

Answers

1 Algebra

1.1 Operations with polynomials (Page 2)

1	Expression	Polynomial Yes / No?	Order
	$2x - x^2$	Yes	2
	$\frac{x}{2} - \frac{2}{x}$	No	
	0	Yes	0
	$x^{23} + 2x^{15} + 1$	Yes	23
	$x^3 - 2x^2 + \sqrt{x}$	No	
	$x^2 + \sqrt{2}$	Yes	2
	$5 + x + \pi x^{45}$	Yes	45
	$1 - 3x$	Yes	1

2 $x^3 - 4 \equiv (x - 1)(Ax^2 + Bx + C) + D$

$A = 1$

$B - A = 0 \Rightarrow B = 1$

$C - B = 0 \Rightarrow C = 1$

$D - C = -4 \Rightarrow D = -3$

3 (i) $4x^3 + 2x^2 + 5x$

(ii) $3x^6 - 2x^4 + x^3 - 15x^3 + 10x - 5$
 $= 3x^6 - 2x^4 - 14x^3 + 10x - 5$

4 (i)
$$\begin{array}{r} x+1 \\ x+2 \overline{) x^2+3x-1} \\ \underline{-(x^2+2x)} \\ x-1 \\ \underline{-(x+2)} \\ -3 \end{array}$$

Quotient $x + 1$, remainder -3

(ii)
$$\begin{array}{r} x^3 - x^2 + 8x - 4 \\ 2x+1 \overline{) 2x^4 - x^3 + 15x^2 + 0x + 12} \\ \underline{-(2x^4 + x^3)} \\ -2x^3 + 15x^2 \\ \underline{-(-2x^3 - x^2)} \\ 16x^2 + 0x \\ \underline{-(16x^2 + 8x)} \\ -8x + 12 \\ \underline{-(-8x - 4)} \\ 16 \end{array}$$

Quotient $x^3 - x^2 + 8x - 4$, remainder 16

(iii)
$$\begin{array}{r} 3 \\ x^2+1 \overline{) 3x^2+5x-1} \\ \underline{-(3x^2+0x+3)} \\ 5x-4 \end{array}$$

Quotient 3, remainder $5x - 4$

(iv)
$$\begin{array}{r} x^2+2x-1 \\ x^2-x \overline{) x^4+x^3-3x^2+2x-4} \\ \underline{-(x^4-x^3)} \\ 2x^3-3x^2 \\ \underline{-(2x^3-2x^2)} \\ -x^2+2x \\ \underline{-(x^2+x)} \\ x-4 \end{array}$$

Quotient $x^2 + 2x - 1$, remainder $x - 4$

5
$$\begin{array}{r} x \\ x^2-b \overline{) x^3+5x+a} \\ \underline{-(x^3-bx)} \\ (5+b)x+a \\ 5+b=0 \Rightarrow b=-5 \\ a=2 \end{array}$$

1.2 Solution of polynomial equations (Page 4)

1 (i) (a) 10 (b) -10
(c) $\frac{81}{8}$ (d) 0

(ii) $f(1) = 0 \Rightarrow (x - 1)$ is a factor of $f(x)$.

Dividing to find the other factors:

$$\begin{array}{r} x^2-2x-8 \\ x-1 \overline{) x^3-3x^2-6x+8} \\ \underline{x^3-x^2} \\ -2x^2-6x \\ \underline{-2x^2-2x} \\ -8x-8 \\ \underline{-8x+8} \\ 0 \end{array}$$

$$\begin{aligned} f(x) &= (x - 1)(x^2 - 2x - 8) \\ &= (x - 1)(x - 4)(x + 2) \end{aligned}$$

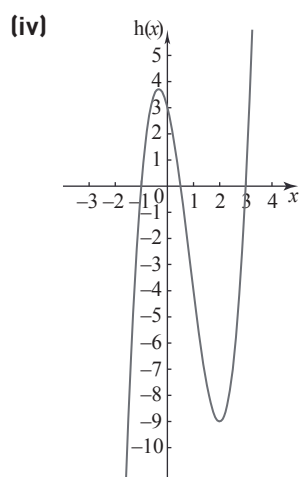
(iii) $x = -2$ or 1 or 4

2 (i) $h(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$
 $= -2 - 5 + 4 + 3$
 $= 0$

$$\begin{array}{r}
 \text{(ii)} \quad \frac{2x^2 - 7x + 3}{x+1} \overline{) 2x^3 - 5x^2 - 4x + 3} \\
 \underline{2x^3 + 2x^2} \\
 -7x^2 - 4x \\
 \underline{-7x^2 - 7x} \\
 3x + 3 \\
 \underline{3x + 3} \\
 0
 \end{array}$$

$$\begin{aligned}
 h(x) &= (x+1)(2x^2 - 7x + 3) \\
 &= (x+1)(2x-1)(x-3)
 \end{aligned}$$

$$\text{(iii)} \quad h(x) = 0 \text{ when } x = -1 \text{ or } \frac{1}{2} \text{ or } 3$$



$$\begin{aligned}
 \text{3} \quad f(x) &= x^3 + 4x^2 - 11x + k \\
 f(-5) &= 0 \Rightarrow (-5)^3 + 4(-5)^2 - 11(-5) + k = 0 \\
 &\quad -125 + 100 + 55 + k = 0 \\
 &\quad 30 + k = 0 \\
 &\quad k = -30
 \end{aligned}$$

$$\begin{aligned}
 \text{4} \quad f(x) &= x^4 - 6x^3 + 7x^2 + px + q \\
 f(-1) &= 0 \Rightarrow \\
 (-1)^4 - 6(-1)^3 + 7(-1)^2 + p(-1) + q &= 0 \\
 1 + 6 + 7 - p + q &= 0 \\
 p - q &= 14
 \end{aligned}$$

$$\begin{aligned}
 f(-2) &= 72 \Rightarrow \\
 (-2)^4 - 6(-2)^3 + 7(-2)^2 + p(-2) + q &= 72 \\
 16 + 48 + 28 - 2p + q &= 72 \\
 2p - q &= 20
 \end{aligned}$$

Solving simultaneously, $p = 6$, $q = -8$

$$\begin{aligned}
 \text{5} \quad h(-1) &= (-1)^4 - 8(-1)^3 + k(-1)^2 - 6(-1) - 9 \\
 &= k + 6 \\
 g(-1) &= k(-1)^3 + 3(-1)^2 + 7(-1) + 13 \\
 &= -k + 9
 \end{aligned}$$

$$\begin{aligned}
 k + 6 &= -k + 9 \\
 2k &= 3 \\
 k &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{6} \quad p(x) &= x^3 + ax^2 + 3x \\
 p(2) &= 26 \Rightarrow 2^3 + a \times 2^2 + 3 \times 2 = 26 \\
 8 + 4a + 6 &= 26 \\
 4a + 14 &= 26 \\
 a &= 3
 \end{aligned}$$

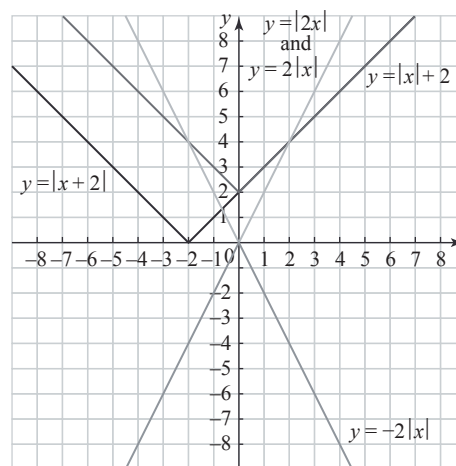
$$\text{So } p(x) = x^3 + 3x^2 + 3x$$

$$p(-4) = (-4)^3 + 3(-4)^2 + 3(-4) = -28$$

Remainder is -28

1.3 The modulus function (Page 6)

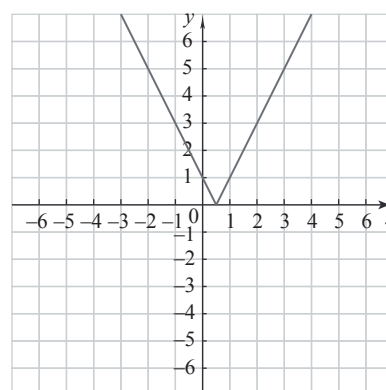
1 (i)

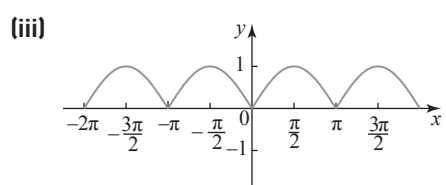
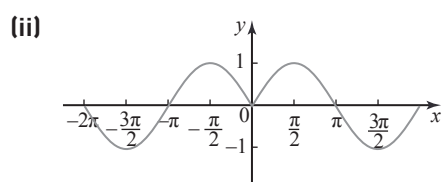
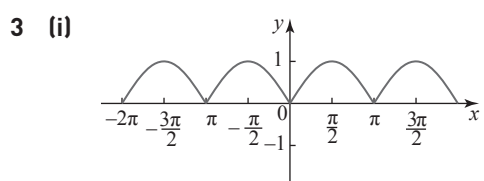
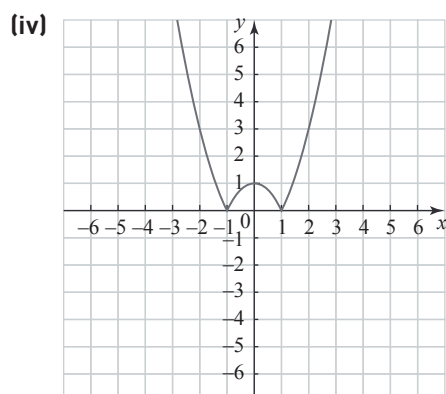
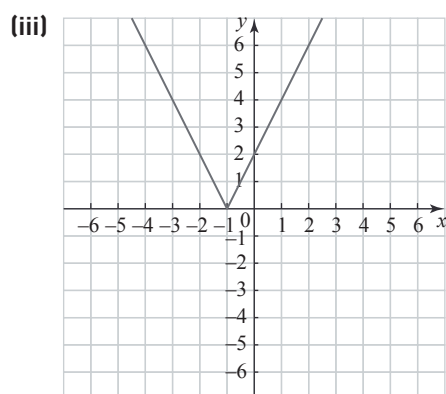
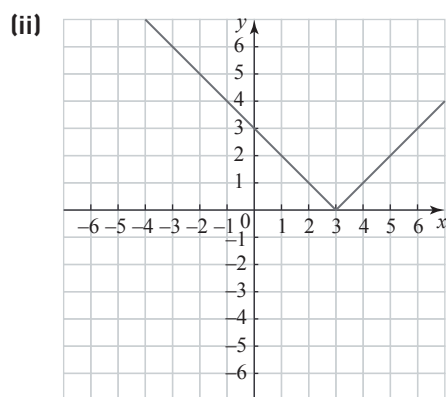


(ii)

Equation	Description
$y = x + a $	Shift a units to the left or right
$y = x + b$	Shift b units up or down
$y = cx $	Makes the line steeper or shallower
$y = d x $	Makes the line steeper or shallower
$y = -e x $	Makes the line steeper or shallower and turns it upside-down

2 (i)

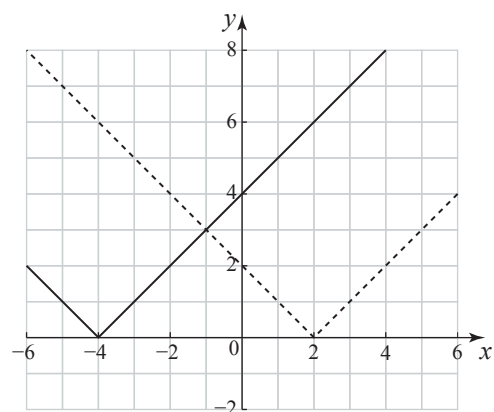




4 (i) $|x - 1| = 2 \Rightarrow x - 1 = 2$ or $x - 1 = -2$
 $x = 3$ or $x = -1$

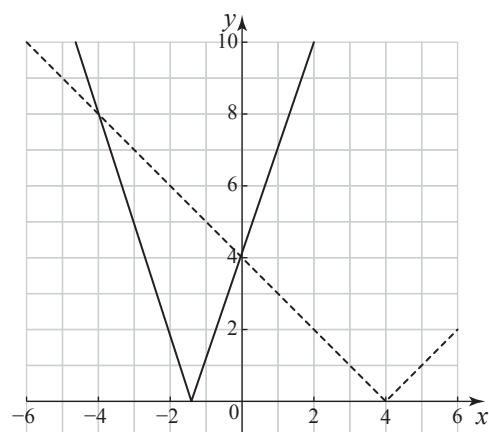
(ii) $|2x + 3| = 7 \Rightarrow 2x + 3 = 7$ or $2x + 3 = -7$
 $x = 2$ or $x = -5$

