 - 2p = 2p - 12$$

$$4p = 16$$

$$p = 4$$

$$\overrightarrow{OA} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix}$$

$$|\overrightarrow{OA}| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

$$6 \text{ (i)} \quad \overrightarrow{OP} = \begin{pmatrix} 0 \\ p \\ 2 \end{pmatrix}, \overrightarrow{XP} = \begin{pmatrix} -4 \\ p-5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ p \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ p-5 \\ 2 \end{pmatrix} = 0$$

$$p(p-5) + 4 = 0$$

$$p^2 - 5p + 4 = 0$$

$$(p-1)(p-4) = 0$$

$$p = 1 \text{ or } 4$$

$$\text{(ii)} \quad \overrightarrow{XP} = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix}$$

Unit vector is

$$\frac{1}{\sqrt{(-4)^2 + 4^2 + 2^2}} \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\text{(iii)} \quad \overrightarrow{AG} = \begin{pmatrix} -4 \\ 15 \\ 2 \end{pmatrix}$$

$$\overrightarrow{XQ} = k\overrightarrow{AG} = k \begin{pmatrix} -4 \\ 15 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 10 \\ b \end{pmatrix}$$

$$\Rightarrow k = \frac{2}{3}$$

$$\text{So } \overrightarrow{XQ} = \frac{2}{3} \begin{pmatrix} -4 \\ 15 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{8}{3} \\ 10 \\ \frac{4}{3} \end{pmatrix} \text{ or } -\frac{8}{3}\mathbf{i} + 10\mathbf{j} + \frac{4}{3}\mathbf{k}$$

Stretch and challenge (Page 64)

$$1 \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = 0 \Rightarrow -a + b + 3c = 0 \quad (1)$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = 0 \Rightarrow 2a + 5c = 0 \quad (2)$$

$$-2a + 2b + 6c = 0 \quad (1) \times 2 - (3)$$

$$2b + 11c = 0 \quad (2) + (3)$$

There are infinitely many solutions to this system of 2 equations with 3 unknowns.

Choosing the whole-number solutions,

$$b = 11, c = -2, a = 5$$

The vector is $\begin{pmatrix} 5 \\ 11 \\ -2 \end{pmatrix}$ (or any scalar multiple).

$$2 \quad |p| = |q| \Rightarrow \sqrt{a^2 + b^2 + 3^2} = \sqrt{4^2 + a^2 + 3^2}$$

$$a^2 + b^2 + 9 = 16 + a^2 + 9$$

$$b^2 = 16$$

$$b = \pm 4$$

$$\begin{pmatrix} a \\ b \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ a \\ 3 \end{pmatrix} = 0 \Rightarrow 4a + ab + 9 = 0$$

$$\text{If } b = 4, 4a + 4a + 9 = 0 \Rightarrow a = -\frac{9}{8}$$

If $b = -4$, there are no solutions for a .

$$\text{So } a = -\frac{9}{8} \text{ and } b = 4$$

$$3 \quad |\mathbf{v}_B| = \sqrt{120^2 + (-60)^2 + 40^2} = 140 \text{ m s}^{-1}$$

$$|\mathbf{v}_A| = 140 + 35 = 175 \text{ m s}^{-1}$$

$$|\mathbf{v}_A| = k|\mathbf{v}_B| = k \begin{pmatrix} 120 \\ -60 \\ 40 \end{pmatrix} = \begin{pmatrix} 120k \\ -60k \\ 40k \end{pmatrix}$$

$$|\mathbf{v}_A| = \sqrt{(120k)^2 + (-60k)^2 + (40k)^2} = 175$$

$$19\,600k^2 = 175^2$$

$$k^2 = \frac{25}{16}$$

$$k = \frac{5}{4}$$

$$\mathbf{v}_A = \frac{5}{4} \begin{pmatrix} 120 \\ -60 \\ 40 \end{pmatrix} = \begin{pmatrix} 150 \\ -75 \\ 50 \end{pmatrix} = 150\mathbf{i} - 75\mathbf{j} + 50\mathbf{k}$$

$$4 \text{ (i) } \mathbf{r}_1 = \begin{pmatrix} -1 \\ -2-t \\ -1-t \end{pmatrix} \quad \mathbf{r}_2 = \begin{pmatrix} -2-s \\ -4-3s \\ -1-3s \end{pmatrix}$$

$$\overrightarrow{\text{PQ}} = \begin{pmatrix} -1-s \\ -2+t-3s \\ t-3s \end{pmatrix}$$

$$\overrightarrow{\text{PQ}} \cdot \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = 0$$

$$\begin{aligned} 2-t+3s-t+3s &= 0 \\ -2t+6s &= -2 \\ t-3s &= 1 \end{aligned} \quad (1)$$

$$\overrightarrow{\text{PQ}} \cdot \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix} = 0$$

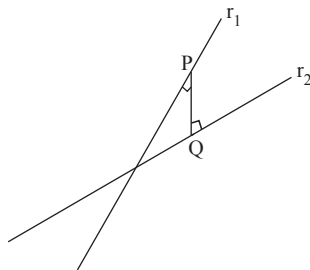
$$\begin{aligned} 1+s+6-3t+9s-3t+9s &= 0 \\ 7+19s-6t &= 0 \\ -6t+19s &= -7 \end{aligned} \quad (2)$$

Solving gives $s = -1$, $t = -2$

P is $(-1, -2 - (-2), -1 - (-2)) = (-1, 0, 1)$

Q is $(-2 - (-1), -4 - 3(-1), -1 - 3(-1)) = (-1, -1, 2)$

$$\begin{aligned} \text{(iii) } |\overrightarrow{\text{PQ}}| &= \sqrt{(-1 - (-1))^2 + (-1 - 0)^2 + (2 - 1)^2} \\ &= \sqrt{2} \approx 1.41 \text{ km} \end{aligned}$$



- 5 Let Q be the point on the line that gives the minimum distance.

$$\mathbf{Q} = \begin{pmatrix} -1+2t \\ a-t \\ 3-t \end{pmatrix}$$

$$\overrightarrow{\text{PQ}} = \begin{pmatrix} -2+2t \\ -t \\ -1-t \end{pmatrix}$$

$$\overrightarrow{\text{PQ}} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2+2t \\ -t \\ -1-t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 0$$

$$-4 + 4t + t + 1 + t = 0$$

$$-3 + 6t = 0$$

$$t = \frac{1}{2}$$

$$\mathbf{Q} = \left(-1 + 2 \times \frac{1}{2}, a - \frac{1}{2}, 3 - \frac{1}{2}\right)$$

$$= \left(0, a - \frac{1}{2}, \frac{5}{2}\right)$$

$$|\overrightarrow{\text{PQ}}| = \sqrt{(1-0)^2 + \left(a - \frac{1}{2} - a\right)^2 + \left(4 - \frac{5}{2}\right)^2}$$

$$= \frac{\sqrt{14}}{2}$$

a is any real number.

5 Complex numbers

5.1 Working with complex numbers (Page 65)

1 (i) $i^3 = i^2 \times i = -i$

(ii) $-2i^6 = -2(i^2)^3$
 $= -2(-1)^3$
 $= 2$

(iii) $(3i^3)^2 = (-3i)^2$
 $= 9i^2$
 $= -9$

(iv) $i^4 - i^3 + i^2 - i = 1 - (-i) + (-1) - i$
 $= 0$

2 (i) $9i + 2$

(ii) $12i^2 = -12$

(iii) $7i + i^2 = -1 + 7i$

(iv) $15 - 5i + 6i - 2i^2 = 15 + i + 2$
 $= 17 + i$

3 (i) $z^2 + 9 = 0$

$$z^2 = -9$$

$$z^2 = 9i^2$$

$$z = \pm 3i$$

(ii) $z^2 - 4z + 8 = 0$

$$z = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-16}}{2}$$

$$= \frac{4 \pm \sqrt{16}i^2}{2}$$

$$= \frac{4 \pm 4i}{2}$$

$$= 2 \pm 2i$$

(iii) $2z^2 - 6z + 5 = 0$

$$z = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$= \frac{6 \pm \sqrt{-4}}{4}$$

$$= \frac{6 \pm \sqrt{4}i^2}{4}$$

$$= \frac{6 \pm 2i}{4}$$

$$= \frac{3}{2} \pm \frac{1}{2}i$$

(iv) $z^2 + 2iz - 5 = 0$

$$z = \frac{-2i \pm \sqrt{(2i)^2 - 4(1)(-5)}}{2(1)}$$

$$= \frac{-2i \pm \sqrt{4i^2 + 20}}{2}$$

$$= \frac{-2i \pm \sqrt{16}}{2}$$

$$= \frac{-2i \pm 4}{2}$$

$$= -i \pm 2$$

So $z = 2 - i$ or $-2 - i$

4 $z = 5 - 2i$, $w = 3 + 7i$

(i) $4z - 3w = 4(5 - 2i) - 3(3 + 7i)$
 $= 20 - 8i - 9 - 21i$
 $= 11 - 29i$

(ii) $z^*w = (5 + 2i)(3 + 7i)$
 $= 15 + 35i + 6i + 14i^2$
 $= 15 + 41i - 14$
 $= 1 + 41i$

(iii) $z^*z = (5 + 2i)(5 - 2i)$
 $= 25 - 10i + 10i - 4i^2$
 $= 25 + 4$
 $= 29$

(iv) $\operatorname{Re}(z - z^*) = \operatorname{Re}(5 - 2i - (5 + 2i))$
 $= \operatorname{Re}(-4i)$
 $= 0$

(v) $\operatorname{Im}[(w^*)^2] = \operatorname{Im}[(3 - 7i)^2]$
 $= \operatorname{Im}[9 - 42i + 49i^2]$
 $= \operatorname{Im}[-40 - 42i]$
 $= -42$

(vi) $[(z^* + w)^*]^2 = [(5 + 2i + 3 + 7i)^*]^2$
 $= [(8 + 9i)^*]^2$
 $= [8 - 9i]^2$
 $= 64 - 144i + 81i^2$
 $= -17 - 144i$