(iii) $|x + 4| = |x - 2|$



$x = -1$

(iv) $|3x + 4| = |4 - x|$

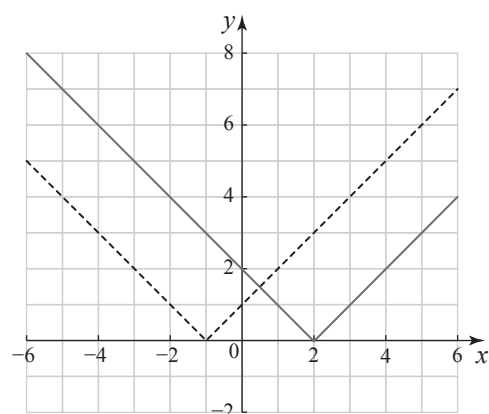


$3x + 4 = -4 + x \Rightarrow x = -4$

$-3x - 4 = -4 + x \Rightarrow x = 0$

5 (i) $|x + 2| < 5 \Rightarrow -5 < x + 2 < 5 \Rightarrow -7 < x < 3$

(ii) $|x + 1| \leq |x - 2|$



Solving, $x + 1 = -(x - 2)$

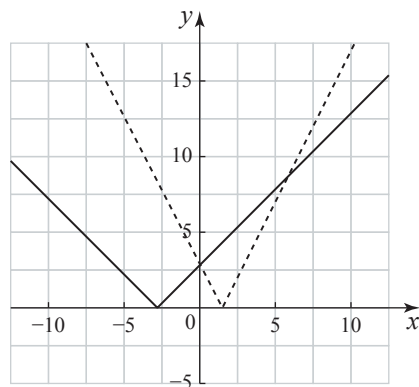
$x + 1 = -x + 2$

$2x = 1$

$x = \frac{1}{2}$

From the graph, $x \leq \frac{1}{2}$

(iii) $|2x - 3| > |x + 3|$



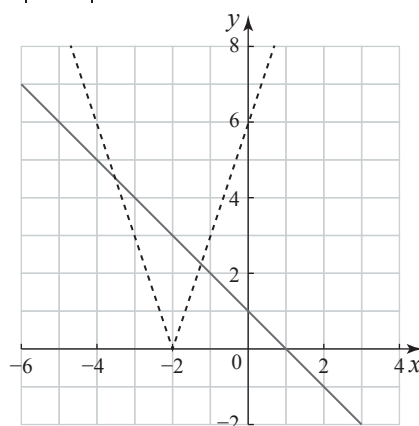
Solving, $-(2x - 3) = x + 3 \Rightarrow$

$$-2x + 3 = x + 3 \Rightarrow x = 0$$

$$2x - 3 = x + 3 \Rightarrow x = 6$$

From the graph, $x < 0$ or $x > 6$

(iv) $3|x + 2| < 1 - x$



Solving, $-3(x + 2) = 1 - x$

$$-3x - 6 = 1 - x$$

$$-2x = 7$$

$$x = -\frac{7}{2}$$

$$3(x + 2) = 1 - x$$

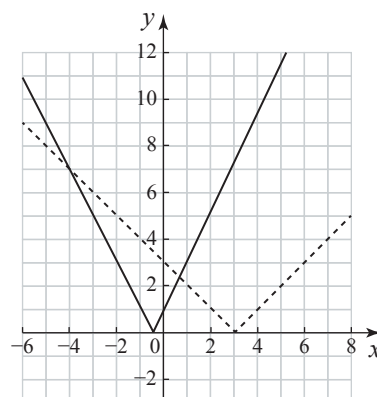
$$3x + 6 = 1 - x$$

$$4x = -5$$

$$x = -\frac{5}{4}$$

From the graph, $-\frac{7}{2} < x < -\frac{5}{4}$

6 (i) $|2x + 1| \leq |x - 3|$



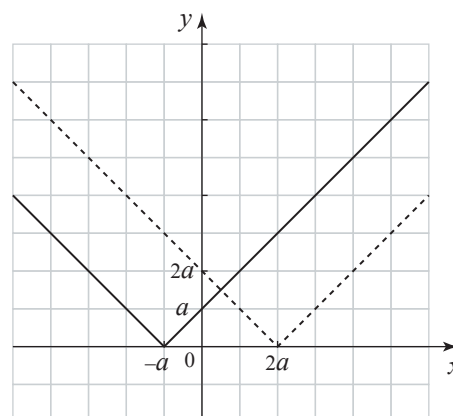
Solving, $-2x - 1 = -x + 3 \Rightarrow x = -4$

$$2x + 1 = -x + 3 \Rightarrow x = \frac{2}{3}$$

From the graph, $-4 \leq x \leq \frac{2}{3}$

(ii) Greatest value is when $x = \frac{2}{3}$,
$$\left| \frac{2}{3} + 2 \right| = 2\frac{2}{3}$$

7 $|x - 2a| < |x + a|$



Solving, $-(x - 2a) = x + a$

$$-x + 2a = x + a$$

$$a = 2x$$

$$x = \frac{1}{2}a$$

From the graph, $x > \frac{1}{2}a$ **Further practice (Page 10)**

1 $2x^3 - 3x^2 + x - 2 \equiv (x + 2)(Ax^2 + Bx + C) + D$

$$A = 2$$

$$2A + B = -3 \Rightarrow B = -7$$

$$2B + C = 1 \Rightarrow C = 15$$

$$2C + D = -2 \Rightarrow D = -32$$

2 (i) $x^4 - x^3 - x^2 + 10x + 1$

(ii) $(x^2 - 1)^2 - (x^2 - 2)^2$

$$= ((x^2 - 1) - (x^2 - 2))((x^2 - 1) + (x^2 - 2))$$

$$= 1 \times (2x^2 - 3)$$

$$= 2x^2 - 3$$

$$\begin{array}{r}
 \text{3 (i)} \quad \frac{x^2 + x - 3}{x - 1} \overline{) x^3 + 0x^2 - 4x + 3} \\
 \underline{-(x^3 - x^2)} \\
 x^2 - 4x \\
 \underline{-(x^2 - x)} \\
 -3x + 3 \\
 \underline{-(-3x + 3)} \\
 0
 \end{array}$$

Quotient $x^2 + x - 3$, remainder 0

$$\begin{array}{r}
 \text{(ii)} \quad \frac{4x + 1}{x^2 - 2x + 1} \overline{) 4x^3 - 7x^2 + 8x + 14} \\
 \underline{-(4x^3 - 8x^2 + 4x)} \\
 x^2 + 4x + 14 \\
 \underline{-(x^2 - 2x + 1)} \\
 6x + 13
 \end{array}$$

Quotient $4x + 1$, remainder $6x + 13$

$$\begin{aligned}
 \text{4 } f(x) &= x^3 - 3x^2 + px \\
 f(4) &= 0 \Rightarrow 4^3 - 3(4)^2 + p(4) = 0 \\
 64 - 48 + 4p &= 0 \\
 16 + 4p &= 0 \\
 p &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{5 } f(x) &= 2x^4 - 5x^3 + mx^2 - x + 4 \\
 f\left(\frac{1}{2}\right) &= 0 \Rightarrow \\
 2\left(\frac{1}{2}\right)^4 - 5\left(\frac{1}{2}\right)^3 + m\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 4 &= 0 \\
 \frac{1}{8} - \frac{5}{8} + \frac{m}{4} - \frac{1}{2} + 4 &= 0 \\
 \frac{m}{4} + 3 &= 0 \\
 m &= -12
 \end{aligned}$$

$$\begin{array}{r}
 \frac{x^3 - 2x^2 - 7x - 4}{2x - 1} \overline{) 2x^4 - 5x^3 - 12x^2 - x + 4} \\
 \underline{(2x^4 - x^3)} \\
 -4x^3 - 12x^2 \\
 \underline{-(-4x^3 + 2x^2)} \\
 -14x^2 - x \\
 \underline{-(-14x^2 + 7x)} \\
 -8x + 4 \\
 \underline{-(-8x + 4)} \\
 0
 \end{array}$$

So $f(x) = (2x - 1)(x^3 - 2x^2 - 7x - 4)$

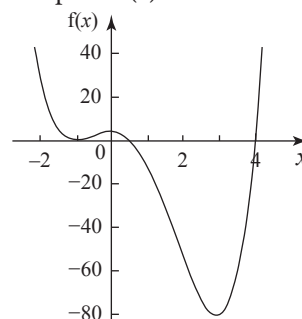
$$\text{Let } g(x) = x^3 - 2x^2 - 7x - 4$$

$$g(-1) = 0 \Rightarrow x + 1 \text{ is a factor}$$

$$\begin{array}{r}
 \frac{x^2 - 3x - 4}{x + 1} \overline{) x^3 - 2x^2 - 7x - 4} \\
 \underline{(x^3 + x^2)} \\
 -3x^2 - 7x \\
 \underline{-(-3x^2 - 3x)} \\
 -4x - 4 \\
 \underline{-(-4x - 4)} \\
 0
 \end{array}$$

$$\begin{aligned}
 f(x) &= (2x - 1)(x^3 - 2x^2 - 7x - 4) \\
 &= (2x - 1)(x + 1)(x^2 - 3x - 4) \\
 &= (2x - 1)(x + 1)(x - 4)(x + 1) \\
 &= (2x - 1)(x + 1)^2(x - 4)
 \end{aligned}$$

Graph of $f(x)$:



$$\begin{aligned}
 \text{6 } f(x) &= x^4 + px^3 + 9x^2 + qx - 12 \\
 f(2) &= 0 \Rightarrow \\
 (2)^4 + p(2)^3 + 9(2)^2 + q(2) - 12 &= 0 \\
 16 + 8p + 36 + 2q - 12 &= 0 \\
 8p + 2q &= -40 \\
 4p + q &= -20
 \end{aligned}$$

$$\begin{aligned}
 f(-1) &= 0 \Rightarrow \\
 (-1)^4 + p(-1)^3 + 9(-1)^2 + q(-1) - 12 &= 0 \\
 1 - p + 9 - q - 12 &= 0 \\
 -p - q &= 2
 \end{aligned}$$

Solving simultaneously, $p = -6$, $q = 4$

$$\begin{aligned}
 \text{7 (i)} \quad p(x) &= 2x^3 + 6x^2 + ax + b \\
 p(-3) &= -25 \Rightarrow \\
 2(-3)^3 + 6(-3)^2 + a(-3) + b &= -25 \\
 -54 + 54 - 3a + b &= -25 \\
 3a - b &= 25 \\
 p(2) &= 55 \Rightarrow \\
 2(2)^3 + 6(2)^2 + a(2) + b &= 55 \\
 16 + 24 + 2a + b &= 55 \\
 2a + b &= 15
 \end{aligned}$$

Solving simultaneously, $a = 8$, $b = -1$

$$\begin{array}{r}
 \text{(ii)} \quad \frac{2x+6}{x^2+1} \overline{) 2x^3+6x^2+8x-1} \\
 \underline{-(2x^3+0x^2+2x)} \\
 6x^2+6x-1 \\
 \underline{-(6x^2+0x+6)} \\
 6x-7
 \end{array}$$

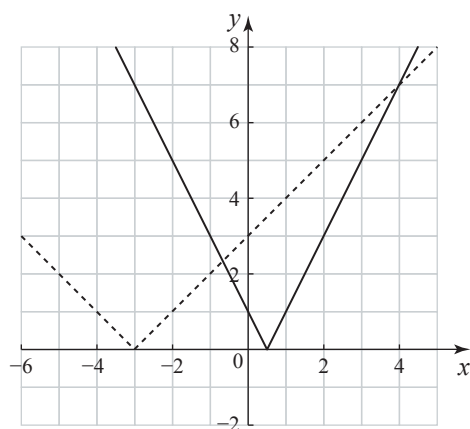
Quotient $2x+6$, remainder $6x-7$

$$\begin{array}{r}
 \text{8} \quad \frac{x^2-4x+2}{x^2+2x-1} \overline{) x^4-2x^3-7x^2+7x+a} \\
 \underline{-(x^4+2x^3-x^2)} \\
 -4x^3-6x^2+7x \\
 \underline{-(-4x^3-8x^2+4x)} \\
 2x^2+3x+a \\
 \underline{-(2x^2+4x-2)} \\
 -x+(a+2)
 \end{array}$$

$$a+2=7 \Rightarrow a=5, b=-4, c=-1$$

$$\begin{aligned}
 \text{9 (i)} \quad 7-|3+4x| &= 2 \\
 5 &= |3+4x| \\
 3+4x &= 5 \text{ or } 3+4x = -5 \\
 x &= \frac{1}{2} \text{ or } -2
 \end{aligned}$$

$$\text{(ii)} \quad |2x-1| = |x+3|$$

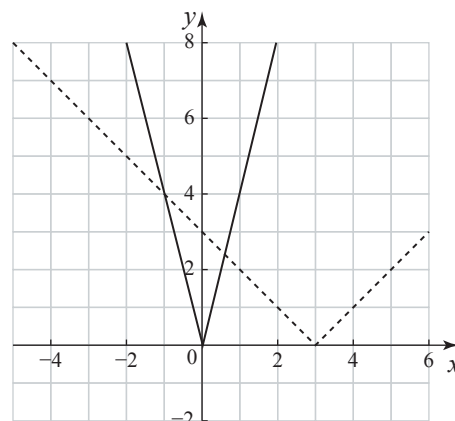


$$2x-1 = x+3 \Rightarrow x = 4$$

$$-2x+1 = x+3 \Rightarrow -3x = 2 \Rightarrow x = -\frac{2}{3}$$

$$\begin{aligned}
 \text{10 (i)} \quad |3x-1| &\geq 8 \Rightarrow 3x-1 \geq 8 \text{ or } 3x-1 \leq -8 \\
 x &\geq 3 \text{ or } x \leq -\frac{7}{3}
 \end{aligned}$$

$$\text{(ii)} \quad |4x| \leq |3-x|$$



Solving,

$$-4x = 3-x \Rightarrow -3x = 3 \Rightarrow x = -1$$

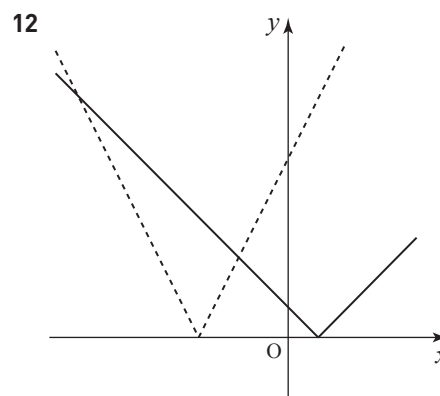
$$4x = 3-x \Rightarrow 5x = 3 \Rightarrow x = \frac{3}{5}$$

From the graph, $-1 \leq x \leq \frac{3}{5}$

$$\text{11} \quad |3x+4a| = 5a$$

$$3x+4a = 5a \Rightarrow x = \frac{a}{3}$$

$$3x+4a = -5a \Rightarrow x = -3a$$



Solving, $a-x = -2(x+3a)$

$$a-x = -2x-6a$$

$$x = -7a$$

Also $a-x = 2(x+3a)$

$$a-x = 2x+6a$$

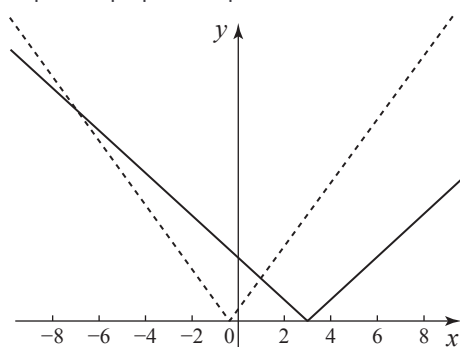
$$-5a = 3x$$

$$x = -\frac{5}{3}a$$

From the graph, $x < -7a$ or $x > -\frac{5}{3}a$

Past exam questions (Page 10)

1 $2|x-3| > |3x+1|$



Finding intersection points,

$$\begin{aligned} -3x-1 &= -2(x-3) & 3x+1 &= -2(x-3) \\ -3x-1 &= -2x+6 & 3x+1 &= -2x+6 \\ -x &= 7 & 5x &= 5 \\ x &= -7 & x &= 1 \end{aligned}$$