5 (i) Let $z = a + bi$, $a, b \in \mathbb{R}$

$$zz^* = (a + bi)(a - bi)$$

$$= a^2 - abi + abi - b^2i^2$$

$$= a^2 + b^2$$

(ii) $z + z^* = a + bi + a - bi$
 $= 2a$

5.2 Dividing and finding square roots of complex numbers (Page 67)

$$1 \text{ (i)} \quad \frac{2}{i} \times \frac{i}{i} = \frac{2i}{i^2} = -2i$$

$$\text{(ii)} \quad \left(\frac{15}{i}\right)^2 = \frac{225}{i^2} = -225$$

$$\begin{aligned} \text{(iii)} \quad \frac{3}{1-i} \times \frac{1+i}{1+i} &= \frac{3+3i}{1-i^2} \\ &= \frac{3+3i}{2} \\ &= \frac{3}{2} + \frac{3}{2}i \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad \frac{2i}{3-i} \times \frac{3+i}{3+i} &= \frac{6i+2i^2}{9-i^2} \\ &= \frac{-2+6i}{10} \\ &= -\frac{1}{5} + \frac{3}{5}i \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad \frac{5+3i}{4+3i} \times \frac{4-3i}{4-3i} &= \frac{20-15i+12i-9i^2}{16-9i^2} \\ &= \frac{29-3i}{25} \\ &= \frac{29}{25} - \frac{3}{25}i \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad \frac{(2-3i)^2}{3+i} &= \frac{4-12i+9i^2}{3+i} \\ &= \frac{-5-12i}{3+i} \times \frac{3-i}{3-i} \\ &= \frac{-15+15i-36i+12i^2}{9-i^2} \\ &= \frac{-27-21i}{10} \\ &= -\frac{27}{10} - \frac{21}{10}i \end{aligned}$$

$$\begin{aligned} 2 \text{ (i)} \quad (a+bi)^2 &= -5-12i \\ a^2+2abi+b^2i^2 &= -5-12i \\ (a^2-b^2)+2abi &= -5-12i \\ \Rightarrow a^2-b^2 &= -5 \text{ and } 2ab = -12 \Rightarrow b = \frac{-6}{a} \\ a^2 - \left(\frac{-6}{a}\right)^2 &= -5 \\ a^2 - \frac{36}{a^2} &= -5 \\ a^4 - 36 &= -5a^2 \\ a^4 + 5a^2 - 36 &= 0 \\ (a^2+9)(a^2-4) &= 0 \\ \Rightarrow a^2 &= 4 \Rightarrow a = 2 \text{ or } -2 \\ \Rightarrow b &= -3 \text{ or } 3 \\ \text{So } (a, b) &= (2, -3) \text{ or } (-2, 3) \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad (a+bi)^2 &= -24+10i \\ a^2-b^2+2abi &= -24+10i \\ \Rightarrow a^2-b^2 &= -24 \text{ and } 2ab=10 \Rightarrow a = \frac{5}{b} \\ \left(\frac{5}{b}\right)^2 - b^2 &= -24 \\ \frac{25}{b^2} - b^2 &= -24 \\ 25 - b^4 &= -24b^2 \\ b^4 - 24b^2 - 25 &= 0 \\ (b^2-25)(b^2+1) &= 0 \\ \Rightarrow b^2 &= 25 \Rightarrow b = 5 \text{ or } -5 \\ a &= 1 \text{ or } -1 \end{aligned}$$

$$\text{So } (a, b) = (1, 5) \text{ or } (-1, -5)$$

$$\begin{aligned} 3 \text{ (i)} \quad 2z - z^* &= 3+6i \\ 2(x+iy) - (x-iy) &= 3+6i \\ 2x+2yi - x+iy &= 3+6i \\ x+3yi &= 3+6i \\ \Rightarrow x=3, 3y=6 &\Rightarrow y=2 \\ \text{So } z &= 3+2i \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad iz + (2+i)z^* &= 10-2i \\ i(x+iy) + (2+i)(x-iy) &= 10-2i \\ ix+i^2y+2x-2yi+ix-i^2y &= 10-2i \\ ix-y+2x-2yi+ix+y &= 10-2i \\ 2x+(2x-2y)i &= 10-2i \\ 2x=10 &\Rightarrow x=5 \\ 2x-2y=-2 &\Rightarrow 2 \times 5 - 2y = -2 \\ -2y &= -12 \\ y &= 6 \end{aligned}$$

$$\text{So } z = 5+6i$$

$$\begin{aligned} 4 \quad iz^2 &= 4z^* \\ i(a+bi)^2 &= 4(a-bi) \\ i(a^2+2abi+b^2i^2) &= 4(a-bi) \\ i(a^2-b^2+2abi) &= 4a-4bi \\ a^2i-b^2i+2abi^2 &= 4a-4bi \\ -2ab+(a^2-b^2)i &= 4a-4bi \\ -2ab &= 4a \text{ and } a^2-b^2 = -4b \\ ab &= -2a \\ b &= -2 \\ a^2 - (-2)^2 &= -4(-2) \\ a^2 - 4 &= 8 \\ a^2 &= 12 \\ a &= \pm\sqrt{12} \end{aligned}$$

$$\text{So } z = \sqrt{12} - 2i \text{ or } -\sqrt{12} - 2i$$

$$-2ab = 4a \Rightarrow a = 0$$

$$\text{So } -b^2 = -4b$$

$$0 = b^2 - 4b$$

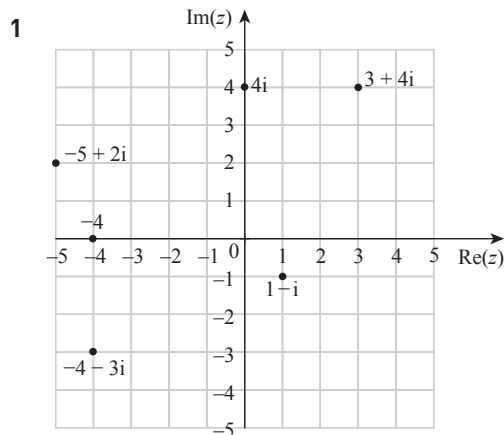
$$0 = b(b - 4)$$

$$b = 0 \text{ or } 4$$

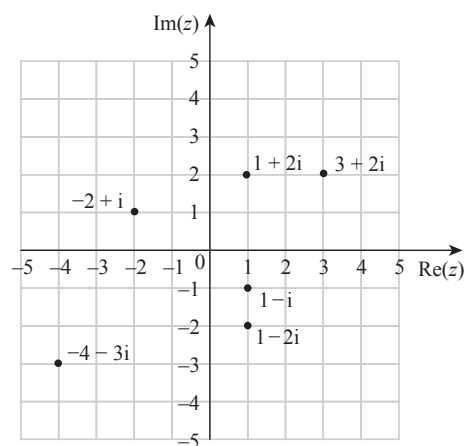
$$\text{So } z = 0 + 0i \text{ or } 4i$$

$$\begin{aligned} 5 \quad u &= \frac{4i}{a-3i} \times \frac{a+3i}{a+3i} \\ &= \frac{4ai + 12i^2}{a^2 - 9i^2} \\ &= \frac{-12 + 4ai}{a^2 + 9} \\ &= \frac{-12}{a^2 + 9} + \frac{4a}{a^2 + 9}i \end{aligned}$$

5.3 Representing complex numbers geometrically and finding the modulus (Page 69)



- 2 (i) $z = 1 + 2i$
 (ii) $z + 2 = 3 + 2i$
 (iii) $z - 3i = 1 - i$
 (iv) $z^* = 1 - 2i$
 (v) $iz = -2 + i$
 (vi) $iz^2 = i(1 + 4i + 4i^2)$
 $= i(-3 + 4i)$
 $= -4 - 3i$



$$3 \quad \text{(i)} \quad |\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

$$\text{(ii)} \quad |3 - 4i| = \sqrt{3^2 + (-4)^2} = 5$$

$$\text{(iii)} \quad |5 + 12i| = \sqrt{5^2 + 12^2} = 13$$

$$\text{(iv)} \quad |-4i| = 4$$

$$\text{(v)} \quad |-\sqrt{2} + \sqrt{5}i| = \sqrt{(-\sqrt{2})^2 + (\sqrt{5})^2} = \sqrt{7}$$

$$\text{(vi)} \quad |-\sqrt{12}| = \sqrt{12}$$

$$4 \quad \text{(i)} \quad |u| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\begin{aligned} \text{(ii)} \quad |u + v| &= |-3 - 2i| \\ &= \sqrt{(-3)^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\text{(iii)} \quad |w^*| = |-2i| = 2$$

$$\begin{aligned} \text{(iv)} \quad \left| \frac{u}{w^*} \right| &= \left| \frac{1+i}{-2i} \right| \\ &= \left| \frac{1+i}{-2i} \times \frac{2i}{2i} \right| \\ &= \left| \frac{2i + 2i^2}{-4i^2} \right| \\ &= \left| \frac{-2 + 2i}{4} \right| \\ &= \left| -\frac{1}{2} + \frac{1}{2}i \right| \\ &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}$$

$$5 \text{ (i) } |3 - 4i| = \sqrt{3^2 + (-4)^2} \\ = 5$$

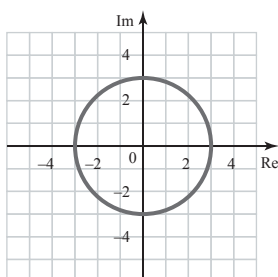
$$\text{(ii) } |x - 1 + i| = \sqrt{(x - 1)^2 + 1^2} \\ = \sqrt{x^2 - 2x + 1 + 1} \\ = \sqrt{x^2 - 2x + 2}$$

$$\text{(iii) } |(x + 1) + i(y + 1)| \\ = \sqrt{(x + 1)^2 + (y + 1)^2} \\ = \sqrt{x^2 + 2x + 1 + y^2 + 2y + 1} \\ = \sqrt{x^2 + y^2 + 2x + 2y + 2}$$

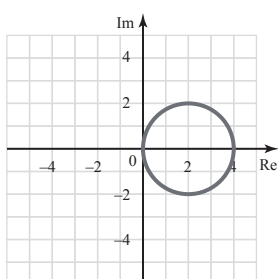
$$\text{(iv) } |\cos \theta + i \sin \theta| \\ = \sqrt{\cos^2 \theta + \sin^2 \theta} \\ = \sqrt{1} \\ = 1$$

5.4 Sets of points in an Argand diagram (Page 71)

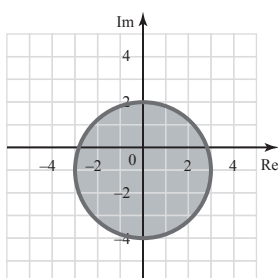
1 (i)



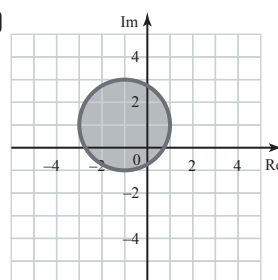
(ii)



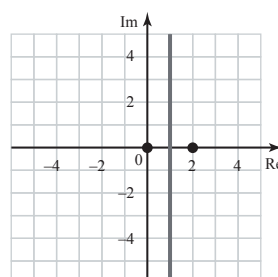
(iii)



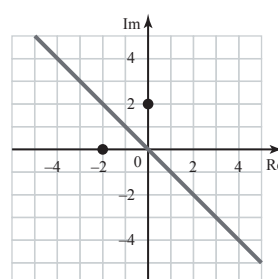
(iv)



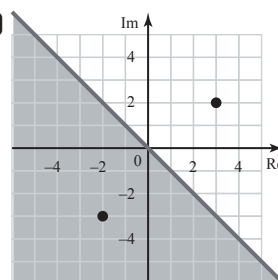
2 (i)



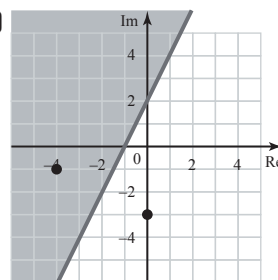
(ii)



(iii)



(iv)



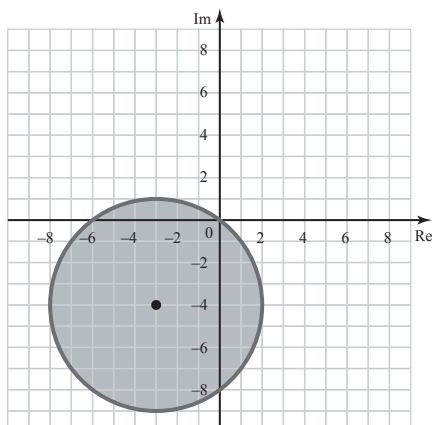
3 If the $<$ or $>$ inequality signs had been used, the boundary curves or lines would need to be dashed, representing the fact that the actual boundary curve or line is not included in the region.

$$4 \quad w = 1 - 2i$$

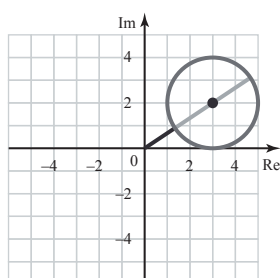
$$\text{(i) } w^2 = (1 - 2i)(1 - 2i) \\ = 1 - 4i + 4i^2 \\ = 1 - 4i - 4 \\ = -3 - 4i \\ |w^2| = \sqrt{(-3)^2 + (-4)^2} = 5$$

(ii) $|z - w^2| \leq |w^2|$

$$|z - (-3 - 4i)| \leq 5$$



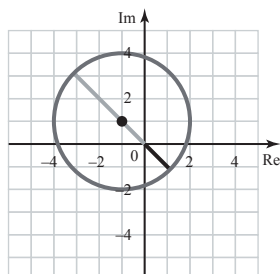
5 (i)



Minimum value of $|z|$ is $\sqrt{13} - 2$

Maximum value of $|z|$ is $\sqrt{13} + 2$

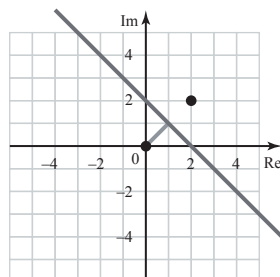
(ii)



Minimum value of $|z|$ is $3 - \sqrt{2}$

Maximum value of $|z|$ is $3 + \sqrt{2}$

6



$$|z| = |z - (2 + 2i)|$$

Minimum value of $|z|$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$

5.5 The modulus-argument form of complex numbers

(Page 74)

1 (i) $-2i = 2\left(\cos -\frac{\pi}{2} + i\sin -\frac{\pi}{2}\right)$

(ii) $4 - 3i = 5(\cos -0.644 + i\sin -0.644)$

(iii) $1 + \sqrt{3}i = 2\left(\cos \frac{\pi}{3} + i\sin \frac{\pi}{3}\right)$

(iv) $8 = 8(\cos 0 + i\sin 0)$

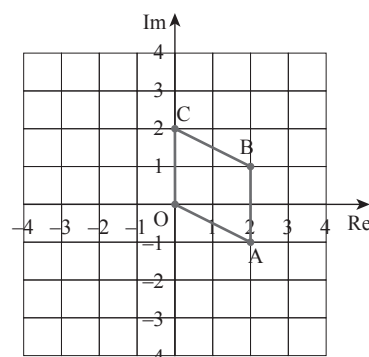
2 (i) $2(\cos 45^\circ + i\sin 45^\circ) = \sqrt{2} + \sqrt{2}i$

(ii) $\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

(iii) $0.5(\cos -90^\circ + i\sin -90^\circ) = -0.5i$

(iv) $\sqrt{2}(\cos 135^\circ + i\sin 135^\circ) = -1 + i$

3



OABC is a parallelogram.

$$\begin{aligned} \text{(ii)} \quad \frac{u^*}{u} &= \frac{2+i}{2-i} \\ &= \frac{2+i}{2-i} \times \frac{2+i}{2+i} \\ &= \frac{4+4i+i^2}{4-i^2} \\ &= \frac{3+4i}{5} \\ &= \frac{3}{5} + \frac{4}{5}i \end{aligned}$$

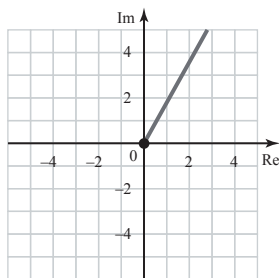
(iii) $\arg\left(\frac{u^*}{u}\right) = \arg u^* - \arg u$

$$\tan^{-1}\left(\frac{\frac{4}{5}}{\frac{3}{5}}\right) = \tan^{-1}\left(\frac{1}{2}\right) - \left[-\tan^{-1}\left(\frac{1}{2}\right)\right]$$

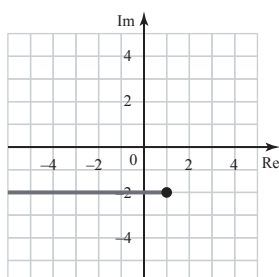
$$\tan^{-1}\left(\frac{4}{3}\right) = 2\tan^{-1}\left(\frac{1}{2}\right)$$

5.6 Sets of points using the polar form (Page 76)

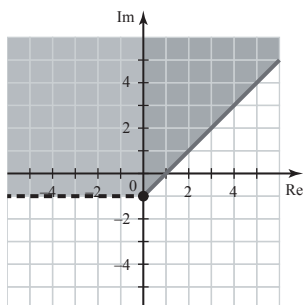
1 (i)



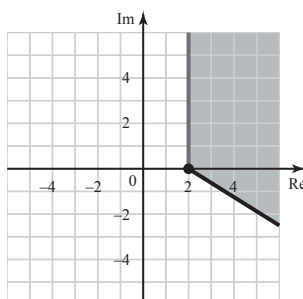
(ii)



(iii)



(iv)



2 (i) $|z - 4 - 2i| = 2$

(ii) $\arg(z - 4 - 2i) = 0$

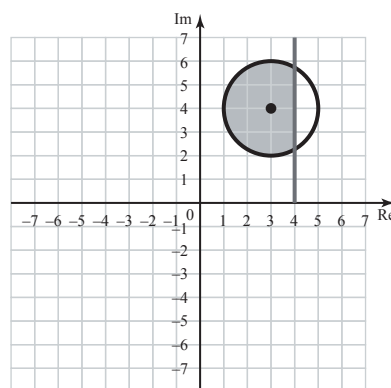
(iii) $a = 4 - 2\cos\frac{\pi}{4} = 4 - \sqrt{2}$

$b = 2 + 2\sin\frac{\pi}{4} = 2 + \sqrt{2}$

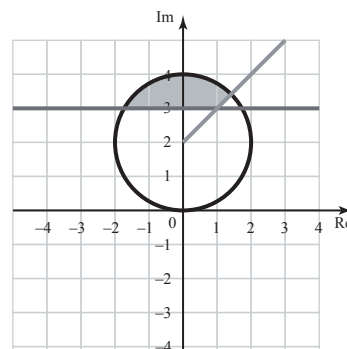
P is $(4 - \sqrt{2}, 2 + \sqrt{2})$

(iv) $0 < \arg(z - 4 - 2i) < \frac{3\pi}{4}$ and $|z - 4 - 2i| < 2$

3

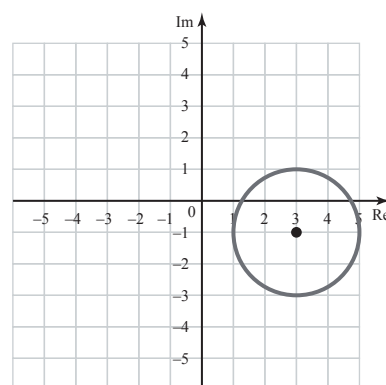


4 (i)



(ii) $\cos\frac{\pi}{4} = \frac{x}{2} \Rightarrow x = \sqrt{2}$

5 (i)