From the graph, $-7 < x < 1$

OR

$$\begin{aligned} (2(x-3))^2 &> (3x+1)^2 \\ 4(x^2-6x+9) &> 9x^2+6x+1 \\ 4x^2-24x+36 &> 9x^2+6x+1 \\ 0 &> 5x^2+30x-35 \\ 0 &> x^2+6x-7 \\ 0 &> (x-1)(x+7) \\ -7 &< x < 1 \end{aligned}$$

2 $p(x) = x^3 + 4x^2 + ax + 2$

(i) $p(-1) = p(2)$

$$\begin{aligned} (-1)^3 + 4(-1)^2 + a(-1) + 2 &= 2^3 + 4(2)^2 + a(2) + 2 \\ 5 - a &= 26 + 2a \\ -21 &= 3a \\ a &= -7 \end{aligned}$$

(ii) $p(1) = 1^3 + 4 \times 1^2 - 7 \times 1 + 2 = 0$

By long division or synthetic division,

1	1	4	-7	2
		1	5	-2
	1	5	-2	0

Quotient is $x^2 + 5x - 2$

3 $p(x) = ax^3 - 3x^2 - 11x + b$

(i) $p(-2) = 0 \Rightarrow$

$$\begin{aligned} a(-2)^3 - 3(-2)^2 - 11(-2) + b &= 0 \\ -8a - 12 + 22 + b &= 0 \\ -8a + b &= -10 \quad (1) \end{aligned}$$

$$\begin{aligned} p(-1) &= 12 \Rightarrow \\ a(-1)^3 - 3(-1)^2 - 11(-1) + b &= 12 \\ -a - 3 + 11 + b &= 12 \\ -a + b &= 4 \quad (2) \end{aligned}$$

Solving simultaneously, $a = 2, b = 6$

(ii) $p(x) = 2x^3 - 3x^2 - 11x + 6$

-2	2	-3	-11	6
		-4	14	-6
	2	-7	3	0

$$\begin{aligned} p(x) &= (x+2)(2x^2-7x+3) \\ &= (x+2)(2x-1)(x-3) \end{aligned}$$

4 (i) $(x+2)$ is a factor $\Rightarrow p(-2) = 0$

$$\begin{aligned} 4(-2)^3 + a(-2)^2 - (a+1)(-2) - 18 &= 0 \\ -32 + 4a + 2a + 2 - 18 &= 0 \\ 6a - 48 &= 0 \\ a &= 8 \end{aligned}$$

(ii) Factorise $p(x) = 4x^3 + 8x^2 - 9x - 18$

-2	4	8	-9	-18
		-8	0	18
	4	0	-9	0

$$\begin{aligned} p(x) &= (x+2)(4x^2-9) \\ &= (x+2)(2x-3)(2x+3) \end{aligned}$$

$$\begin{array}{r} 2x-3 \\ x^2-2x+5 \overline{) 2x^3-7x^2-9x+3} \\ \underline{-(2x^3-4x^2+10x)} \\ 3x^2-19x+3 \\ \underline{-(-3x^2+6x-15)} \\ -25x+18 \end{array}$$

Quotient is $2x - 3$, remainder $-25x + 18$

(ii) Subtracting the remainder from the original expression,

$$\begin{aligned} (2x^3 - 7x^2 - 9x + 3) - (-25x + 18) \\ = 2x^3 - 7x^2 + 16x - 15 \\ \text{So } p = 16, q = -15 \end{aligned}$$

6 $|x+a| = |2x-5a|$

Solving, $x+a = -2x+5a$

$3x = 4a$

$x = \frac{4a}{3}$

$x+a = 2x-5a$

$x = 6a$

Stretch and challenge (Page 11)

$$\begin{array}{r}
 1 \quad \frac{x^2 + (a-5)x + (-5a+37)}{x+5} \overline{) x^3 + ax^2 + 12x + b} \\
 \underline{-(x^3 + 5x^2)} \\
 (a-5)x^2 + 12x \\
 \underline{-((a-5)x^2 + (5a-25)x)} \\
 (-5a+37)x + b \\
 \underline{-((-5a+37)x - 25a + 185)} \\
 b + 25a - 185
 \end{array}$$

$$a - 5 = 10 \Rightarrow a = 15$$

$$-5 \times 15 + 37 = -38 \Rightarrow c = -38$$

$$b + 25a - 185 = 150$$

$$b + 25 \times 15 - 185 = 150$$

$$b = -40$$

- 2 One solution is $a = 3, b = 4, c = 1, d = 4$

$$|3x - 4| \leq |x - 4|$$

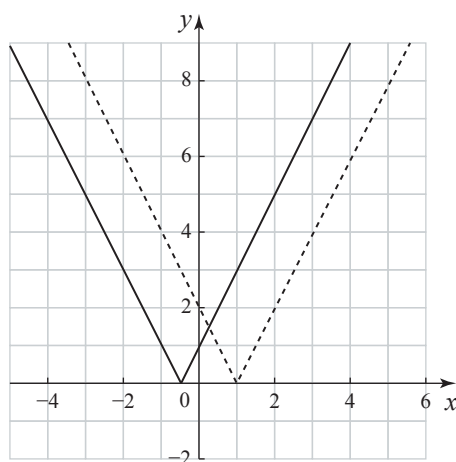
$$\text{Also } a = 5, b = 4, c = -1, d = 4$$

$$\text{and } a = -8, b = -6, c = 2, d = -6$$

3 $\left| \frac{2x+1}{x-1} \right| \leq 2$

$$\frac{|2x+1|}{|x-1|} \leq 2$$

$$|2x+1| \leq 2|x-1|$$



$$\text{Solving, } 2x + 1 = -2(x - 1)$$

$$2x + 1 = -2x + 2$$

$$4x = 1$$

$$x = \frac{1}{4}$$

$$\text{From the graph, } x \leq \frac{1}{4}$$

$$\text{OR squaring both sides, } \left(\frac{2x+1}{x-1} \right)^2 \leq 2^2$$

$$(2x+1)^2 \leq 4(x-1)^2$$

$$4x^2 + 4x + 1 \leq 4(x^2 - 2x + 1)$$

$$4x^2 + 4x + 1 \leq 4x^2 - 8x + 4$$

$$12x \leq 3$$

$$x \leq \frac{1}{4}$$

$$\begin{array}{l}
 4 \text{ (i)} \quad \frac{ax^2 + (b+ar)x + (c+br+ar^2)}{x-r} \overline{) ax^3 + bx^2 + cx + d} \\
 \underline{-(ax^3 - arx^2)} \\
 (b+ar)x^2 + cx \\
 \underline{-((b+ar)x^2 - r(b+ar)x)} \\
 (c+br+ar^2)x + d \\
 \underline{-((c+br+ar^2)x - r(c+br+ar^2))} \\
 d + rc + br^2 + ar^3
 \end{array}$$

$$\begin{aligned}
 \text{(ii)} \quad x &= \frac{-(b+ar) \pm \sqrt{(b+ar)^2 - 4 \times a \times (c+br+ar^2)}}{2a} \\
 &= \frac{-(b+ar) \pm \sqrt{b^2 + 2abr + a^2r^2 - 4ac - 4abr - 4a^2r^2}}{2a} \\
 &= \frac{-(b+ar) \pm \sqrt{b^2 - 2abr - 3a^2r^2 - 4ac}}{2a}
 \end{aligned}$$

5 (i) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+ac+bc)$

$$5^2 = 9 + 2(ab+ac+bc)$$

$$ab+ac+bc = \frac{25-9}{2} = 8$$

$$\text{Also, } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2$$

$$\text{Multiplying by } abc,$$

$$bc+ac+ab = 2abc$$

$$8 = 2abc$$

$$abc = 4$$

- (ii) If a, b and c are the roots then

$$(x-a)(x-b)(x-c) = 0$$

$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = 0$$

Hence, by comparing coefficients,

$$a+b+c = 5, ab+ac+bc = 8 \text{ and } abc = 4$$

(as above)

- (iii) $a = 1, b = 2, c = 2$

- 6 Let the polynomial be $f(x)$. When $f(x)$ is divided by $(x-1)(x-2)(x-4)$ we get a polynomial, $Q(x)$, with degree 42 and a remainder, which must be a quadratic as the divisor is a cubic polynomial.

$$\frac{f(x)}{(x-1)(x-2)(x-4)} = Q(x) + \frac{ax^2 + bx + c}{(x-1)(x-2)(x-4)}$$

$$f(x) = Q(x)(x-1)(x-2)(x-4) + ax^2 + bx + c$$

$$f(1) = a + b + c = 4 \quad \text{①}$$

$$f(2) = 4a + 2b + c = 7 \quad \text{②}$$

$$f(4) = 16a + 4b + c = 25 \quad \text{③}$$

$$\text{Subtracting ① from ②, } 3a + b = 3$$

$$\text{Subtracting ② from ③, } 12a + 2b = 18$$

Solving simultaneously,

$$a = 2, b = -3, c = 5$$

The remainder is $2x^2 - 3x + 5$.

2 Logarithms and exponentials

2.1 Exponential functions and logarithms (Page 12)

- 1 (i) $\log_2 16 = 4$
 (ii) $\log_3 27 = 3$
 (iii) $\log_4 \frac{1}{16} = -2$
- 2 (i) $2^{-1} = \frac{1}{2}$
 (ii) $3^2 = 9$
 (iii) $4^{\frac{1}{2}} = 2$
- 3 (i) $\log_2 8 = 3$
 (ii) Let $\log_6 \frac{1}{216} = x \Rightarrow \frac{1}{216} = \frac{1}{6^3} = 6^{-3}$ so $x = -3$
 (iii) Let $\log_{\frac{1}{3}} 27 = x \Rightarrow \left(\frac{1}{3}\right)^x = 3^3 \Rightarrow \left(\frac{1}{3}\right)^x = \left(\frac{1}{3}\right)^{-3} \Rightarrow x = -3$
 (iv) Let $\log_{16} 4 = x \Rightarrow 16^x = 4 \Rightarrow (4^2)^x = 4^1 \Rightarrow 4^{2x} = 4^1$
 $2x = 1 \Rightarrow x = \frac{1}{2}$
 (v) $\log_5 5\sqrt{5} = \log_5 5^{\frac{3}{2}} = \frac{3}{2}$
 (vi) $\log_{30} 30 = 1$
 (vii) $\log_{\sqrt{2}} 4 = x \Rightarrow (\sqrt{2})^x = 4 \Rightarrow \left(2^{\frac{1}{2}}\right)^x = 2^2$
 $2^{\frac{1}{2}x} = 2^2 \Rightarrow \frac{1}{2}x = 2 \Rightarrow x = 4$
 (viii) $\log_8 \sqrt{2} = x \Rightarrow 8^x = \sqrt{2} \Rightarrow (2^3)^x = 2^{\frac{1}{2}}$
 $2^{3x} = 2^{\frac{1}{2}} \Rightarrow 3x = \frac{1}{2} \Rightarrow x = \frac{1}{6}$
 (ix) $\frac{\log 9}{\log 27} = \frac{\log 3^2}{\log 3^3} = \frac{2\log 3}{3\log 3} = \frac{2}{3}$
- 4 (i) $\log_2 a = 5 \Rightarrow 2^5 = a \Rightarrow a = 32$
 (ii) $\log_c 27 = 3 \Rightarrow c^3 = 27 \Rightarrow c = 3$
 (iii) $\log_5 \frac{1}{25} = e \Rightarrow 5^e = \frac{1}{25} \Rightarrow e = -2$
- 5 (i) 1
 (ii) 0
 (iii) $-\frac{1}{2}$
- 6 (i) $\log 20$
 (ii) $\log 7$
 (iii) $\frac{1}{2}\log 36 = \log 36^{\frac{1}{2}} = \log 6$
 (iv) $2\log 3 + 3\log 2 = \log 3^2 + \log 2^3$
 $= \log 9 + \log 8$
 $= \log 72$
 (v) $\frac{1}{2}\log 100 - 2\log 5 = \log 100^{\frac{1}{2}} - \log 5^2$
 $= \log \sqrt{100} - \log 25$
 $= \log \frac{10}{25} = \log \frac{2}{5}$
- (vi) $\log 8 - \log 2 + \log 5 = \log\left(\frac{8}{2}\right) + \log 5$
 $= \log 4 + \log 5 = \log 20$
- 7 (i) $\log 6 = \log(2 \times 3) = \log 2 + \log 3 = a + b$
 (ii) $\log \frac{9}{4} = \log \frac{3^2}{2^2} = \log 3^2 - \log 2^2$
 $= 2\log 3 - 2\log 2$
 $= 2b - 2a$
 (iii) $\log 0.2 = \log \frac{2}{9} = \log 2 - \log 9$
 $= \log 2 - 2\log 3$
 $= a - 2b$
- 8 (i) $\log_{10} A = 2\log_{10} b + 1$
 $\log_{10} A - 2\log_{10} b = 1$
 $\log_{10} A - \log_{10} b^2 = 1$
 $\log_{10} \left(\frac{A}{b^2}\right) = 1$
 $\frac{A}{b^2} = 10$ or $A = 10b^2$
 (ii) $2\log_5 D = \log_5 (E - 1) + \log_5 3$
 $\log_5 D^2 = \log_5 3(E - 1)$
 $D^2 = 3E - 3$
- 9 (i) $2^x = 12$
 $\log 2^x = \log 12$
 $x \log 2 = \log 12$
 $x = \frac{\log 12}{\log 2} = 3.58$
 (ii) $6^{2x} = 4^{x-1}$
 $\log 6^{2x} = \log 4^{x-1}$
 $2x \log 6 = (x - 1) \log 4$
 $2x \log 6 = x \log 4 - \log 4$
 $2x \log 6 - x \log 4 = -\log 4$
 $x(2\log 6 - \log 4) = -\log 4$
 $x = \frac{-\log 4}{2\log 6 - \log 4} = -0.631$
 (iii) $2^{x-1} 3^x = 16$
 $\log(2^{x-1} 3^x) = \log 16$
 $\log(2^{x-1}) + \log(3^x) = \log 16$
 $(x - 1) \log 2 + x \log 3 = \log 16$
 $x \log 2 - \log 2 + x \log 3 = \log 16$
 $x(\log 2 + \log 3) = \log 16 + \log 2$
 $x = \frac{\log 16 + \log 2}{\log 2 + \log 3} = 1.93(3 \text{ s.f.})$
 (iv) $\log_{10} x + 3 = \log_{10} (x + 3)$
 $\log_{10} x - \log_{10} (x + 3) = -3$
 $\log_{10} \frac{x}{x + 3} = -3$
 $\frac{x}{x + 3} = 10^{-3} = \frac{1}{1000}$

$$\begin{aligned} 1000x &= x + 3 \\ 999x &= 3 \\ x &= \frac{3}{999} \end{aligned}$$

OR

$$\begin{aligned} 10^{\log_{10} x + 3} &= 10^{\log_{10}(x+3)} \\ 10^{\log_{10} x} \times 10^3 &= x + 3 \\ x \times 1000 &= x + 3 \\ x &= \frac{3}{999} \end{aligned}$$

(v) $\log_2(x-1) = \log_2 x - 1$

$$\begin{aligned} 2^{\log_2(x-1)} &= 2^{\log_2 x - 1} \\ x - 1 &= 2^{\log_2 x} \times 2^{-1} \\ x - 1 &= x \times \frac{1}{2} \\ 2x - 2 &= x \\ x &= 2 \end{aligned}$$

(vi) $7^{x+2} = 7^x + 7^2$

$$\begin{aligned} 7^{x+2} - 7^x &= 7^2 \\ 7^{x+2}(1 - 7^{-2}) &= 7^2 \\ 7^{x+2} &= \frac{7^2}{1 - 7^{-2}} \\ 7^{x+2} &= \frac{2401}{48} \\ (x+2)\log 7 &= \log \frac{2401}{48} \\ x &= \frac{\log \frac{2401}{48}}{\log 7} - 2 = 0.0106 \end{aligned}$$

10 (i) $2^x < 5$

$$\log 2^x < \log 5$$

$$x \log 2 < \log 5$$

$$x < \frac{\log 5}{\log 2}$$

$$x < 2.32$$

(ii) $3 \times 2^{3x-1} < 5$

$$2^{3x-1} < \frac{5}{3}$$

$$(3x-1)\log 2 < \log\left(\frac{5}{3}\right)$$

$$3x-1 < \frac{\log\left(\frac{5}{3}\right)}{\log 2}$$

$$x < \frac{1}{3} \left(\frac{\log\left(\frac{5}{3}\right)}{\log 2} + 1 \right)$$

$$x < 0.579$$

11 (i) $5 = a(2^0) + b \Rightarrow 5 = a + b$

$$14 = a(2^2) + b \Rightarrow 14 = 4a + b$$

$$3a = 9 \Rightarrow a = 3, b = 2$$

(ii) Since graph is shifted 2 to the right, $a = -2$

$$y = \log_{10}(x-2) + b$$

$$2 = \log_{10}(12-2) + b$$

$$b = 1$$

12 (i) $8.3 = \log_{10} \frac{I_M}{S} \Rightarrow 8.3 = \log_{10} I_M - \log_{10} S$

$$7.1 = \log_{10} \frac{I_C}{S} \Rightarrow 7.1 = \log_{10} I_C - \log_{10} S$$

$$1.2 = \log_{10} I_M - \log_{10} I_C$$

$$1.2 = \log_{10} \frac{I_M}{I_C}$$

$$\frac{I_M}{I_C} = 10^{1.2} = 15.8 \approx 16$$

(ii) $8.3 = \log_{10} \frac{I_M}{S} \Rightarrow 8.3 = \log_{10} I_M - \log_{10} S$

$$x = \log_{10} \frac{\frac{1}{2} I_M}{S} \Rightarrow x = \log_{10} \frac{1}{2} I_M - \log_{10} S$$

$$8.3 - x = \log_{10} I_M - \log_{10} \frac{1}{2} I_M$$

$$8.3 - x = \log_{10} \frac{I_M}{\frac{1}{2} I_M}$$

$$8.3 - x = \log_{10} 2$$

$$x = 8.3 - \log_{10} 2 = 8$$

(iii) $7.1 = \log_{10} \frac{I_C}{S} \Rightarrow 7.1 = \log_{10} I_C - \log_{10} S$

$$x = \log_{10} \frac{2I_C}{S} \Rightarrow x = \log_{10} 2I_C - \log_{10} S$$

$$x - 7.1 = \log_{10} 2I_C - \log_{10} I_C$$

$$x - 7.1 = \log_{10} \frac{2I_C}{I_C}$$

$$x - 7.1 = \log_{10} 2$$

$$x = 7.1 + \log_{10} 2 = 7.4$$

13 $xy^2 = 80 \Rightarrow \log(xy^2) = \log 80$

$$\Rightarrow \log_{10} x + 2\log_{10} y = \log_{10} 80$$

Solving simultaneously,

$$2\log_{10} x + \log_{10} y = 2$$

$$\log_{10} x + 2\log_{10} y = \log_{10} 80$$

Multiply first equation by 2:

$$4\log_{10} x + 2\log_{10} y = 4$$

Subtract:

$$3\log_{10} x = 4 - \log_{10} 80$$

$$\log_{10} x = \frac{4 - \log_{10} 80}{3}$$

$$x = 10^{\frac{4 - \log_{10} 80}{3}}$$

$$x = 5$$

$$xy^2 = 80 \Rightarrow 5y^2 = 80$$

$$y^2 = 16$$

$$y = 4$$

2.2 Modelling curves (Page 16)

1 (i) $y = kx^p$

$$\log y = \log(kx^p)$$

$$\log y = \log k + \log x^p$$

$$\log y = p \log x + \log k$$

(ii) Gradient is p , y -intercept is $\log k$.

2 (i)

m	W	$\log_{10} m$	$\log_{10} W$
1	8.00	0	0.903
2	5.66	0.301	0.753
5	3.58	0.699	0.554
10	2.53	1	0.403

(ii) $\log W = b \log m + \log A$

Gradient is b , y -intercept is $\log A$.

$$b = -\frac{0.903 - 0.403}{1 - 0} = -0.5$$

$$\log A = 0.903 \Rightarrow A = 10^{0.903} = 8$$

(iii) (a) $W = 8m^{-0.5} = 8 \times (12)^{-0.5} = 2.31 \text{ kg}$

(b) $8m^{-0.5} < 1$

$$m^{-0.5} < \frac{1}{8}$$

$$\frac{1}{\sqrt{m}} < \frac{1}{8}$$

$$8 < \sqrt{m}$$

$$m > 64$$

3 (i)

t	L	$\log_{10} L$
0	50	1.70
5	57	1.76
10	66	1.82
15	77	1.89
20	87	1.94
36	128	2.11

(ii) $L = Ab^t$

$$\log L = \log(Ab^t)$$

$$\log L = \log A + \log b^t$$

$$\log L = t \log b + \log A$$

$$\log L = (\log b)t + \log A$$

$$y = mx + c$$

(iii) Gradient of the line = $\log b$

$$\log b = \frac{2.11 - 1.70}{36} = 0.0113...$$

$$b = 10^{0.0113...} = 1.026$$

$$\log A = 1.70 \Rightarrow A = 10^{1.70} = 50$$

$$L = 50(1.026)^t$$

(iv) 18 years = 216 months

$$L = 50(1.026)^{216} = 12\,788 \text{ cm}$$

which is about 128 m.

The answer is clearly unreasonable because the model does not account for the fact that the rate of growth changes as we age. This model may only be useful from birth to 36 months.

4

$$y = kb^{-x}$$

$$\log y = \log(kb^{-x})$$

$$\log y = \log k + \log b^{-x}$$

$$\log y = \log k - x \log b$$

$$\log y = (-\log b)x + \log k$$

$$\text{Gradient} = -\log b = -\frac{3.4 - 0.7}{4.2 - 1.3} = -0.931...$$

$$\Rightarrow \log b = 0.931... \Rightarrow b = 8.53$$

$$\log y = (-\log b)x + \log k$$

$$3.4 = (-0.931) \times 1.3 + \log k$$

$$\Rightarrow \log k = 4.61... \Rightarrow k = 40\,800 \text{ (3.s.f.)}$$

2.3 The natural logarithm and exponential functions (Page 19)

1 (i)

$$e^{2x+1} = 4$$

$$\ln(e^{2x+1}) = \ln 4$$

$$2x + 1 = \ln 4$$

$$x = \frac{\ln 4 - 1}{2}$$

(ii)

$$\ln x - \ln 4 = \ln(x - 4)$$

$$\ln x - \ln(x - 4) = \ln 4$$

$$\ln \frac{x}{x - 4} = \ln 4$$

$$\frac{x}{x - 4} = 4$$

$$x = 4(x - 4)$$

$$x = 4x - 16$$

$$16 = 3x$$

$$x = \frac{16}{3}$$

(iii)

$$e^{2x} + e^x = 30$$

$$(e^x)^2 + e^x - 30 = 0$$

$$\text{Let } y = e^x \Rightarrow$$

$$y^2 + y - 30 = 0$$

$$(y + 6)(y - 5) = 0$$

$$y = -6 \text{ or } 5$$

$$e^x = -6 \text{ (no solutions)}$$

$$e^x = 5 \Rightarrow x = \ln 5$$

(iv) $\ln x = \ln(x+1) + 1$

$$\ln x - \ln(x+1) = 1$$

$$\ln \frac{x}{x+1} = 1$$

$$\frac{x}{x+1} = e$$

$$x = ex + e$$

$$x - ex = e$$

$$x(1 - e) = e$$

$$x = \frac{e}{1 - e}$$

2 $\ln A = 3 \ln B + 2 \ln 3$

$$\ln A - 3 \ln B = 2 \ln 3$$

$$\ln A - \ln B^3 = \ln 3^2$$

$$\ln \frac{A}{B^3} = \ln 9$$

$$\frac{A}{B^3} = 9 \Rightarrow A = 9B^3$$

3 (i) $\ln(e^{x+y})^2 = \ln(e^{2(x+y)}) = 2(x+y) = 2x + 2y$

(ii) $e^{2 \ln x + 3 \ln y} = e^{\ln x^2 + \ln y^3} = e^{\ln x^2} \times e^{\ln y^3} = x^2 y^3$

4 $y = Ax^b$

$$\ln y = b \ln x + \ln A$$

The gradient is b and the y -intercept is $\ln A$.

$$b = \frac{12.3 - 0.5}{8.1 - 1.2} = 1.71 \text{ (3 s.f.)}$$

$$\ln y = b \ln x + \ln A$$

$$0.5 = 1.71 \dots \times 1.2 + \ln A$$

$$\ln A = -1.552 \dots \Rightarrow A = 0.212 \text{ (3 s.f.)}$$

$$y = 0.212x^{1.71}$$

5 (i) $N_0 = 1000$

(ii) $N = 1000e^{0.4 \times 5} = 7389 = 7390 \text{ (3 s.f.)}$

(iii) $2000 = 1000e^{0.4t} \Rightarrow e^{0.4t} = 2$

$$0.4t = \ln 2 \Rightarrow t = \frac{\ln 2}{0.4} = 1.73 \text{ hours}$$

$$= 1 \text{ hr } 44 \text{ mins}$$

(iv) $10^6 = 1000e^{0.4t} \Rightarrow e^{0.4t} = 1000$

$$0.4t = \ln 1000$$

$$\Rightarrow t = \frac{\ln 1000}{0.4} = 17.3 \text{ hours} = 17 \text{ hr } 16 \text{ mins}$$

6 $e^{x-1} = e^{-x}$

$$e^{x-1} = \frac{1}{e^x}$$

$$e^x \times e^{x-1} = 1$$

$$e^{2x-1} = 1$$

$$2x - 1 = 0$$

$$x = \frac{1}{2}$$

7 (i) $p = e^{280}$ and $q = e^{300}$

$$p = e^{280} \Rightarrow \ln p = 280$$

$$q = e^{300} \Rightarrow \ln q = 300$$

$$\ln\left(\frac{ep^2}{q}\right) = \ln e + 2 \ln p - \ln q$$

$$= 1 + 2 \times 280 - 300$$

$$= 261$$

(ii) $5^n > pq$

$$n \ln 5 > \ln(pq)$$

$$n \ln 5 > \ln p + \ln q$$

$$n \ln 5 > 280 + 300$$

$$n > \frac{580}{\ln 5}$$

$$n > 360.37 \dots$$

Smallest integer is 361

Further practice (Page 21)

1 (i) 2

(ii) $\log_{10} 0.1 = \log_{10} \frac{1}{10} = -1$

(iii) $\log_7 1 = 0$

(iv) $\frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2} = \frac{3 \log 2}{\log 2} = 3$

(v) $\log_9 \sqrt{3} = x \Rightarrow 9^x = \sqrt{3} \Rightarrow (3^2)^x = 3^{\frac{1}{2}}$
 $3^{2x} = 3^{\frac{1}{2}} \Rightarrow 2x = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

(vi) $\frac{\log 0.2}{\log 25} = \frac{\log 5^{-1}}{\log 5^2} = \frac{-\log 5}{2 \log 5} = -\frac{1}{2}$