(ii) (a) Maximum value of $|z|$ is $\sqrt{10} + 2$

Minimum value of $|z|$ is $\sqrt{10} - 2$

(b) Maximum value of $|z - 3|$ is 3

Minimum value of $|z - 3|$ is 1

(c) $\sin\theta = \frac{2}{\sqrt{10}} \Rightarrow \theta = 39.23^\circ$

$\tan\alpha = \frac{1}{3} \Rightarrow \alpha = 18.43^\circ$

Maximum value of $\arg(z)$ is

$39.23^\circ - 18.43^\circ = 20.8^\circ$

Minimum value of $\arg(z)$ is

$-(39.23^\circ + 18.43^\circ) = -57.7^\circ$

5.7 Working with complex numbers in polar form

(Page 80)

1 $u = 4 \cos 30^\circ$, $v = 2 \cos 60^\circ$, $w = \cos(-90^\circ)$

(i) $u^* = 4[\cos(-30^\circ) + i \sin(-30^\circ)]$

(ii) $v \times w = 2[\cos(-30^\circ) + i \sin(-30^\circ)]$

(iii) $u \times w^*$
 $= 4[\cos 30^\circ + i \sin 30^\circ] \times [\cos 90^\circ + i \sin 90^\circ]$
 $= 4[\cos 120^\circ + i \sin 120^\circ]$

(iv) $\frac{v}{u} = \frac{1}{2}[\cos 30^\circ + i \sin 30^\circ]$

(v) $\frac{u}{v^*} = \frac{4[\cos 30^\circ + i \sin 30^\circ]}{2\cos(-60^\circ) + i \sin(-60^\circ)}$
 $= 2[\cos 90^\circ + i \sin 90^\circ]$

(vi) $\frac{1}{v} = \frac{1[\cos 0^\circ + i \sin 0^\circ]}{2[\cos 60^\circ + i \sin 60^\circ]}$

(vii) $w^4 = \cos(-90^\circ) + i \sin(-90^\circ)$
 $= \cos(-360^\circ) + i \sin(-360^\circ)$
 $= 1$

(viii) $\frac{iv}{u} = \frac{2i(\cos 60^\circ + i \sin 60^\circ)}{4(\cos 30^\circ + i \sin 30^\circ)}$
 $= \frac{1}{2}(\cos 30^\circ + i \sin 30^\circ)$
 $= \frac{1}{2}(i \cos 30^\circ - \sin 30^\circ)$

2 (i) T (ii) F

(iii) T (iv) T

(v) T (vi) T

3 $\alpha = -\sqrt{3} - i = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)\right)$

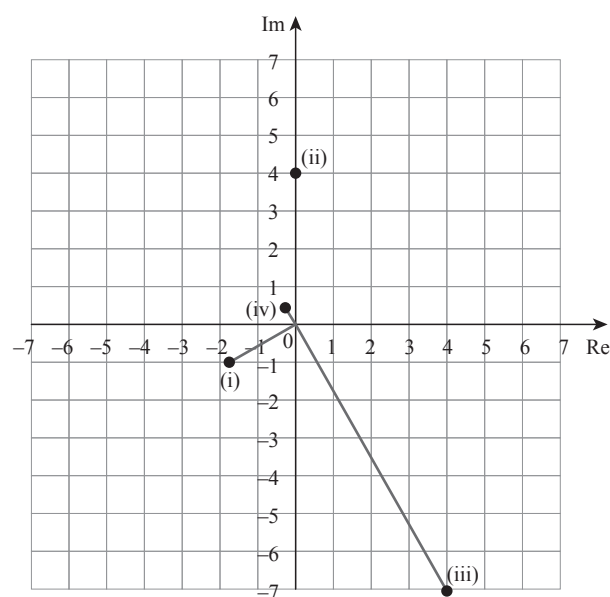
$\beta = 4\left(\cos\frac{1}{2}\pi + i \sin\frac{1}{2}\pi\right) = 4i$

(i) $|\alpha| = 2$
 $\arg \alpha = -\frac{5\pi}{6}$

(ii) $|\beta| = 4$
 $\arg \beta = \frac{1}{2}\pi$

(iii) $\alpha\beta$
 $= 2\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)\right) \times 4\left(\cos\frac{1}{2}\pi + i \sin\frac{1}{2}\pi\right)$
 $= 8\left(\cos\left(-\frac{1}{3}\pi\right) + i \sin\left(-\frac{1}{3}\pi\right)\right)$
 $|\alpha\beta| = 8$
 $\arg|\alpha\beta| = -\frac{1}{3}\pi$

(iv) $\frac{\alpha}{\beta} = \frac{2\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)\right)}{4\left(\cos\frac{1}{2}\pi + i \sin\frac{1}{2}\pi\right)}$
 $= \frac{1}{2}\left(\cos\left(-\frac{4}{3}\pi\right) + i \sin\left(-\frac{4}{3}\pi\right)\right)$
 $= \frac{1}{2}\left(\cos\left(\frac{2}{3}\pi\right) + i \sin\left(\frac{2}{3}\pi\right)\right)$
 $\left|\frac{\alpha}{\beta}\right| = \frac{1}{2}$
 $\arg\left(\frac{\alpha}{\beta}\right) = \frac{2}{3}\pi$



4 Let $z = x + iy$

$$\begin{aligned} \frac{1}{z} &= \frac{1}{x + iy} \times \frac{x - iy}{x - iy} \\ &= \frac{x - iy}{x^2 - i^2 y^2} \\ &= \frac{x - iy}{x^2 + y^2} \\ &= \frac{z^*}{|z|^2} \end{aligned}$$

5 $z^2 = 5 - 12i$

(i) $z^2 = 13[\cos(-1.176) + i \sin(-1.176)]$ or $13 \operatorname{cis}(-67.4^\circ)$

(ii) $z^2 = 5 - 12i = 13[\cos(-1.176) + i \sin(-1.176)]$
 $z = (13 \cos(-1.176) + i \sin(-1.176))^{\frac{1}{2}}$

(iii) $z = \sqrt{13} [\cos(-0.588) + i \sin(-0.588)]$

(iv) $z = \sqrt{13} \cos 2.554 + i \sin 2.554$

(v) $z = -3 + 2i$ or $3 - 2i$

$$6 \quad u = \frac{3}{2+ai}$$

$$\begin{aligned} \text{(i)} \quad u &= \frac{3}{2+ai} \times \frac{2-ai}{2-ai} \\ &= \frac{6-3ai}{4-a^2i^2} \\ &= \frac{6-3ai}{4+a^2} \\ &= \frac{6}{4+a^2} - \frac{3a}{4+a^2}i \end{aligned}$$

$$\text{(ii) (a)} \quad u^* = \frac{6}{4+a^2} + \frac{3a}{4+a^2}i$$

$$\arg u^* = \tan^{-1} \left(\frac{\frac{3a}{4+a^2}}{\frac{6}{4+a^2}} \right)$$

$$= \tan^{-1} \left(\frac{3a}{6} \right)$$

$$= \tan^{-1} \left(\frac{a}{2} \right)$$

$$\tan^{-1} \left(\frac{a}{2} \right) = \frac{\pi}{4}$$

$$\frac{a}{2} = \tan \frac{\pi}{4}$$

$$\frac{a}{2} = 1$$

$$a = 2$$

$$\begin{aligned} \text{(b)} \quad |u| &= \sqrt{\left(\frac{6}{4+a^2} \right)^2 + \left(\frac{3a}{4+a^2} \right)^2} \\ &= \sqrt{\frac{36+9a^2}{(4+a^2)^2}} \end{aligned}$$

$$\sqrt{\frac{36+9a^2}{(4+a^2)^2}} = \sqrt{2}$$

$$36+9a^2 = 2(4+a^2)^2$$

$$36+9a^2 = 2(16+8a^2+a^4)$$

$$36+9a^2 = 32+16a^2+2a^4$$

$$0 = 2a^4 + 7a^2 - 4$$

$$0 = (2a^2 - 1)(a^2 + 4)$$

$$\Rightarrow 2a^2 - 1 = 0 \text{ or } a^2 + 4 = 0$$

$$a^2 = \frac{1}{2} \quad (\text{no solution})$$

$$\text{So } a = \pm \frac{1}{\sqrt{2}}$$

5.8 Complex exponents (Page 83)

$$1 \quad \text{(i)} \quad 2e^{\frac{\pi}{2}i}$$

$$\text{(ii)} \quad -1+i = \sqrt{2} e^{\frac{3\pi}{4}i}$$

$$\text{(iii)} \quad 2-3i = \sqrt{13} e^{-0.983i}$$

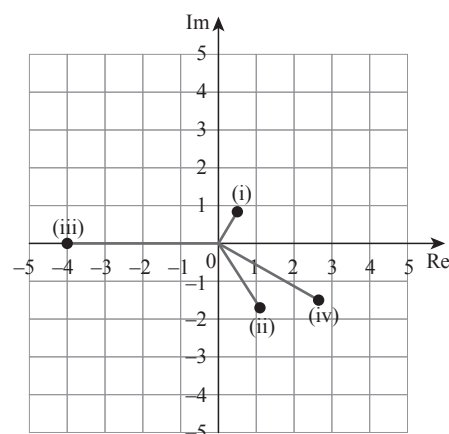
$$\text{(iv)} \quad -5i = 5e^{\frac{-\pi}{2}i}$$

$$2 \quad \text{(i)} \quad e^{\frac{\pi}{3}i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\text{(ii)} \quad 2e^{-i} = 1.08 - 1.68i$$

$$\text{(iii)} \quad 4e^{i\pi} = -4$$

$$\text{(iv)} \quad 3e^{\frac{-\pi}{6}i} = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$$



$$\begin{aligned} 3 \quad \text{(i)} \quad & 4e^{\frac{\pi}{3}i} \times 3 \left(\cos \left(\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{6} \right) \right) \\ &= 4e^{\frac{\pi}{3}i} \times 3e^{\frac{\pi}{6}i} \\ &= 12e^{\frac{\pi}{2}i} \end{aligned}$$

$$\text{(ii) (a)} \quad -2i + \sqrt{3} + i = \sqrt{3} - i$$

$$\text{(b)} \quad \sqrt{3} - i = 2 \left[\cos \left(\frac{-\pi}{6} \right) + i \sin \left(\frac{-\pi}{6} \right) \right]$$