2 (i) $\log_3 b = -2 \Rightarrow 3^{-2} = b \Rightarrow b = \frac{1}{9}$

(ii) $\log_d 8 = \frac{1}{2} \Rightarrow d^{\frac{1}{2}} = 8 \Rightarrow d = 64$

(iii) $\log_{36} 6 = f \Rightarrow 36^f = 6 \Rightarrow f = \frac{1}{2}$

3 (i) 2 (ii) -1 (iii) $\frac{1}{3}$

4 (i) $\log 144$

(ii) $3 \log 4 = \log 4^3 = \log 64$

(iii) $2 \log 4 - \log 32 = \log 4^2 - \log 32$
 $= \log 16 - \log 32$

$$= \log \frac{16}{32}$$

$$= \log \frac{1}{2}$$

(iv) $\frac{1}{2} \log 9 - \frac{1}{3} \log 8 + \frac{1}{4} \log 256$

$$= \log \sqrt{9} - \log \sqrt[3]{8} + \log \sqrt[4]{256}$$

$$= \log 3 - \log 2 + \log 4$$

$$= \log \frac{3}{2} + \log 4$$

$$= \log 6$$

$$(v) \quad 2\log 5 + \frac{1}{3}\log 64 - \frac{1}{2}\log 121$$

$$= \log 5^2 + \log 64^{\frac{1}{3}} - \log 121^{\frac{1}{2}}$$

$$= \log 25 + \log 4 - \log 11$$

$$= \log 100 - \log 11$$

$$= \log\left(\frac{100}{11}\right)$$

$$\begin{aligned} 5 \quad (i) \quad \log 12 &= \log(4 \times 3) = \log 2^2 + \log 3 \\ &= 2\log 2 + \log 3 \\ &= 2a + b \end{aligned}$$

$$(ii) \quad \log 8 = \log 2^3 = 3\log 2 = 3a$$

$$\begin{aligned} (iii) \quad \log \frac{4}{\sqrt{3}} &= \log \frac{2^2}{3^{\frac{1}{2}}} = \log 2^2 - \log 3^{\frac{1}{2}} \\ &= 2\log 2 - \frac{1}{2}\log 3 \\ &= 2a - \frac{1}{2}b \end{aligned}$$

$$6 \quad (i) \quad \log_{10} M = \log_{10} \left(\frac{P}{Q} \right) - \log_{10} R^2$$

$$\log_{10} M = \log_{10} \left(\frac{P}{Q} \right) - \log_{10} R^2$$

$$\log_{10} M = \log_{10} \left(\frac{P}{QR^2} \right)$$

$$M = \frac{P}{QR^2}$$

$$\begin{aligned} (ii) \quad \log_3 S &= 2 - \log_3 \sqrt{T} - 3\log_3 U \\ \log_3 S &= \log_3 9 - \log_3 \sqrt{T} - \log_3 U^3 \end{aligned}$$

$$\log_3 S = \log_3 \frac{9}{\sqrt{T}} - \log_3 U^3$$

$$\log_3 S = \log_3 \frac{9}{U^3 \sqrt{T}}$$

$$S = \frac{9}{U^3 \sqrt{T}}$$

$$7 \quad (i) \quad 3^{x+1} = 24$$

$$\log 3^{x+1} = \log 24$$

$$(x+1)\log 3 = \log 24$$

$$x+1 = \frac{\log 24}{\log 3}$$

$$x = \frac{\log 24}{\log 3} - 1 = 1.89$$

$$(ii) \quad 3^{x+1} = 4^{2x-1}$$

$$\log 3^{x+1} = \log 4^{2x-1}$$

$$(x+1)\log 3 = (2x-1)\log 4$$

$$x\log 3 + \log 3 = 2x\log 4 - \log 4$$

$$\log 4 + \log 3 = 2x\log 4 - x\log 3$$

$$\log 4 + \log 3 = x(2\log 4 - \log 3)$$

$$x = \frac{\log 4 + \log 3}{2\log 4 - \log 3}$$

$$= \frac{\log 12}{\log\left(\frac{16}{3}\right)} = 1.48 \quad (3 \text{ s.f.})$$

$$(iii) \quad \log_5(x-1) = \log_5 x - 1$$

$$5^{\log_5(x-1)} = 5^{\log_5 x - 1}$$

$$x-1 = 5^{\log_5 x} \times 5^{-1}$$

$$x-1 = x \times \frac{1}{5}$$

$$5x-5 = x$$

$$4x = 5$$

$$x = \frac{5}{4}$$

$$(iv) \quad \log_3(x+4) - \log_3(x-4) = 1$$

$$\log_3\left(\frac{x+4}{x-4}\right) = 1$$

$$\left(\frac{x+4}{x-4}\right) = 3$$

$$x+4 = 3(x-4)$$

$$x+4 = 3x-12$$

$$16 = 2x$$

$$x = 8$$

$$(v) \quad 3 - 4^x = \frac{2}{4^x}$$

$$3(4^x) - (4^x)^2 = 2$$

$$0 = (4^x)^2 - 3(4^x) + 2$$

$$\text{Let } t = 4^x \Rightarrow$$

$$t^2 - 3t + 2 = 0$$

$$(t-1)(t-2) = 0$$

$$t = 1 \text{ or } 2$$

$$4^x = 1 \text{ or } 4^x = 2$$

$$x = 0 \text{ or } \frac{1}{2}$$

$$8 \quad (i) \quad 4^{x+2} - 2 < 18$$

$$4^{x+2} < 20$$

$$\log 4^{x+2} < \log 20$$

$$(x+2)\log 4 < \log 20$$

$$x < \frac{\log 20}{\log 4} - 2$$

$$x < 0.161$$

$$(ii) \quad |3^x - 2| \geq 1 \Rightarrow 3^x - 2 \geq 1 \text{ or } 3^x - 2 \leq -1$$

$$3^x - 2 \geq 1 \Rightarrow 3^x \geq 3 \Rightarrow x \geq 1$$

$$3^x \leq 1 \Rightarrow x \leq 0$$

$$x \leq 0 \text{ and } x \geq 1$$

$$9 \quad 125 = 20 \log_{10} \left(\frac{P_{\text{plane}}}{P_0} \right)$$

$$125 = 20(\log_{10} P_{\text{plane}} - \log_{10} P_0)$$

$$110 = 20 \log_{10} \left(\frac{P_{\text{band}}}{P_0} \right)$$

$$110 = 20(\log_{10} P_{\text{band}} - \log_{10} P_0)$$

$$15 = 20(\log_{10} P_{\text{plane}} - \log_{10} P_{\text{band}})$$

$$\frac{15}{20} = \log_{10} \frac{P_{\text{plane}}}{P_{\text{band}}}$$

$$\frac{P_{\text{plane}}}{P_{\text{band}}} = 10^{\frac{15}{20}} = 5.62$$

$$10 \text{ (i)} \quad \ln 2x = \frac{1}{2} \ln 16 + \frac{2}{3} \ln 8$$

$$\ln 2x = \ln 16^{\frac{1}{2}} + \ln 8^{\frac{2}{3}}$$

$$\ln 2x = \ln 4 + \ln 4$$

$$\ln 2x = \ln 16$$

$$2x = 16$$

$$x = 8$$

$$\text{(ii)} \quad 3e^{-x} + 1 = 2e^x$$

$$\frac{3}{e^x} + 1 = 2e^x$$

$$3 + e^x = 2(e^x)^2$$

$$0 = 2(e^x)^2 - e^x - 3$$

$$\text{Let } y = e^x$$

$$0 = 2y^2 - y - 3$$

$$0 = (2y - 3)(y + 1)$$

$$y = -1 \text{ or } \frac{3}{2}$$

$$e^x = -1 \text{ (no solutions)}$$

$$e^x = \frac{3}{2} \Rightarrow x = \ln \frac{3}{2}$$

$$\text{(iii)} \quad \ln(x+1) + 1 = 3$$

$$\ln(x+1) = 2$$

$$x+1 = e^2$$

$$x = e^2 - 1$$

$$\text{(iv)} \quad e^{3x} - e^{2x} = 2e^x$$

$$(e^x)^3 - (e^x)^2 = 2(e^x)$$

$$e^x((e^x)^2 - e^x - 2) = 0$$

$$\text{Let } y = e^x \Rightarrow$$

$$y(y^2 - y - 2) = 0$$

$$y(y+1)(y-2) = 0$$

$$y = 0 \text{ or } -1 \text{ or } 2$$

$$e^x = 0 \text{ or } -1 \text{ (no solutions)}$$

$$e^x = 2 \Rightarrow x = \ln 2$$

$$11 \text{ (i)} \quad \log_4(x-4) = 2 - \log_4 x$$

$$\log_4(x-4) + \log_4 x = 2$$

$$\log_4 x(x-4) = 2$$

$$x(x-4) = 4^2$$

$$x^2 - 4x - 16 = 0$$

$$\text{(ii)} \quad x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times -16}}{2 \times 1} = -2.47 \text{ or } 6.47$$

Since $x > 0$ ($\ln x$ is only defined for positive values of x), $x = 6.47$

$$12 \quad |8 - 3^x| = 15$$

$$8 - 3^x = 15 \Rightarrow 3^x = -7 \text{ (no solutions)}$$

$$8 - 3^x = -15 \Rightarrow 3^x = 23 \Rightarrow x = \frac{\log 23}{\log 3} = 2.85$$

$$13 \quad y = Ab^x$$

$$\ln y = \ln(Ab^x)$$

$$\ln y = \ln A + x \ln b$$

$$\ln y = (\ln b)x + \ln A$$

$$\ln b = \frac{9.4 - 4.5}{5.9 - 4.1} = 2.72 \Rightarrow b = 15.2$$

$$4.5 = 2.72 \times 4.1 + \ln A$$

$$\ln A = -6.652 \Rightarrow A = 0.00129$$

$$14 \quad 6^x = 6^{x-1} + 6$$

$$6^x - 6^{x-1} = 6$$

$$6^x(1 - 6^{-1}) = 6$$

$$6^x \left(1 - \frac{1}{6}\right) = 6$$

$$6^x \left(\frac{5}{6}\right) = 6$$

$$6^x = \frac{36}{5}$$

$$\log 6^x = \log \left(\frac{36}{5}\right)$$

$$x \log 6 = \log \left(\frac{36}{5}\right)$$

$$x = \frac{\log \left(\frac{36}{5}\right)}{\log 6} = 1.10 \text{ (3 s.f.)}$$

$$15 \quad \ln(3x+2) = 2\ln x + \ln 2$$

$$\ln(3x+2) - 2\ln x = \ln 2$$

$$\ln(3x+2) - \ln x^2 = \ln 2$$

$$\ln\left(\frac{3x+2}{x^2}\right) = \ln 2$$

$$\frac{3x+2}{x^2} = 2$$

$$3x+2 = 2x^2$$

$$0 = 2x^2 - 3x - 2$$

$$0 = (2x+1)(x-2)$$

$$x = -\frac{1}{2} \text{ or } 2$$

But since $\ln x$ is only defined for positive values of x , the only solution is $x = 2$.

$$16 \quad 5^{x-1} = 7^{2x-1}$$

$$\log 5^{x-1} = \log 7^{2x-1}$$

$$(x-1)\log 5 = (2x-1)\log 7$$

$$x\log 5 - \log 5 = 2x\log 7 - \log 7$$

$$\log 7 - \log 5 = 2x\log 7 - x\log 5$$

$$\log 7 - \log 5 = x(2\log 7 - \log 5)$$

$$x = \frac{\log 7 - \log 5}{2\log 7 - \log 5} = \frac{\log\left(\frac{7}{5}\right)}{\log\left(\frac{49}{5}\right)} = 0.147 \quad (3 \text{ s.f.})$$

$$17 \quad 10^{\log_{10} y} = 10^{0.7x+3.5}$$

$$10^{\log_{10} y} = 10^{0.7x} \times 10^{3.5}$$

$$y = (10^{0.7})^x \times 3162.277...$$

$$y = 3160(5.01)^x$$

OR

Starting with $y = A(b)^x$ and applying logarithms,

$$\log y = (\log b)x + \log A$$

$$\log b = 0.7 \Rightarrow b = 5.01$$

$$\log A = 3.5 \Rightarrow A = 3160$$

Past exam questions (Page 23)

$$1 \quad (i) \quad 2^x + 3(2^{-x}) = 4$$

$$2^x + \frac{3}{2^x} = 4$$

$$y + \frac{3}{y} = 4$$

$$y^2 + 3 = 4y$$

$$y^2 - 4y + 3 = 0$$

$$(ii) \quad (y-1)(y-3) = 0$$

$$y = 1 \quad \text{or} \quad y = 3$$

$$2^x = 1 \quad \text{or} \quad 2^x = 3$$

$$x = 0 \quad \text{or} \quad x = \frac{\log 3}{\log 2} = 1.58 \quad (3 \text{ s.f.})$$

$$2 \quad \ln(1+x^2) = 1 + 2\ln x$$

$$e^{\ln(1+x^2)} = e^{1+2\ln x}$$

$$1+x^2 = e^1 \times e^{\ln x^2}$$

$$1+x^2 = ex^2$$

$$1 = (e-1)x^2$$

$$x^2 = \frac{1}{e-1}$$

$$x = \sqrt{\frac{1}{e-1}} = 0.763 \quad (3 \text{ s.f.})$$

(since $\ln x$ is only defined for positive values of x)

$$3 \quad y = A(b^x)$$

$$\ln y = \ln A(b^x)$$

$$\ln y = \ln A + \ln b^x$$

$$\ln y = (\ln b)x + \ln A$$

$$\text{Gradient} = \ln b$$

$$\frac{1.2 - 0.8}{2.2 - 1.4} = \ln b$$

$$\ln b = 0.5$$

$$b = e^{0.5}$$

$$b = 1.65 \quad (2 \text{ d.p.})$$

$$y = A(b^x) \Rightarrow e^{0.8} = A(1.65^{1.4}) \Rightarrow A = 1.11$$

$$\ln y = 0.8 \Rightarrow y = e^{0.8}$$

OR

$$0.8 = \ln A + \ln 1.65^{1.4}$$

$$\ln A = 0.1 \Rightarrow A = e^{0.1} = 1.11 \quad (2 \text{ d.p.})$$

$$4 \quad 3^x = 2^{x+2}$$

$$\ln 3^x = \ln 2^{x+2}$$

$$x \ln 3 = (x+2) \ln 2$$

$$x \ln 3 = x \ln 2 + 2 \ln 2$$

$$x(\ln 3 - \ln 2) = 2 \ln 2$$

$$x = \frac{2 \ln 2}{\ln 3 - \ln 2}$$

$$x = 3.42 \quad (3 \text{ s.f.})$$

5 (i) $2\ln(x-4) - \ln x = \ln 2$

$$\ln(x-4)^2 - \ln x = \ln 2$$

$$\ln\left(\frac{(x-4)^2}{x}\right) = \ln 2$$

$$\frac{(x-4)^2}{x} = 2$$

$$x^2 - 8x + 16 = 2x$$

$$x^2 - 10x + 16 = 0$$

$$(x-2)(x-8) = 0$$

$$x = 2 \text{ or } 8$$

Since $x = 2$ creates a negative logarithm, the only solution is $x = 8$

(ii) $1.4^y > 10^{10}$

$$y \ln 1.4 > 10 \ln 10$$

$$y > \frac{10 \ln 10}{\ln 1.4}$$

$$y > 68.433 \dots$$

Smallest integer is 69

6 $3^x + 3^{2x} = 3^{3x}$

$$u + u^2 = u^3$$

$$u^3 - u^2 - u = 0$$

$$u(u^2 - u - 1) = 0$$

$$u = 0 \text{ or } -0.618 \text{ or } 1.618$$

$$3^x = 0, 3^x = -0.618, 3^x = 1.618$$

No solutions for first two, so

$$3^x = 1.618 \Rightarrow x = \frac{\log 1.618}{\log 3} = 0.438 \text{ (3 s.f.)}$$

7 $5^{3x} = 7^{4y}$

$$3x \log 5 = 4y \log 7$$

$$\frac{x}{y} = \frac{4 \log 7}{3 \log 5} = 1.612 \text{ (4 s.f.)}$$

8 $y = \frac{K}{a^{2x}} = Ka^{-2x}$

$$\ln y = \ln(Ka^{-2x})$$

$$\ln y = \ln K - 2x \ln a$$

$$\ln y = (-2 \ln a)x + \ln K$$

$$\text{Gradient of line} = \frac{1.39 - 1.81}{1.4 - 0.6} = -0.525$$

$$\text{So } -2 \ln a = -0.525$$

$$\ln a = 0.2625$$

$$a = e^{0.2625} = 1.3 \text{ (2 s.f.)}$$

$$1.39 = (-2 \times 0.2625)1.4 + \ln K$$

$$\ln K = 2.125$$

$$K = 8.4 \text{ (2 s.f.)}$$

Stretch and challenge (Page 24)

1 (i) Let $x = \log_4 \sqrt[5]{\frac{32}{1024}} = \log_4 \frac{2}{4} = \log_4 \frac{1}{2}$

$$4^x = \frac{1}{2} \Rightarrow 2^{2x} = 2^{-1} \Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$$

(ii) $\log_2 \left[\frac{\sqrt{256} \left(\frac{1}{8}\right)^6}{32 \left(\frac{1}{2}\right)^3} \right] = \log_2 \left[\frac{\sqrt{2^8} \left(\frac{1}{2^3}\right)^6}{2^5 \left(\frac{1}{2}\right)^3} \right]$

$$= \log_2 \left[\frac{2^4 \left(\frac{1}{2^{18}}\right)}{2^5 \left(\frac{1}{2^3}\right)} \right] = \log_2 \left[\frac{2^{-14}}{2^2} \right]$$

$$= \log_2 [2^{-16}] = -16$$

(iii) $e^{\ln e^{\ln \pi}} = e^{\ln \pi} = \pi$

(iv) $36^{\frac{1}{2} - \log_6 \sqrt{3}} = 36^{\frac{1}{2}} \times 36^{-\log_6 \sqrt{3}}$
 $= \sqrt{36} \times 36^{\log_6 \left(\frac{1}{3^2}\right)^{-1}}$
 $= 6 \times (6)^{\log_6 \left(\frac{1}{3^2}\right)^{-1}}$
 $= 6 \times (6)^{\log_6 \left(\frac{1}{3^2}\right)^{-2}}$
 $= 6 \times (6)^{\log_6 (3^{-1})}$
 $= 6 \times 3^{-1}$
 $= 2$

2 (i) Sometimes – when $a = b$ or when $b = \frac{1}{a}$ (provided both a and b are positive)

(ii) Sometimes – when $a = 1$

(iii) Never

(iv) Sometimes – when $a = 1$

(v) Sometimes – if the logarithms are base n then a solution is $a = n^4, b = n^2$

$$\frac{\log_n n^4}{\log_n n^2} = \frac{4 \log_n n}{2 \log_n n} = 2$$

$$\log_n n^4 - \log_n n^2 = 4 \log_n n - 2 \log_n n = 2$$

(vi) Never

(vii) Sometimes – when $x = 1$

$$3 \quad 3(2^x) - 4^x = 2$$

$$3(2^x) - (2^2)^x = 2$$

$$3(2^x) - (2^x)^2 = 2$$

$$0 = (2^x)^2 - 3(2^x) + 2$$

$$\text{Let } t = 2^x \Rightarrow$$

$$t^2 - 3t + 2 = 0$$

$$(t-1)(t-2) = 0$$

$$t = 1 \text{ or } 2$$

$$2^x = 1 \text{ or } 2^x = 2$$

$$x = 0 \text{ or } 1$$

$$4 \quad \text{(i)} \quad \text{Let } y = \log_a b \Rightarrow a^y = b$$

$$\log_c a^y = \log_c b$$

$$y \log_c a = \log_c b$$

$$y = \frac{\log_c b}{\log_c a}$$

$$\text{(ii)} \quad \log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\text{Hence } \log_a b = \frac{\log_c b}{\log_c a} = \frac{1}{\frac{\log_c a}{\log_c b}} = \frac{1}{\log_b a}$$

$$5 \quad \log_3 x + \log_4 x - 1 = \log_5 x$$

$$\log_3 x + \log_4 x - \log_5 x = 1$$

$$\frac{\log x}{\log 3} + \frac{\log x}{\log 4} - \frac{\log x}{\log 5} = 1$$

$$\log x \left(\frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} \right) = 1$$

$$\log x = \frac{1}{\frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5}}$$

$$\log x = 0.429887...$$

$$x = 2.69 \text{ (3 s.f.)}$$

$$6 \quad 3 \log_x y + 3 \log_y x = 10$$

$$\text{Let } t = \log_x y \Rightarrow \frac{1}{t} = \log_y x$$

$$3t + \frac{3}{t} = 10$$

$$3t^2 + 3 = 10t$$

$$3t^2 - 10t + 3 = 0$$

$$(3t-1)(t-3) = 0$$

$$t = \frac{1}{3} \text{ or } 3$$

$$\log_x y = \frac{1}{3} \Rightarrow y = x^{\frac{1}{3}} \Rightarrow y = \sqrt[3]{x} \Rightarrow x = y^3$$

$$\log_x y = 3 \Rightarrow y = x^3$$

$$7 \quad 2 \log(x-2y) = \log x + \log y$$

$$\log(x-2y)^2 = \log xy$$

$$\log(x^2 - 4xy + 4y^2) = \log xy$$

$$x^2 - 4xy + 4y^2 = xy$$

$$x^2 - 5xy + 4y^2 = 0$$

$$\frac{x^2}{y^2} - \frac{5xy}{y^2} + \frac{4y^2}{y^2} = \frac{0}{y^2}$$

$$\left(\frac{x}{y}\right)^2 - 5\left(\frac{x}{y}\right) + 4 = 0$$

$$\text{Let } t = \frac{x}{y} \Rightarrow$$

$$t^2 - 5t + 4 = 0$$

$$(t-1)(t-4) = 0$$

$$t = 1 \text{ or } 4$$

$$\frac{x}{y} = 1 \text{ or } \frac{x}{y} = 4$$

$$8 \quad \text{(i)} \quad y = mx + c \Rightarrow xy = mx^2 + c$$

$$m = \frac{21-6}{9-4} = 3$$

$$xy = 3x^2 + c \Rightarrow 6 = 3 \times 4 + c \Rightarrow c = -6$$

$$xy = 3x^2 - 6$$

$$\text{When } x = 6, y = \frac{3x^2 - 6}{x} = \frac{3(6)^2 - 6}{6} = 17$$

$$\text{(ii)} \quad \text{When } y = 7, x(7) = 3x^2 - 6$$

$$\Rightarrow 3x^2 - 7x - 6 = 0$$

$$(3x+2)(x-3) = 0$$

$$x = -\frac{2}{3} \text{ or } 3$$

$$9 \quad 3^{\log_{10} x^2} = 2(3^{1+\log_{10} x}) + 27$$

$$3^{2\log_{10} x} = 2(3^1 \times 3^{\log_{10} x}) + 27$$

$$3^{2\log_{10} x} = 6 \times 3^{\log_{10} x} + 27$$

$$\text{Let } y = 3^{\log_{10} x} \Rightarrow$$

$$y^2 = 6y + 27$$

$$y^2 - 6y - 27 = 0$$

$$(y+3)(y-9) = 0$$

$$y = -3 \text{ or } y = 9$$

$$3^{\log_{10} x} = -3 \text{ (no solutions)}$$

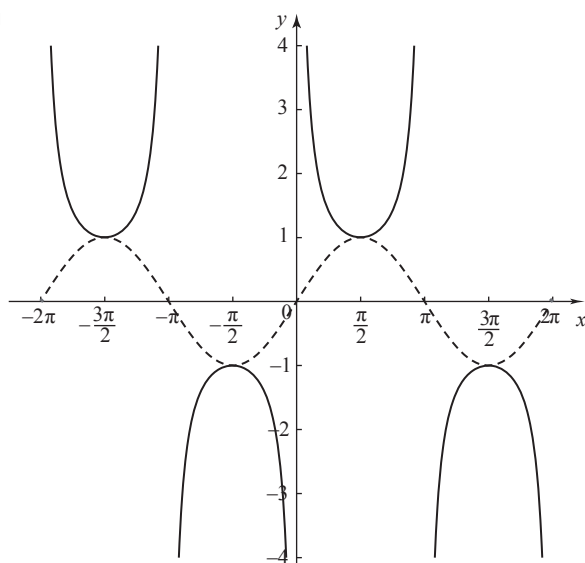
$$3^{\log_{10} x} = 9$$

$$\log_{10} x = 2 \Rightarrow x = 100$$

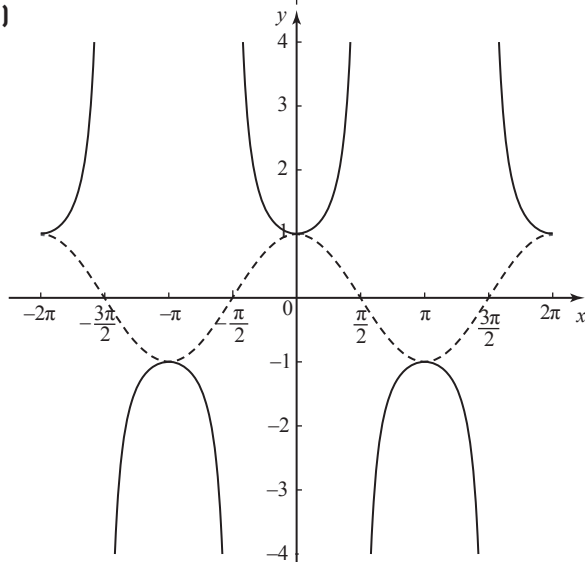
3 Trigonometry

3.1 Reciprocal trigonometrical functions (Page 25)

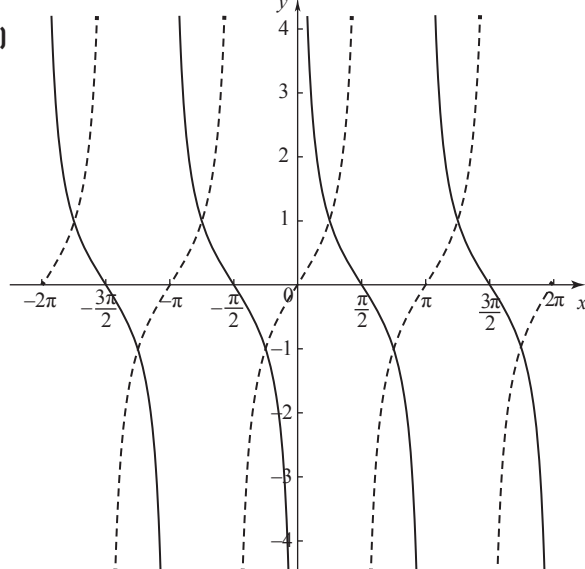
1 (i)



(ii)



(iii)



$$2 \text{ (i)} \quad \operatorname{cosec} 150^\circ = \frac{1}{\sin 150^\circ} = \frac{1}{\frac{1}{2}} = 2$$

$$\text{(ii)} \quad \sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\text{(iii)} \quad \cot 300^\circ = \frac{\cos 300^\circ}{\sin 300^\circ} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

OR

$$\cot 300^\circ = \frac{1}{\tan 300^\circ} = -\frac{1}{\sqrt{3}}$$

$$\text{(iv)} \quad \operatorname{cosec} \frac{4\pi}{3} = \frac{1}{\sin \frac{4\pi}{3}} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$3 \text{ (i)} \quad x^2 = 4 \operatorname{cosec}^2 \theta \quad y^2 = 9 \cot^2 \theta$$

$$\frac{x^2}{4} = \operatorname{cosec}^2 \theta \quad \frac{y^2}{9} = \cot^2 \theta$$

$$1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta \Rightarrow 1 + \frac{y^2}{9} = \frac{x^2}{4}$$

$$36 + 4y^2 = 9x^2$$

$$9x^2 - 4y^2 = 36$$

$$\text{(ii)} \quad x^2 = \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta$$

$$= 1 - 2 \sin \theta \cos \theta$$

$$y^2 = \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta$$

$$= 1 + 2 \sin \theta \cos \theta$$

$$\text{Adding the equations gives } x^2 + y^2 = 2$$

$$4 \text{ (i)} \quad \operatorname{cosec} \theta = 4$$

$$\frac{1}{\sin \theta} = 4$$

$$\sin \theta = \frac{1}{4}$$

$$\theta = \sin^{-1}\left(\frac{1}{4}\right) = 14.5^\circ \text{ or } 165.5^\circ$$

$$\text{(ii)} \quad \tan \theta + \cot \theta = -4$$

$$\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = -4$$

$$\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = -4$$

$$\frac{1}{\sin \theta \cos \theta} = -4$$

$$-\frac{1}{4} = \sin \theta \cos \theta$$

$$-\frac{1}{2} = 2 \sin \theta \cos \theta$$

$$-\frac{1}{2} = \sin 2\theta$$

$$2\theta = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$2\theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

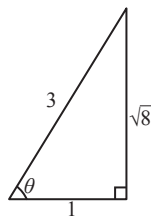
$$\theta = -\frac{\pi}{12}, -\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

5 (i) $\sec \theta = 3 \Rightarrow \cos \theta = \frac{1}{3}$

(ii) $\sin \theta = \frac{\sqrt{8}}{3}$

(iii) $\operatorname{cosec} \theta = \frac{3}{\sqrt{8}} = \frac{3\sqrt{8}}{8}$

(iv) $\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\frac{1}{3}}{\frac{\sqrt{8}}{3}} = \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8}$



6 (i) $\frac{1}{\tan \theta + \cot \theta} \equiv \sin \theta \cos \theta$

$$\begin{aligned} \text{LHS} &= \frac{1}{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}} \\ &= \frac{1}{\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}} \\ &= \frac{1}{\frac{1}{\sin \theta \cos \theta}} \\ &= \sin \theta \cos \theta \\ &= \text{RHS} \end{aligned}$$

(ii) $\sec^4 \theta - \tan^4 \theta \equiv \sec^2 \theta + \tan^2 \theta$

$$\begin{aligned} \text{LHS} &= (\sec^2 \theta - \tan^2 \theta)(\sec^2 \theta + \tan^2 \theta) \\ &= (\tan^2 \theta + 1 - \tan^2 \theta)(\sec^2 \theta + \tan^2 \theta) \\ &= (1)(\sec^2 \theta + \tan^2 \theta) \\ &= \sec^2 \theta + \tan^2 \theta \\ &= \text{RHS} \end{aligned}$$

7 (i) $m = \frac{1 + \cos \theta}{\sin \theta}$

$$\begin{aligned} \frac{1}{m} &= \frac{\sin \theta}{1 + \cos \theta} \\ &= \frac{\sin \theta}{1 + \cos \theta} \times \frac{1 - \cos \theta}{1 - \cos \theta} \\ &= \frac{\sin \theta(1 - \cos \theta)}{1 - \cos^2 \theta} \\ &= \frac{\sin \theta(1 - \cos \theta)}{\sin^2 \theta} \\ &= \frac{1 - \cos \theta}{\sin \theta} \end{aligned}$$