5.9 Complex numbers and equations (Page 85)

$$1 \quad 2z^3 - z^2 + 4z + k = 0$$

$$z = 1 + 2i \text{ is a root}$$

$$\text{So } 1 - 2i \text{ is a root.}$$

$$(z - (1 + 2i))(z - (1 - 2i))(2z - c) = 0$$

$$(z - 1 - 2i)(z - 1 + 2i)(2z - c) = 0$$

$$((z - 1)^2 - 4i^2)(2z - c) = 0$$

$$(z^2 - 2z + 5)(2z - c) = 0$$

$$2z^3 - cz^2 - 4z^2 + 2cz + 10z - 5c = 0$$

$$2z^3 + (-4 - c)z^2 + (2c + 10)z - 5c = 0$$

$$-4 - c = -1 \Rightarrow c = -3$$

$$2c + 10 = 4 \Rightarrow c = -3$$

$$k = -5 \times -3 = 15$$

2 $z^3 - 10z^2 + 37z + p = 0$

$$6^3 - 10 \times 6^2 + 37 \times 6 + p = 0$$

$$p = -78$$

$$\Rightarrow \frac{z^3 - 10z^2 + 37z - 78}{z - 6} = z^2 - 4z + 13$$

$$z^2 - 4z + 13 = 0 \Rightarrow z = \frac{4 \pm \sqrt{(4)^2 - 4 \times 1 \times 13}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

The other roots are $2 + 3i$ and $2 - 3i$

3 (i) $iz^2 - z + 2i = 0$

$$z = \frac{1 \pm \sqrt{(-1)^2 - 4 \times i \times 2i}}{2i}$$

$$= \frac{1 \pm \sqrt{1 - 8i^2}}{2i}$$

$$= \frac{1 \pm 3}{2i}$$

$$= \frac{4}{2i} \text{ or } \frac{-2}{2i}$$

$$= \frac{2}{i} \text{ or } -\frac{1}{i} \times \frac{i}{i}$$

$$= -2i \text{ or } i$$

(ii) The coefficients of the polynomial are not real numbers.

4 (i) $z^4 - 16 = 0$

$$(z^2 - 4)(z^2 + 4) = 0$$

$$z^2 = 4 \quad z^2 = -4$$

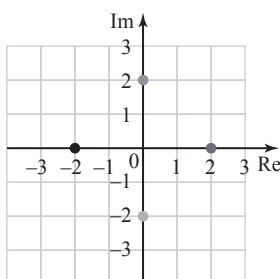
$$z = \pm 2 \quad z = \pm \sqrt{-4}$$

$$= \pm \sqrt{4i^2}$$

$$= \pm 2i$$

$$\Rightarrow z = 2, -2, 2i, -2i$$

(ii)



The solutions are all 90° apart.

(iii) $z^4 = k$

$$(1 + i)^4 = (1 + i)^2(1 + i)^2$$

$$= (1 + 2i + i^2)(1 + 2i + i^2)$$

$$= 2i \times 2i$$

$$= 4i^2$$

$$= -4$$

$$\Rightarrow k = -4$$

Other solutions are $-1 + i$, $-1 - i$, $1 - i$

5 $x^4 + Ax^3 + Bx^2 + Cx + D = 0$

(i) Roots $2 + i$, $2 - i$, $-2i$, $2i$

(ii) $(x - (2 + i))(x - (2 - i))(x - (-2i))(x - 2i) = 0$

$$(x - 2 - i)(x - 2 + i)(x + 2i)(x - 2i) = 0$$

$$((x - 2)^2 - i^2)(x - 4i^2) = 0$$

$$(x^2 - 4x + 5)(x^2 + 4) = 0$$

$$x^4 + 4x^2 - 4x^3 - 16x + 5x^2 + 20 = 0$$

$$x^4 - 4x^3 + 9x^2 - 16x + 20 = 0$$

$$A = -4, B = 9, C = -16, D = 20$$

6 (i) $3^3 + 3^2 - 7 \times 3 - 15$

$$= 27 + 9 - 21 - 15$$

$$= 0$$

So $z = 3$ is a root.

By long division or synthetic division,

3	1	1	-7	15
		3	12	15
	1	4	5	0

the quotient is $x^2 + 4x + 5$.

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2 \times 1}$$

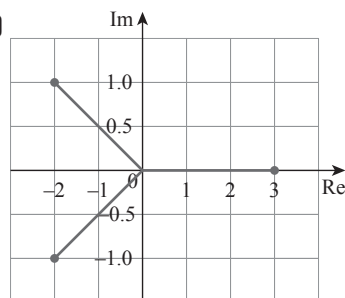
$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm \sqrt{4i^2}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

(ii)



7 $\frac{13\pi}{24} - \frac{\pi}{24} = \frac{12\pi}{24} = \frac{\pi}{2}$

So the roots are $\frac{\pi}{2}$ apart.

$$\Rightarrow n = \frac{2\pi}{\frac{\pi}{2}} = 4$$

$$\left(\cos \frac{\pi}{24} + i \sin \frac{\pi}{24}\right)^4 = \cos\left(4 \times \frac{\pi}{24}\right) + i \sin\left(4 \times \frac{\pi}{24}\right)$$

$$= \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$a = \frac{\sqrt{3}}{2}, \quad b = \frac{1}{2}$$

Further practice (Page 88)

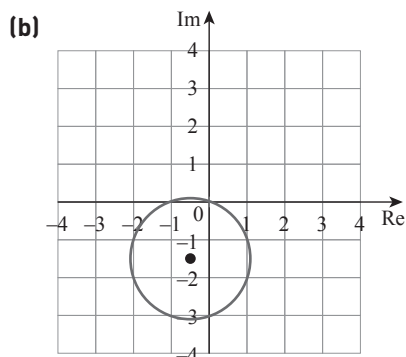
$$1 \text{ (i) (a) } z - 3w = (1 - 2i) - 3(3 - i) \\ = 1 - 2i - 9 + 3i \\ = -8 + i$$

$$\text{(b) } zw = (1 - 2i)(3 - i) \\ = 3 - i - 6i + 2i^2 \\ = 3 - 7i - 2 \\ = 1 - 7i$$

$$\text{(c) } \frac{1}{w} = \frac{1}{3 - i} \times \frac{3 + i}{3 + i} \\ = \frac{3 + i}{9 - i^2} \\ = \frac{3 + i}{10} \\ = \frac{3}{10} + \frac{1}{10}i$$

$$\text{(d) } \frac{w}{z^*} = \frac{3 - i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} \\ = \frac{3 - 6i - i + 2i^2}{1 - 4i^2} \\ = \frac{1 - 7i}{5} \\ = \frac{1}{5} - \frac{7}{5}i$$

$$\text{(ii) (a) } u = \frac{z^2}{w} = \frac{(1 - 2i)^2}{3 - i} \\ = \frac{1 - 4i + 4i^2}{3 - i} \\ = \frac{-3 - 4i}{3 - i} \times \frac{3 + i}{3 + i} \\ = \frac{-9 - 3i - 12i - 4i^2}{9 - i^2} \\ = \frac{-5 - 15i}{10} \\ = -\frac{1}{2} - \frac{3}{2}i$$

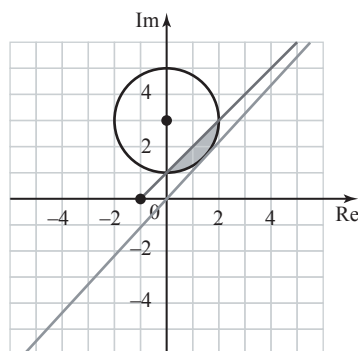


$$2 \quad u - w = 6i \quad uw = 13$$

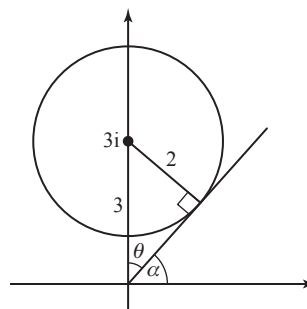
$$u(u - 6i) = 13 \\ u^2 - 6iu - 13 = 0 \\ u = \frac{6i \pm \sqrt{(-6i)^2 - 4 \times 1 \times -13}}{2} \\ = \frac{6i \pm \sqrt{36i^2 + 52}}{2} \\ = \frac{6i \pm \sqrt{16}}{2} \\ = \frac{6i \pm 4}{2} \\ = 3i \pm 2$$

$$\Rightarrow u = 2 + 3i, w = 2 - 3i \\ \text{and } u = -2 + 3i, w = -2 - 3i$$

3 (i), (ii) and (iii) (a)



$$\text{(iii)(b) } \sin \theta = \frac{2}{3} \Rightarrow \theta = 41.8^\circ \\ \Rightarrow \alpha = 90^\circ - 41.8^\circ = 48.2^\circ$$



$$4 \quad u = 1 + 2i$$

$$\text{(i) } (1 + 2i)^2 = 1 + 4i + 4i^2 = -3 + 4i \\ (1 + 2i)^3 = (1 + 2i)(-3 + 4i) \\ = (-3 - 2i + 8i^2) = -11 - 2i \\ (1 + 2i)^4 = (-3 + 4i)(-3 + 4i) \\ = 9 - 24i + 16i^2 = -7 - 24i - 7 - 24i \\ = -14 - 48i \\ -3(-11 - 2i) + 5(-3 + 4i) - (1 + 2i) - 10 \\ = 33 + 6i - 15 + 20i - 1 - 2i - 10 \\ = 0$$

So $u = 1 + 2i$ is a root.

Also $1 - 2i$ is a root.