(ii) $m = \frac{1 + \cos \theta}{\sin \theta}$

$$\begin{aligned} m^2 &= \frac{(1 + \cos \theta)^2}{\sin^2 \theta} \\ &= \frac{(1 + \cos \theta)^2}{1 - \cos^2 \theta} \\ &= \frac{(1 + \cos \theta)(1 + \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} \end{aligned}$$

$$m^2(1 - \cos \theta) = 1 + \cos \theta$$

$$m^2 - 1 = (m^2 + 1)\cos \theta$$

$$\cos \theta = \frac{m^2 - 1}{m^2 + 1}$$

3.2 Compound-angle formulae (Page 27)

1 (i) $\begin{aligned} \cos 75^\circ &= \cos(45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} \\ &= \frac{\sqrt{2}\sqrt{3}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$

(ii) $\begin{aligned} \sin 15^\circ &= \sin(60^\circ - 45^\circ) \\ &= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ \\ &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$

2 (i) $\begin{aligned} \sin(\theta - 30^\circ) &= \sin \theta \cos 30^\circ - \cos \theta \sin 30^\circ \\ &= \frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta \end{aligned}$

(ii) $\begin{aligned} \cos\left(\frac{\pi}{4} - \theta\right) &= \cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta \\ &= \frac{\sqrt{2}}{2} \cos \theta + \frac{\sqrt{2}}{2} \sin \theta \end{aligned}$

3 (i) $\begin{aligned} \sin \theta \cos 2\beta + \sin 2\beta \cos \theta &= \sin \theta \cos 2\beta + \cos \theta \sin 2\beta \\ &= \sin(\theta + 2\beta) \end{aligned}$

(ii) $\begin{aligned} \cos 3\theta \cos \theta + \sin 3\theta \sin \theta &= \cos(3\theta - \theta) \\ &= \cos 2\theta \end{aligned}$

4 $\sin x = \frac{8}{10} = \frac{4}{5}, \cos x = \frac{6}{10} = \frac{3}{5}$

$$\sin y = \frac{24}{26} = \frac{12}{13}, \cos y = \frac{10}{26} = \frac{5}{13}$$

$$\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \\ &= \frac{4}{5} \times \frac{5}{13} + \frac{3}{5} \times \frac{12}{13} \\ &= \frac{56}{65} \end{aligned}$$

5 (i) $\begin{aligned} \cos(P - Q) &= \cos P \cos Q + \sin P \sin Q \\ &= \frac{2}{5} \times \frac{3}{\sqrt{58}} + \frac{\sqrt{21}}{5} \times \frac{7}{\sqrt{58}} \\ &= \frac{6 + 7\sqrt{21}}{5\sqrt{58}} \end{aligned}$

(ii) $\begin{aligned} \sin(P + Q) &= \sin P \cos Q + \cos P \sin Q \\ &= \frac{\sqrt{21}}{5} \times \frac{3}{\sqrt{58}} + \frac{2}{5} \times \frac{7}{\sqrt{58}} \\ &= \frac{3\sqrt{21} + 14}{5\sqrt{58}} \end{aligned}$

6 (i) $\sin(45^\circ - \theta) = \cos(30^\circ + \theta)$

$$\sin 45^\circ \cos \theta - \cos 45^\circ \sin \theta = \cos 30^\circ \cos \theta - \sin 30^\circ \sin \theta$$

$$\frac{\sqrt{2}}{2} \cos \theta - \frac{\sqrt{2}}{2} \sin \theta = \frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta$$

$$\frac{1}{2} \sin \theta - \frac{\sqrt{2}}{2} \sin \theta = \frac{\sqrt{3}}{2} \cos \theta - \frac{\sqrt{2}}{2} \cos \theta$$

$$\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) \sin \theta = \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right) \cos \theta$$

$$\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) \frac{\sin \theta}{\cos \theta} = \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$\tan \theta = \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right) \div \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{3} - \sqrt{2}}{1 - \sqrt{2}}$$

$$\theta = -37.5^\circ \text{ or } 142.5^\circ$$

(ii) $\tan\left(\frac{\pi}{3} + \theta\right) = 2 \tan\left(\frac{\pi}{6} - \theta\right)$

$$\frac{\tan \frac{\pi}{3} + \tan \theta}{1 - \tan \frac{\pi}{3} \times \tan \theta} = \frac{2\left(\tan \frac{\pi}{6} - \tan \theta\right)}{1 + \tan \frac{\pi}{6} \times \tan \theta}$$

$$\frac{\sqrt{3} + \tan \theta}{1 - \sqrt{3} \tan \theta} = \frac{2\left(\frac{1}{\sqrt{3}} - \tan \theta\right)}{1 + \frac{1}{\sqrt{3}} \tan \theta}$$

$$(\sqrt{3} + \tan \theta)\left(1 + \frac{1}{\sqrt{3}} \tan \theta\right) = \left(\frac{2}{\sqrt{3}} - 2 \tan \theta\right)(1 - \sqrt{3} \tan \theta)$$

$$\sqrt{3} + 2 \tan \theta + \frac{1}{\sqrt{3}} \tan^2 \theta = \frac{2}{\sqrt{3}} - 4 \tan \theta + 2 \sqrt{3} \tan^2 \theta$$

$$3 + 2\sqrt{3} \tan \theta + \tan^2 \theta = 2 - 4\sqrt{3} \tan \theta + 6 \tan^2 \theta$$

$$0 = 5 \tan^2 \theta - 6\sqrt{3} \tan \theta - 1$$

$$\tan \theta = 2.17 \text{ or } -0.0921$$

$$\theta = 1.14 \text{ or } 3.05$$

7 $\cos(A+B)\cos(A-B) \equiv \cos^2 A - \sin^2 B$

LHS

$$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$$

$$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$

$$= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$$

$$= \cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B$$

$$= \cos^2 A - \sin^2 B$$

$$= RHS$$

8 (i) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$5 = \frac{\frac{1}{3} - \tan B}{1 + \frac{1}{3} \tan B}$$

$$5 + \frac{5}{3} \tan B = \frac{1}{3} - \tan B$$

$$\frac{8}{3} \tan B = \frac{1}{3} - 5$$

$$\tan B = -\frac{7}{4}$$

(ii) $\tan A = \frac{1}{3} \Rightarrow \sin A = \frac{1}{\sqrt{10}}$ and $\cos A = \frac{3}{\sqrt{10}}$

$$\tan B = \frac{-7}{4} \Rightarrow \sin B = \frac{7}{\sqrt{65}}$$
 and $\cos B = \frac{-4}{\sqrt{65}}$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{1}{\sqrt{10}} \times -\frac{4}{\sqrt{65}} + \frac{3}{\sqrt{10}} \times \frac{7}{\sqrt{65}}$$

$$= -\frac{4}{\sqrt{650}} + \frac{21}{\sqrt{650}}$$

$$= \frac{17}{\sqrt{650}}$$

3.3 Double-angle formulae (Page 30)

1 (i) $\sin \frac{2\pi}{3} = \sin 2\left(\frac{\pi}{3}\right)$

$$= 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$$

$$= 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2}$$

(ii) $\cos \frac{2\pi}{3} = \cos 2\left(\frac{\pi}{3}\right)$

$$= 2 \cos^2 \left(\frac{\pi}{3}\right) - 1$$

$$= 2 \times \left(\frac{1}{2}\right)^2 - 1$$

$$= -\frac{1}{2}$$

(iii) $\tan \frac{2\pi}{3} = \tan 2\left(\frac{\pi}{3}\right)$

$$= \frac{2 \tan \frac{\pi}{3}}{1 - \tan^2 \frac{\pi}{3}}$$

$$= \frac{2\sqrt{3}}{1 - (\sqrt{3})^2}$$

$$= \frac{2\sqrt{3}}{-2}$$

$$= -\sqrt{3}$$

$$2 \quad (i) \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2 \times \frac{3}{5} \times \frac{4}{5}$$

$$= \frac{24}{25}$$

$$(ii) \quad \cos 2\theta = 1 - 2\sin^2\theta$$

$$= 1 - 2 \times \left(\frac{3}{5}\right)^2$$

$$= \frac{7}{25}$$

$$3 \quad (i) \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2 \times -\frac{2}{3} \times \frac{\sqrt{5}}{3}$$

$$= -\frac{4\sqrt{5}}{9}$$

$$(ii) \quad \cos 2\theta = 1 - 2\sin^2\theta$$

$$= 1 - 2 \times \left(-\frac{2}{3}\right)^2$$

$$= \frac{1}{9}$$

$$(iii) \quad \sin 4\theta = 2\sin 2\theta\cos 2\theta$$

$$= 2 \times -\frac{4\sqrt{5}}{9} \times \frac{1}{9}$$

$$= -\frac{8\sqrt{5}}{81}$$

$$4 \quad (i) \quad 5 + 4\sec^2\theta = 12\tan\theta$$

$$5 + 4(1 + \tan^2\theta) = 12\tan\theta$$

$$5 + 4 + 4\tan^2\theta = 12\tan\theta$$

$$4\tan^2\theta - 12\tan\theta + 9 = 0$$

$$(2\tan\theta - 3)(2\tan\theta - 3) = 0$$

$$\tan\theta = \frac{3}{2}$$

$$(ii) \quad (a) \quad \tan(\theta + 45^\circ) = \frac{\tan\theta + \tan 45^\circ}{1 - \tan\theta \tan 45^\circ}$$

$$= \frac{\tan\theta + 1}{1 - \tan\theta}$$

$$= \frac{\frac{3}{2} + 1}{1 - \frac{3}{2}}$$

$$= -5$$

$$(b) \quad \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2 \times \frac{3}{2}}{1 - \left(\frac{3}{2}\right)^2}$$

$$= -\frac{12}{5}$$

$$5 \quad (i) \quad \sin 2\theta = \sin\theta$$

$$2\sin\theta\cos\theta = \sin\theta$$

$$2\sin\theta\cos\theta - \sin\theta = 0$$

$$\sin\theta(2\cos\theta - 1) = 0$$

$$\sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{3}$$

$$(ii) \quad 3\tan 2\theta + 2\tan\theta = 0$$

$$3\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right) + 2\tan\theta = 0$$

$$6\tan\theta + 2\tan\theta(1 - \tan^2\theta) = 0$$

$$6\tan\theta + 2\tan\theta - 2\tan^3\theta = 0$$

$$2\tan^3\theta - 8\tan\theta = 0$$

$$2\tan\theta(\tan^2\theta - 4) = 0$$

$$\tan\theta = 0 \text{ or } \tan^2\theta = 4$$

$$\tan\theta = \pm 2$$

$$\theta = 0^\circ, 63.4^\circ, -116.6^\circ, -63.4^\circ, 116.6^\circ$$

$$(iii) \quad \frac{1 - \sin\theta - \cos 2\theta}{\cos\theta - \sin 2\theta} = 1$$

$$1 - \sin\theta - \cos 2\theta = \cos\theta - \sin 2\theta$$

$$1 - \sin\theta - (1 - 2\sin^2\theta) = \cos\theta - 2\sin\theta\cos\theta$$

$$2\sin^2\theta - \sin\theta = \cos\theta(1 - 2\sin\theta)$$

$$-\sin\theta(1 - 2\sin\theta) = \cos\theta(1 - 2\sin\theta)$$

$$-\sin\theta = \cos\theta$$

$$\tan\theta = -1$$

$$\theta = 135^\circ, 315^\circ$$

$$6 \quad (i) \quad \sin(45^\circ + \theta)\sin(45^\circ - \theta) \equiv \frac{1}{2}\cos 2\theta$$

$$LHS = (\sin 45^\circ\cos\theta + \cos 45^\circ\sin\theta) \times$$

$$(\sin 45^\circ\cos\theta - \cos 45^\circ\sin\theta)$$

$$= \left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)\left(\frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta\right)$$

$$= \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta$$

$$= \frac{1}{2}(\cos^2\theta - \sin^2\theta)$$

$$= \frac{1}{2}\cos 2\theta$$

$$= RHS$$

$$(ii) \frac{2\sin\theta\cos\theta}{\cos^4\theta - \sin^4\theta} \equiv \tan 2\theta$$

$$\begin{aligned} LHS &= \frac{\sin 2\theta}{(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)} \\ &= \frac{\sin 2\theta}{(\cos^2\theta - \sin^2\theta)(1)} \\ &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \tan 2\theta \\ &= RHS \end{aligned}$$

$$7 (i) \cot\theta + \tan\theta \equiv 2 \operatorname{cosec} 2\theta$$

$$\begin{aligned} LHS &= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \\ &= \frac{1}{\sin\theta\cos\theta} \\ &= \frac{2}{2\sin\theta\cos\theta} \\ &= \frac{2}{\sin 2\theta} \\ &= 2 \operatorname{cosec} 2\theta \\ &= RHS \end{aligned}$$

$$(ii) \cot\theta + \tan\theta = 8$$

$$2 \operatorname{cosec} 2\theta = 8$$

$$\operatorname{cosec} 2\theta = 4$$

$$\frac{1}{\sin 2\theta} = 4$$

$$\sin 2\theta = \frac{1}{4}$$

$$2\theta = \sin^{-1}\left(\frac{1}{4}\right)$$

$$2\theta = 0.2527, 2.8889, 6.5359, 9.1721$$

$$\theta = 0.126, 1.44, 3.27, 4.59$$

$$8 \sin 25^\circ = k$$

$$(i) \sin 50^\circ = \sin(2 \times 25^\circ) = 2 \sin 25^\circ \cos 25^\circ = 2k\sqrt{1-k^2}$$

$$(ii) \cos 50^\circ = \cos(2 \times 25^\circ) = 1 - 2 \sin^2 25^\circ = 1 - 2k^2$$

$$(iii) \tan 155^\circ = \frac{\sin 155^\circ}{\cos 155^\circ} = \frac{\sin 25^\circ}{-\cos 25^\circ} = -\frac{k}{\sqrt{1-k^2}}$$