(ii) $(x - (1 + 2i))(x - (1 - 2i))$

$$= (x - 1 - 2i)(x - 1 + 2i)$$

$$= (x - 1)^2 - (2i)^2$$

$$= x^2 - 2x + 1 - 4i^2$$

$$= x^2 - 2x + 5$$

$$\Rightarrow p(x) = (x^2 - 2x + 5)Q(x)$$

$$Q(x) = \frac{x^4 - 3x^3 + 5x^2 - x - 10}{x^2 - 2x + 5}$$

$$= x^2 - x - 2$$

$$\Rightarrow p(x) = (x^2 - 2x + 5)(x^2 - x - 2)$$

$$= (x^2 - 2x + 5)(x + 1)(x - 2)$$

The other two roots are $x = -1$ and 2

5 (i) If one of the roots is $z = 3$, then $p(3) = 0$

$$p(z) = z^3 + z^2 + 4z - 48$$

$$p(3) = 3^3 + 3^2 + 4 \times 3 - 48$$

$$= 27 + 9 + 12 - 48$$

$$= 0$$

So the real root is 3.

Using synthetic or long division,

$$\frac{p(z)}{z - 3} = z^2 + 4z + 16$$

$$\text{Solving } z^2 + 4z + 16 = 0,$$

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 16}}{2 \times 1}$$

$$= \frac{-4 \pm \sqrt{-48}}{2}$$

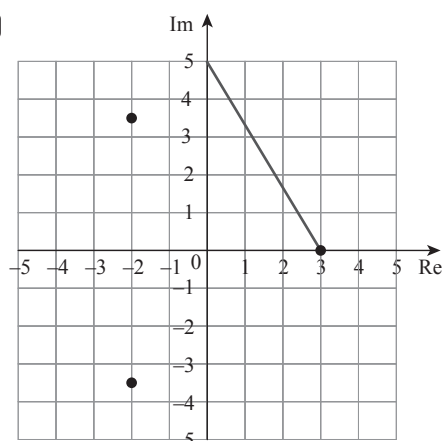
$$= \frac{-4 \pm \sqrt{48i^2}}{2}$$

$$= \frac{-4 \pm 4\sqrt{3}i}{2}$$

$$= -2 \pm 2\sqrt{3}i$$

$$\text{So } \beta = -2 + 2\sqrt{3}i \text{ and } \gamma = -2 - 2\sqrt{3}i$$

(ii) and (iii)



6 (i) $iw^2 = (-3 + 3i)^2$

$$iw^2 = 9 - 18i + 9i^2$$

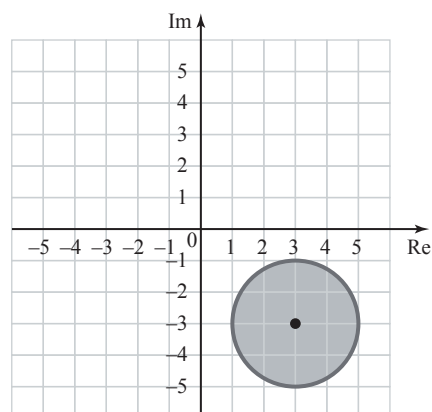
$$iw^2 = -18i$$

$$w^2 = -18$$

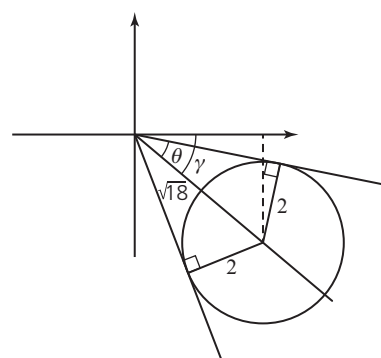
$$w^2 = 18i^2$$

$$w = \pm\sqrt{18}i$$

(ii) (a) $|z - (3 - 3i)| \leq 2$



(b)



$$\sqrt{3^2 + 3^2} = \sqrt{18}$$

$$p = \sqrt{18} - 2, q = \sqrt{18} + 2$$

$$\sin \theta = \frac{2}{\sqrt{18}} \Rightarrow \theta = 28.1^\circ$$

$$\tan \gamma = \frac{3}{3} \Rightarrow \gamma = 45^\circ$$

$$\Rightarrow \beta = -(\gamma - \theta) = -(45^\circ - 28.1^\circ) = -16.9^\circ$$

$$\alpha = -(45^\circ + 28.1^\circ) = -73.1^\circ$$

$$\text{So } \sqrt{18} - 2 \leq |z| \leq \sqrt{18} + 2$$

$$\text{and } -73.1^\circ \leq \arg z \leq -16.9^\circ$$

7 $p(z) = z^3 - z^2 + kz - 27$

(i) $z - 3$ is a factor $\Rightarrow p(3) = 0$

$$3^3 - 3^2 + 3k - 27 = 0$$

$$27 - 9 + 3k - 27 = 0$$

$$3k = 9$$

$$k = 3$$

(ii) (a) By long division or synthetic division,

$$\begin{array}{r|rrrr} 3 & 1 & -1 & 3 & -27 \\ & & 3 & 6 & 27 \\ \hline & 1 & 2 & 9 & 0 \end{array}$$

$$\Rightarrow p(z) = (z - 3)(z^2 + 2z + 9) = 0$$

$$z = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 9}}{2}$$

$$= \frac{-2 \pm \sqrt{32i^2}}{2}$$

$$= -1 \pm 2\sqrt{2}i$$

$$\text{So } z = 3, -1 + 2\sqrt{2}i, -1 - 2\sqrt{2}i$$

(b) $p(z^2) = 0$

$$\Rightarrow z^2 = 3 \quad \text{or} \quad z^2 = -1 + 2\sqrt{2}i \quad \text{or} \quad z^2 = -1 - 2\sqrt{2}i$$

$$z = \pm\sqrt{3} \quad (a+bi)^2 = -1 + 2\sqrt{2}i \quad (a+bi)^2 = -1 - 2\sqrt{2}i$$

$$a^2 - b^2 = -1 \quad a^2 - b^2 = -1$$

$$2ab = 2\sqrt{2} \quad 2ab = -2\sqrt{2}$$

$$b = \frac{\sqrt{2}}{a} \quad b = -\frac{\sqrt{2}}{a}$$

$$\Rightarrow a^2 - \left(\frac{\sqrt{2}}{a}\right)^2 = -1 \quad \Rightarrow a^2 - \left(-\frac{\sqrt{2}}{a}\right)^2 = -1$$

$$a^2 - \frac{2}{a^2} = -1 \quad a^2 - \frac{2}{a^2} = -1$$

$$a^4 - 2 = -a^2$$

$$\text{So } a = \pm 1 \text{ and } b = \mp\sqrt{2}$$

$$a^4 + a^2 - 2 = 0$$

$$(a^2 + 2)(a^2 - 1) = 0$$

$$a^2 = -2 \Rightarrow \text{no solution}$$

$$\text{So } a = \pm 1 \text{ and } b = \pm\sqrt{2}$$

$$\text{So the roots are } \sqrt{3}, -\sqrt{3}, 1 + \sqrt{2}i, -1 - \sqrt{2}i, 1 - \sqrt{2}i, -1 + \sqrt{2}i$$

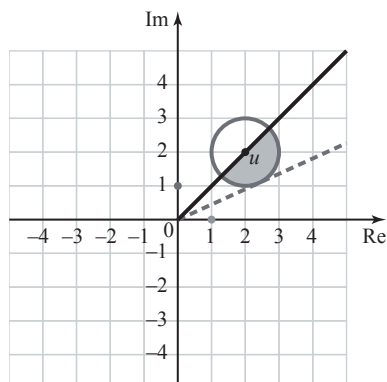
Past exam questions (Page 89)

1 (i) $u = 2 + 2i$

$$|u| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\arg u = \frac{\pi}{4}$$

(ii)



(iii) $a^2 + 1^2 = (\sqrt{8})^2$
 $a = \sqrt{7}$

2 (i) $w = -1 + i = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$

$$w^2 = \left(\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \right)^2 = \sqrt{2}^2 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)$$

$$= 2 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)$$

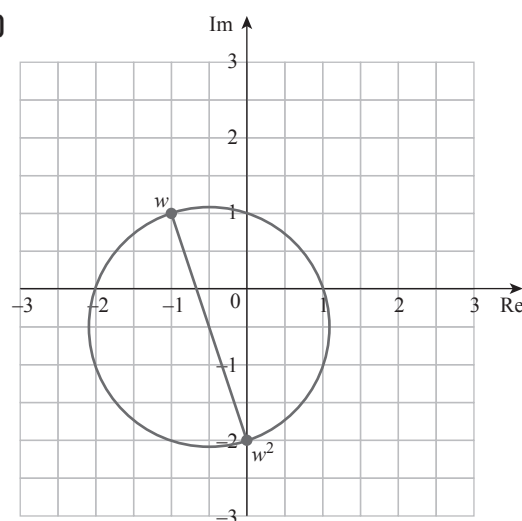
$$w^3 = \left(\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \right)^3 = \sqrt{2}^3 \left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4} \right)$$

$$= 2\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$\Rightarrow |w^2| = 2 \quad \arg w^2 = -\frac{\pi}{2}$$

$$|w^3| = 2\sqrt{2} \quad \arg w^3 = \frac{\pi}{4}$$

(ii)



$$\text{midpoint} = \left(\frac{-1+0}{2}, \frac{-2+1}{2} \right) = \left(-\frac{1}{2}, -\frac{1}{2} \right)$$

$$|w \text{ to } w^2| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\Rightarrow \text{radius} = \frac{\sqrt{10}}{2}$$

$$\text{Equation of the circle is } \left| z - \left(-\frac{1}{2} - \frac{1}{2}i \right) \right| = \frac{\sqrt{10}}{2}$$

3 (i) $iz^2 + 2z - 3i = 0$

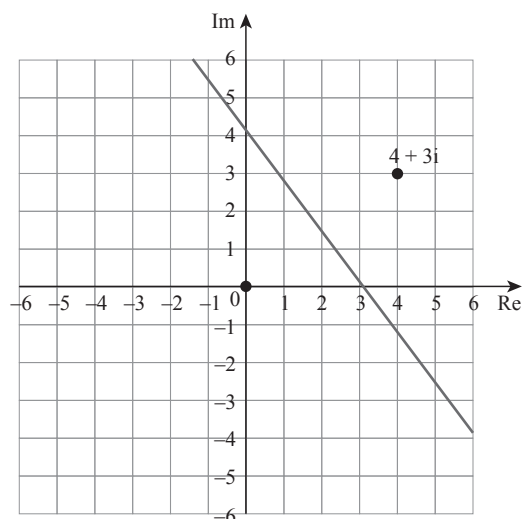
$$\begin{aligned} z &= \frac{-2 \pm \sqrt{2^2 - 4i(-3i)}}{2i} \\ &= \frac{-2 \pm \sqrt{4 + 12i^2}}{2i} \\ &= \frac{-2 \pm \sqrt{-8}}{2i} \\ &= \frac{-2 \pm \sqrt{8}i}{2i} \\ &= \frac{-2 \pm \sqrt{8}i}{2i} \\ &= -\frac{1}{i} \pm \frac{\sqrt{8}}{2} \\ &= -\frac{1}{i} \times \frac{i}{i} \pm \frac{2\sqrt{2}}{2} \\ &= i \pm \sqrt{2} \end{aligned}$$

So $z = \sqrt{2} + i$ or $-\sqrt{2} + i$

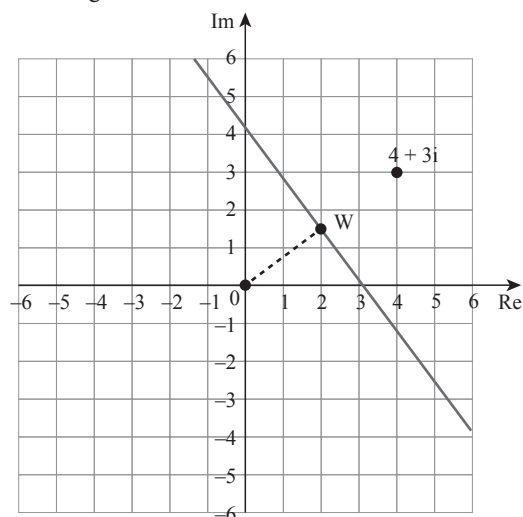
(ii) (a) $|z| = |z - 4 - 3i|$

$|z| = |z - (4 + 3i)|$

which is the perpendicular bisector between (0, 0) and (4, 3)



(b) The point where $|z|$ is least is the point W on the diagram.



W is the midpoint between (0, 0) and (4, 3)

Hence W is (2, 1.5)

$$|w| = \sqrt{2^2 + 1.5^2} = 2.5$$

$$\arg(w) = \tan^{-1}\left(\frac{1.5}{2}\right) = 36.9^\circ \text{ or } 0.64$$

4 $z = (\sqrt{2}) - (\sqrt{6})i$

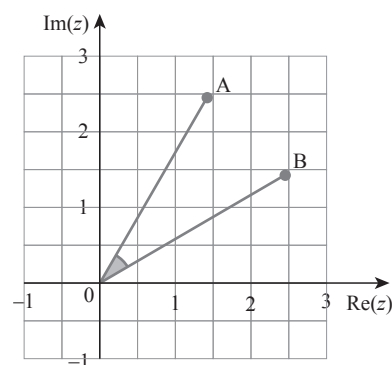
(i) $|z| = \sqrt{(\sqrt{2})^2 + (\sqrt{6})^2} = \sqrt{8} \approx 2.83$

$$\arg(z) = -\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{2}}\right) = -60^\circ \text{ or } -\frac{\pi}{3}$$

(ii) (a) $z + 2z^* = (\sqrt{2}) - (\sqrt{6})i + 2((\sqrt{2}) + (\sqrt{6})i)$
 $= (3\sqrt{2}) + (\sqrt{6})i$

(b) $\frac{z^*}{iz} = \frac{(\sqrt{2}) + (\sqrt{6})i}{i((\sqrt{2}) - (\sqrt{6})i)}$
 $= \frac{(\sqrt{2}) + (\sqrt{6})i}{(\sqrt{2})i - (\sqrt{6})i^2}$
 $= \frac{(\sqrt{2}) + (\sqrt{6})i}{(\sqrt{6}) + (\sqrt{2})i} \times \frac{(\sqrt{6}) - (\sqrt{2})i}{(\sqrt{6}) - (\sqrt{2})i}$
 $= \frac{(\sqrt{12}) + 4i - \sqrt{12}i^2}{6 - 2i^2}$
 $= \frac{(2\sqrt{12}) + 4i}{8}$
 $= \frac{(4\sqrt{3}) + 4i}{8}$
 $= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

(iii) $z^* = (\sqrt{2}) + (\sqrt{6})i$, $iz = (\sqrt{6}) + (\sqrt{2})i$



$$\begin{aligned} \angle AOB &= \arg A - \arg B \\ &= \arg z^* - \arg iz \\ &= \tan^{-1} \frac{\sqrt{6}}{\sqrt{2}} - \tan^{-1} \frac{\sqrt{2}}{\sqrt{6}} \\ &= \tan^{-1} \sqrt{3} - \tan^{-1} \frac{1}{\sqrt{3}} \\ &= \frac{\pi}{3} - \frac{\pi}{6} \\ &= \frac{\pi}{6} \end{aligned}$$

- 5 (i) $u = 2 - i$, $u^* = 2 + i$ is also a root

$$\begin{aligned}x^3 + ax^2 - 3x + b &= 0 \\(x - (2 - i))(x - (2 + i))(x - k) &= 0 \\(x - 2 + i)(x - 2 - i)(x - k) &= 0 \\(x^2 - 2x - ix - 2x + 4 + 2i + ix - 2i - i^2)(x - k) &= 0 \\(x^2 - 4x + 5)(x - k) &= 0\end{aligned}$$

Looking at the x term,

$$4k + 5 = -3 \Rightarrow k = -2$$

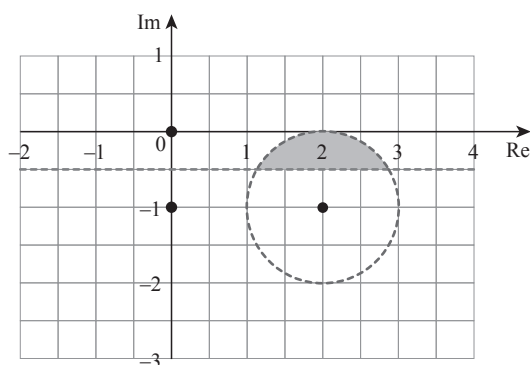
$$\text{So } (x^2 - 4x + 5)(x + 2) = 0$$

$$a = -2, b = 10$$

- (ii) $|z - u| < 1 \Rightarrow |z - (2 - i)| < 1$
 $\Rightarrow$ inside of a circle, centre $2 - i$, radius 1

$$|z| < |z + i| \Rightarrow |z| < |z - (-i)|$$

$\Rightarrow$ all points closer to $(0, 0)$ than $-i$



Stretch and challenge (Page 90)

1 $e^{x+iy} = 3$

$$e^x \times e^{iy} = 3$$

$$e^x (\cos y + i \sin y) = 3$$

$$e^x \cos y + i e^x \sin y = 3$$

$$e^x \cos y = 3 \quad \text{and} \quad e^x \sin y = 0 \Rightarrow y = k\pi, k \in \mathbb{Z}$$

Solving $e^x \cos y = 3$,

- If $y = \dots - 3\pi, -\pi, \pi, 3\pi, 5\pi \dots$ then $\cos y = -1$

$$\Rightarrow -e^x = 3 \text{ which has no solutions}$$

- If $y = -4\pi, -2\pi, 0, 2\pi, 4\pi, 6\pi \dots$ then $\cos y = 1$

$$\Rightarrow e^x = 3 \Rightarrow x = \ln 3$$

2 (i) $z^n + \frac{1}{z^n} = z^n + z^{-n}$

$$\begin{aligned}&= (\cos \theta + i \sin \theta)^n + (\cos \theta + i \sin \theta)^{-n} \\&= \cos n\theta + i \sin n\theta + \cos(-n\theta) + i \sin(-n\theta) \\&= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta \\&= 2 \cos n\theta\end{aligned}$$

$$\begin{aligned}z^n - \frac{1}{z^n} &= z^n - z^{-n} \\&= \cos n\theta + i \sin n\theta - (\cos n\theta - i \sin n\theta) \\&= 2i \sin n\theta\end{aligned}$$

(ii) $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \frac{1}{z^4}$

$$\begin{aligned}&= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} \\&= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \\(2\cos\theta)^4 &= 2\cos 4\theta + 4(2\cos 2\theta) + 6 \\16\cos^4\theta &= 2\cos 4\theta + 8\cos 2\theta + 6 \\ \cos^4\theta &= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} \\&= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)\end{aligned}$$

(iii)(a) $\frac{z - \frac{1}{z}}{i\left(z + \frac{1}{z}\right)} = \frac{2i \sin \theta}{i(2 \cos \theta)}$

$$\begin{aligned}&= \frac{2 \sin \theta}{2 \cos \theta} \\&= \tan \theta\end{aligned}$$

(b) $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \left(\frac{z - z^{-1}}{i(z + z^{-1})}\right)^2}{1 + \left(\frac{z - z^{-1}}{i(z + z^{-1})}\right)^2}$

$$\begin{aligned}&= \frac{1 - \left(\frac{z^2 + \frac{1}{z^2} - 2}{i^2(z^2 + \frac{1}{z^2} + 2)}\right)}{1 + \left(\frac{z^2 + \frac{1}{z^2} - 2}{i^2(z^2 + \frac{1}{z^2} + 2)}\right)} \\&= \frac{1 + \left(\frac{2\cos 2\theta - 2}{2\cos 2\theta + 2}\right)}{1 - \left(\frac{2\cos 2\theta - 2}{2\cos 2\theta + 2}\right)} \\&= \frac{\frac{2\cos 2\theta + 2 + 2\cos 2\theta - 2}{2\cos 2\theta + 2}}{\frac{2\cos 2\theta + 2 - (2\cos 2\theta - 2)}{2\cos 2\theta + 2}} \\&= \frac{4\cos 2\theta}{4} \\&= \cos 2\theta\end{aligned}$$

3 (i) $w = \sqrt{3} - i = 2 \left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right]$

$$\begin{aligned}w^{12} &= \left(2 \left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right] \right)^{12} \\&= 2^{12} \left[\cos\left(12 \times -\frac{\pi}{6}\right) + i \sin\left(12 \times -\frac{\pi}{6}\right) \right] \\&= 4096 [\cos(-2\pi) + i \sin(-2\pi)] \\&= 4096\end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \frac{1}{32}i &= \frac{1}{32}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\
 z^5 &= \frac{1}{32}i = \frac{1}{32}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\
 z &= \left(\frac{1}{32}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right)^{\frac{1}{5}} \\
 &= \left(\frac{1}{32}\right)^{\frac{1}{5}}\left[\cos\left(\frac{1}{5}\times\frac{\pi}{2}\right) + i\sin\left(\frac{1}{5}\times\frac{\pi}{2}\right)\right] \\
 &= \frac{1}{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)
 \end{aligned}$$

The other four solutions are $\frac{2}{5}\pi$ apart

$$\begin{aligned}
 \Rightarrow \frac{1}{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right), \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right), \\
 \frac{1}{2}\left(\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}\right), \\
 \frac{1}{2}\left(\cos\left(-\frac{3\pi}{10}\right) + i\sin\left(-\frac{3\pi}{10}\right)\right), \frac{1}{2}\left(\cos\left(-\frac{7\pi}{10}\right) + i\sin\left(-\frac{7\pi}{10}\right)\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii) (a)} \quad (\cos\theta + i\sin\theta)^5 \\
 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 \\
 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 \\
 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
 - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta
 \end{aligned}$$

$$\text{(b)} (\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

$$\begin{aligned}
 \Rightarrow \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
 &= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta \\
 &= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
 &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
 &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta
 \end{aligned}$$

$$\text{(c)} 16x^5 - 20x^3 + 5x - 1 = 0$$

$$\text{Let } x = \sin\theta$$

$$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 1$$

$$\sin 5\theta = 1$$

$$5\theta = \sin^{-1}(1)$$

$$5\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, -\frac{7\pi}{2}, -\frac{3\pi}{2}$$

$$\theta = \frac{\pi}{10}, \frac{5\pi}{10}, \frac{9\pi}{10}, \frac{-7\pi}{10}, \frac{-3\pi}{10}$$

$$\Rightarrow x = \sin\frac{\pi}{10}, \sin\frac{\pi}{2}, \sin\frac{9\pi}{10}, \sin\left(\frac{-7\pi}{10}\right), \sin\left(\frac{-3\pi}{10}\right)$$

$$4 \quad \text{(i)} \quad \Gamma\left(\frac{1}{2}\right)\Gamma\left(1 - \frac{1}{2}\right) = \frac{\pi}{\sin\frac{1}{2}\pi}$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

(ii) The equation has real coefficients, so $\bar{c} = \sqrt{2} - \sqrt{3}i$ is also a root. Since all the terms are even powers of x , so are

$$-c = -\sqrt{2} - \sqrt{3}i \text{ and } -\bar{c} = -\sqrt{2} + \sqrt{3}i$$

These roots give factors of the polynomial

$$(x - \sqrt{2} - \sqrt{3}i)(x - \sqrt{2} + \sqrt{3}i)(x + \sqrt{2} + \sqrt{3}i)$$

$$(x + \sqrt{2} - \sqrt{3}i)$$

$$= (x^2 - 2\sqrt{2}x + 5)(x^2 + 2\sqrt{2}x + 5)$$

$$= x^4 + 2x^2 + 25$$

By observation, we find that

$$(x^4 + 2x^2 + 25)(x^2 - k) = 0$$

is a factorisation of the original equation.

The roots are $\bar{c}, -c, -\bar{c}, \sqrt{k}$ and $-\sqrt{k}$.

It is possible to find the last two roots by inspection, by substituting $x^2 = k$.

(iii) Some quick geometry gives that the distance between two roots is $\sqrt{2}$ and the centre is at

$$C = a + bi = \left(1 + \frac{1}{\sqrt{2}}\right) + \left(1 + \frac{1}{\sqrt{2}}\right)i$$

Shifting the points to be centred on the origin, the new points are the roots of $x^8 + r^8 = 0$.

$$\begin{aligned}
 r &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1 + \frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + 1 + \sqrt{2} + \frac{1}{2}} \\
 &= \sqrt{2 + \sqrt{2}}
 \end{aligned}$$

From the diagram we find

$$x^8 + (2 + \sqrt{2})^4 = 0$$

Now shift the centre from the origin to

$$C = \left(1 + \frac{1}{\sqrt{2}}\right) + \left(1 + \frac{1}{\sqrt{2}}\right)i$$

$$p(z) = (z - (a + bi))^8 + r^8$$

$$= \left(z - \left(1 + \frac{1}{\sqrt{2}}\right) - \left(1 + \frac{1}{\sqrt{2}}\right)i\right)^8 + (2 + \sqrt{2})^4$$

In the original question, as well as the obvious $n = 8$, we have

$$a + bi = \left(1 + \frac{1}{\sqrt{2}}\right) + \left(1 + \frac{1}{\sqrt{2}}\right)i = (1 + i)\left(1 + \frac{1}{\sqrt{2}}\right)$$

and

$$q = (2 + \sqrt{2})^4 = 68 + 48\sqrt{2}$$

5 $z^5 - 1 = 0$,

$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

Roots of $z^5 - 1 = 0$ are

$$z = \left[\cos\left(\frac{2k\pi}{5}\right) + i \sin\left(\frac{2k\pi}{5}\right) \right], k = 0, 1, 2, 3, 4$$

and $k = 0$ gives $z = 1$, so

$$z^4 + z^3 + z^2 + z + 1$$

$$\begin{aligned} &= \left(z - \left[\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \right] \right) \left(z - \left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right] \right) \left(z - \left[\cos\left(\frac{6\pi}{5}\right) + i \sin\left(\frac{6\pi}{5}\right) \right] \right) \left(z - \left[\cos\left(\frac{8\pi}{5}\right) + i \sin\left(\frac{8\pi}{5}\right) \right] \right) \\ &= \left(z - \left[\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \right] \right) \left(z - \left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right] \right) \left(z - \left[\cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \right] \right) \left(z - \left[\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) \right] \right) \\ &\quad \left(z^2 - \left[\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \right] + \left[\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) \right] \right) z + \left[\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \right] \left[\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) \right] \\ &= \left(z^2 - \left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right] + \left[\cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \right] \right) z + \left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right] \left[\cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \right] \end{aligned}$$

But

$$\begin{aligned} &\left[\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \right] + \left[\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) \right] = 2\cos\left(\frac{2\pi}{5}\right) \text{ and} \\ &\left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right] + \left[\cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \right] = 2\cos\left(\frac{4\pi}{5}\right) \end{aligned}$$

Also

$$\left[\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \right] \left[\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) \right] = \cos(0) = 1$$

$$\text{and } \left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right] \left[\cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \right] = 1$$

So

$$z^4 + z^3 + z^2 + z + 1 =$$

$$\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1 \right) \left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1 \right)$$

Comparing coefficients of z^2 ,

$$1 = 1 + 1 + 4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) \text{ and}$$

$$\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = -\frac{1}{4}$$

6 (ii) $z^2 = \frac{1}{2}(a + \sqrt{a^2 + b^2}) - \frac{1}{2}(-a + \sqrt{a^2 + b^2})$
 $+ 2i\sqrt{\frac{1}{4}(a^2 + b^2 - a^2)}$

$$z^2 = \frac{a}{2} + \frac{a}{2} + 2i\frac{|b|}{2} = a + i|b|$$

(ii) (a) For $c = 1 + \frac{i}{2}$,

$$\begin{aligned} f^2(0) &= c^2 + c = \left(1 + \frac{i}{2}\right)^2 + \left(1 + \frac{i}{2}\right) \\ &= \frac{7}{4} + \frac{3i}{2} \end{aligned}$$

$$|f^2(0)| = \frac{1}{4}(\sqrt{49 + 36}) = \frac{\sqrt{85}}{4} > 2,$$

so $z = 1 + \frac{i}{2}$ is not part of the set.

For $c = i$,

$$f^2(0) = c^2 + c = (i)^2 + i = -1 + i$$

and $|f^2(0)| = \sqrt{2} < 2$, so OK so far.

$$f^3(0) = (-1 + i)^2 + i = -2i + i = -i$$

and $|f^3(0)| = 1 < 2$, so OK so far.

$$f^4(0) = (-i)^2 + i = -1 + i$$

and so the process will loop continually, with

$|f^n(0)| < 2$ for all n , and $z = i$ is part of the set.

(b) $f(0) = c$, $f^2(0) = c^2 + c = a + ib + c$

$$c = \sqrt{\frac{1}{2}(a + \sqrt{4a^2})} + i\sqrt{\frac{1}{2}(-a + \sqrt{4a^2})}$$

$$= \sqrt{\frac{1}{2}(3a)} + i\sqrt{\frac{1}{2}(a)} = \sqrt{\frac{a}{2}}(\sqrt{3} + i)$$

$$f^2(0) = a + ib + \sqrt{\frac{a}{2}}(\sqrt{3} + i)$$

$$= a + \sqrt{\frac{3a}{2}} + i\left(a\sqrt{3} + \sqrt{\frac{a}{2}}\right)$$

and when $a = \frac{1}{8}$, $f^2(0) = \frac{1}{8} + \sqrt{\frac{3}{16}} + i\left(\frac{\sqrt{3}}{8} + \frac{1}{4}\right)$

$$= \frac{1 + 2\sqrt{3}}{8} + i\left(\frac{\sqrt{3} + 2}{8}\right)$$

$$\begin{aligned}\text{So } |f^2(0)| &= \frac{1}{8}\sqrt{(1 + 2\sqrt{3})^2 + (\sqrt{3} + 2)^2} \\ &= \frac{1}{8}\sqrt{(13 + 4\sqrt{3}) + (7 + 4\sqrt{3})} \\ &= \frac{1}{8}\sqrt{20 + 8\sqrt{3}} \\ &= \frac{1}{4}\sqrt{5 + 2\sqrt{3}}\end{aligned}$$