3.4 The forms $r \cos(\theta \pm \alpha)$, $r \sin(\theta \pm \alpha)$ (Page 34)

$$1 (i) \sin\theta - 3\cos\theta \equiv r(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$

$$1 \equiv r\cos\alpha$$

$$-3 \equiv -r\sin\alpha \Rightarrow 3 \equiv r\sin\alpha$$

$$\text{Dividing gives } \tan\alpha = 3 \Rightarrow \alpha = 71.6^\circ$$

$$r^2 = 1^2 + 3^2 = 10 \Rightarrow r = \sqrt{10}$$

$$\text{So } \sin\theta - 3\cos\theta = \sqrt{10} \sin(\theta - 71.6^\circ)$$

$$(ii) 12\cos\theta + 5\sin\theta \equiv r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$12 \equiv r\cos\alpha$$

$$5 \equiv r\sin\alpha$$

$$\text{Dividing gives } \tan\alpha = \frac{5}{12} \Rightarrow \alpha = 22.6^\circ$$

$$r^2 = 12^2 + 5^2 \Rightarrow r = 13$$

$$\text{So } 12\cos\theta + 5\sin\theta \equiv 13\cos(\theta - 22.6^\circ)$$

$$2 (i) 2\cos\theta + \sin\theta \equiv r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$2 \equiv r\cos\alpha$$

$$1 \equiv r\sin\alpha$$

$$\text{Dividing gives } \tan\alpha = \frac{1}{2} \Rightarrow \alpha = 26.6^\circ$$

$$r^2 = 2^2 + 1^2 \Rightarrow r = \sqrt{5}$$

$$\text{So } 2\cos\theta + \sin\theta = \sqrt{5} \cos(\theta - 26.6^\circ)$$

$$(ii) 2\cos\theta + \sin\theta = 1$$

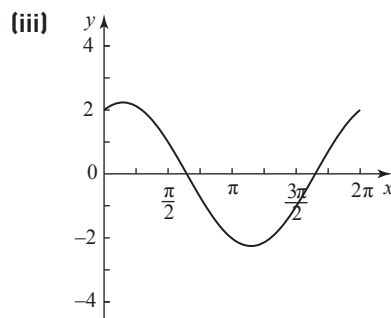
$$\sqrt{5} \cos(\theta - 26.6^\circ) = 1$$

$$\cos(\theta - 26.6^\circ) = \frac{1}{\sqrt{5}}$$

$$\theta - 26.6^\circ = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$\theta - 26.6^\circ = 63.4^\circ, 296.6^\circ$$

$$\theta = 90.0^\circ, 323.2^\circ$$



$$(iv) \text{ Since } 2\cos\theta + \sin\theta \equiv \sqrt{5} \cos(\theta - 26.6^\circ),$$

$$-\sqrt{5} \leq 2\cos\theta + \sin\theta \leq \sqrt{5}$$

$$-\sqrt{5} + 5 \leq 2\cos\theta + \sin\theta + 5 \leq \sqrt{5} + 5$$

$$\begin{aligned} 3 (i) T(\theta) &= 3 \cos(\theta - 60^\circ) + 2 \cos(\theta + 60^\circ) \\ &= 3(\cos\theta\cos 60^\circ + \sin\theta\sin 60^\circ) \\ &\quad + 2(\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ) \\ &= \frac{5}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \end{aligned}$$

$$A = \frac{\sqrt{3}}{2}, B = \frac{5}{2}$$

$$(ii) \frac{\sqrt{3}}{2}\sin\theta + \frac{5}{2}\cos\theta \equiv R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$\frac{\sqrt{3}}{2} \equiv R\cos\alpha$$

$$\frac{5}{2} \equiv R\sin\alpha$$

$$\text{Dividing gives } \tan \alpha = \frac{5}{\sqrt{3}} \Rightarrow \alpha = 70.9^\circ$$

$$R^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{5}{2}\right)^2 \Rightarrow R = \sqrt{7}$$

$$\text{So } \frac{\sqrt{3}}{2} \sin \theta + \frac{5}{2} \cos \theta \equiv \sqrt{7} \sin(\theta + 70.9^\circ)$$

$$\text{(iii) } \sqrt{7} \sin(\theta + 70.9^\circ) + 1 = 0$$

$$\sin(\theta + 70.9^\circ) = -\frac{1}{\sqrt{7}}$$

$$\theta + 70.9^\circ = -22.2^\circ, -157.8^\circ$$

$$\theta + 70.9^\circ = 337.8^\circ, 202.2^\circ$$

$$\theta = 266.9^\circ, 131.3^\circ$$

Smallest positive angle is 131.3°

$$4 \text{ (i) } 3 \sin \theta + 4 \cos \theta \equiv R(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$$

$$3 \equiv R \cos \alpha$$

$$4 \equiv R \sin \alpha$$

$$\text{Dividing gives } \tan \alpha = \frac{4}{3} \Rightarrow \alpha = 53.1^\circ$$

$$R^2 = 3^2 + 4^2 \Rightarrow R = 5$$

$$\text{So } 3 \sin \theta + 4 \cos \theta \equiv 5 \sin(\theta + 53.1^\circ)$$

$$\text{(ii) (a) } 3 \sin \theta + 4 \cos \theta + 1 = 0$$

$$5 \sin(\theta + 53.1^\circ) + 1 = 0$$

$$\sin(\theta + 53.1^\circ) = -\frac{1}{5}$$

$$\theta + 53.1^\circ = -11.54^\circ, 191.54^\circ, \dots$$

$$\theta = -64.6^\circ, 138.4^\circ$$

$$\text{(b) } -37 \leq k(3 \sin \theta + 4 \cos \theta) + c \leq 43$$

$$-37 \leq 5k \sin(\theta + 53.1^\circ) + c \leq 43$$

$$\text{Since } -1 \leq \sin(\theta + 53.1^\circ) \leq 1,$$

$$5k + c = 43$$

$$-5k + c = -37$$

$$2c = 6 \Rightarrow c = 3$$

$$5k + 3 = 43 \Rightarrow k = 8$$

Further practice (Page 37)

$$1 \quad \sin(x - 45^\circ) - \cos(45^\circ - x) = 1$$

$$(\sin x \cos 45^\circ - \cos x \sin 45^\circ) -$$

$$(\cos 45^\circ \cos x + \sin 45^\circ \sin x) = 1$$

$$\frac{\sqrt{2}}{2} \sin x - \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x = 1$$

$$-\sqrt{2} \cos x = 1$$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = 135^\circ, 225^\circ$$

$$2 \quad 2 \sin(x + 30^\circ) = \cos(x - 45^\circ)$$

$$2(\sin x \cos 30^\circ + \cos x \sin 30^\circ)$$

$$= \cos x \cos 45^\circ + \sin x \sin 45^\circ$$

$$\sqrt{3} \sin x + \cos x = \frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \sin x$$

$$\left(\sqrt{3} - \frac{\sqrt{2}}{2}\right) \sin x = \left(\frac{\sqrt{2}}{2} - 1\right) \cos x$$

$$\left(\sqrt{3} - \frac{\sqrt{2}}{2}\right) \tan x = \left(\frac{\sqrt{2}}{2} - 1\right)$$

$$\tan x = \frac{\left(\frac{\sqrt{2}}{2} - 1\right)}{\left(\sqrt{3} - \frac{\sqrt{2}}{2}\right)}$$

$$x = 164.1^\circ, 344.1^\circ$$

$$3 \text{ (i) } 2 \operatorname{cosec} 2\theta \equiv \sec \theta \operatorname{cosec} \theta$$

$$LHS = \frac{2}{\sin 2\theta}$$

$$= \frac{2}{2 \sin \theta \cos \theta}$$

$$= \frac{1}{\sin \theta \cos \theta}$$

$$= \frac{1}{\cos \theta} \times \frac{1}{\sin \theta}$$

$$= \sec \theta \operatorname{cosec} \theta$$

$$= RHS$$

$$\text{(ii) } \sec \theta \operatorname{cosec} \theta = 4$$

$$2 \operatorname{cosec} 2\theta = 4$$

$$\frac{1}{\sin 2\theta} = 2$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = 30^\circ, 150^\circ, \dots$$

$$\theta = 15^\circ, 75^\circ$$

$$4 \quad 4 \operatorname{cosec}^2 \theta - 7 = 4 \cot \theta$$

$$4(\cot^2 \theta + 1) - 7 = 4 \cot \theta$$

$$4 \cot^2 \theta + 4 - 7 = 4 \cot \theta$$

$$4 \cot^2 \theta - 4 \cot \theta - 3 = 0$$

$$(2 \cot \theta + 1)(2 \cot \theta - 3) = 0$$

$$\cot \theta = -\frac{1}{2} \text{ or } \frac{3}{2}$$

$$\tan \theta = -2 \text{ or } \frac{2}{3}$$

$$\theta = 33.7^\circ, 116.6^\circ$$

$$5 \text{ (i) } 8 \sin \theta + 15 \cos \theta \equiv R(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$$

$$8 \equiv R \cos \alpha$$

$$15 \equiv R \sin \alpha$$

$$\tan \alpha = \frac{15}{8} \Rightarrow \alpha = 61.9^\circ$$

$$R^2 = 8^2 + 15^2 \Rightarrow R = 17$$

$$\text{So } 8 \sin \theta + 15 \cos \theta \equiv 17 \sin(\theta + 61.9^\circ)$$

(ii) $8 \sin \theta + 15 \cos \theta = 14$

$$17 \sin(\theta + 61.9^\circ) = 14$$

$$\sin(\theta + 61.9^\circ) = \frac{14}{17}$$

$$\theta + 61.9^\circ = 55.4^\circ, 124.6^\circ$$

$$\theta = -6.5^\circ, 62.7^\circ$$

$$\theta = 353.5^\circ, 62.7^\circ$$

(iii) Since $-17 \leq 8 \sin \theta + 15 \cos \theta \leq 17$,

there are no solutions if $k < -17$ or $k > 17$

6 (i) $\tan(\theta + 60^\circ) \tan(\theta - 60^\circ) \equiv \frac{\tan^2 \theta - 3}{1 - 3 \tan^2 \theta}$

$$LHS = \tan(\theta + 60^\circ) \tan(\theta - 60^\circ)$$

$$= \left(\frac{\tan \theta + \tan 60^\circ}{1 - \tan \theta \tan 60^\circ} \right) \left(\frac{\tan \theta - \tan 60^\circ}{1 + \tan \theta \tan 60^\circ} \right)$$

$$= \left(\frac{\tan \theta + \sqrt{3}}{1 - \sqrt{3} \tan \theta} \right) \left(\frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta} \right)$$

$$= \frac{\tan^2 \theta - 3}{1 - 3 \tan^2 \theta}$$

$$= RHS$$

(ii) $\frac{\tan^2 \theta - 3}{1 - 3 \tan^2 \theta} = 4 \sec^2 \theta - 3$

$$\tan^2 \theta - 3 = (4 \sec^2 \theta - 3)(1 - 3 \tan^2 \theta)$$

$$\tan^2 \theta - 3 = (4(1 + \tan^2 \theta) - 3)(1 - 3 \tan^2 \theta)$$

$$\tan^2 \theta - 3 = (4 + 4 \tan^2 \theta - 3)(1 - 3 \tan^2 \theta)$$

$$\tan^2 \theta - 3 = (4 \tan^2 \theta + 1)(1 - 3 \tan^2 \theta)$$

$$\tan^2 \theta - 3 = 4 \tan^2 \theta - 12 \tan^4 \theta + 1 - 3 \tan^2 \theta$$

$$12 \tan^4 \theta = 4$$

$$\tan^4 \theta = \frac{1}{3}$$

$$\tan \theta = \pm \sqrt[4]{\frac{1}{3}}$$

$$\theta = 37.2^\circ, 142.8^\circ$$

(iii) $\frac{\tan^2 \theta - 3}{1 - 3 \tan^2 \theta} = k^2$

$$\tan^2 \theta - 3 = k^2(1 - 3 \tan^2 \theta)$$

$$\tan^2 \theta - 3 = k^2 - 3k^2 \tan^2 \theta$$

$$(1 + 3k^2) \tan^2 \theta = 3 + k^2$$

$$\tan^2 \theta = \frac{3 + k^2}{1 + 3k^2}$$

$$\tan \theta = \pm \sqrt{\frac{3 + k^2}{1 + 3k^2}}$$

Since $k^2 \geq 0$, $\frac{3 + k^2}{1 + 3k^2} \geq 0$

so there are two roots of the equation.

7 (i) $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

$$\tan 2\alpha \tan \alpha = 8$$

$$\frac{2 \tan \alpha}{1 - \tan^2 \alpha} \times \tan \alpha = 8$$

$$\frac{2 \tan^2 \alpha}{1 - \tan^2 \alpha} = 8$$

$$2 \tan^2 \alpha = 8(1 - \tan^2 \alpha)$$

$$2 \tan^2 \alpha = 8 - 8 \tan^2 \alpha$$

$$10 \tan^2 \alpha = 8$$

$$\tan^2 \alpha = \frac{8}{10}$$

$$\tan \alpha = \pm \sqrt{\frac{8}{10}}$$

$$\alpha = 41.8^\circ, 138.2^\circ$$

(ii) (a) $\operatorname{cosec} \beta = \frac{7}{6}$

(b) $\cot^2 \beta = \frac{\cos^2 \beta}{\sin^2 \beta} = \frac{\left(\frac{\sqrt{13}}{7}\right)^2}{\left(\frac{6}{7}\right)^2} = \frac{13}{36}$

Past exam questions (Page 37)

1 (i) $\tan(x + 45^\circ) = 6 \tan x$

$$\frac{\tan x + \tan 45^\circ}{1 - \tan x \tan 45^\circ} = 6 \tan x$$

$$\frac{\tan x + 1}{1 - \tan x} = 6 \tan x$$

$$\tan x + 1 = 6 \tan x(1 - \tan x)$$

$$\tan x + 1 = 6 \tan x - 6 \tan^2 x$$

$$6 \tan^2 x - 5 \tan x + 1 = 0$$

(ii) $(3 \tan x - 1)(2 \tan x - 1) = 0$

$$\tan x = \frac{1}{3} \quad \text{or} \quad \tan x = \frac{1}{2}$$

$$x = 18.4^\circ \quad \text{or} \quad 26.6^\circ$$

2 $\cos(\theta + 60^\circ) = 2 \sin \theta$

$$\cos \theta \cos 60^\circ - \sin \theta \sin 60^\circ = 2 \sin \theta$$

$$\frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = 2 \sin \theta$$

$$\cos \theta - \sqrt{3} \sin \theta = 4 \sin \theta$$

$$\cos \theta + (-\sqrt{3} - 4) \sin \theta = 0$$

$$1 + (-\sqrt{3} - 4) \tan \theta = 0$$

$$\tan \theta = \frac{-1}{-\sqrt{3} - 4}$$

$$\theta = 9.9^\circ \quad \text{or} \quad 189.9^\circ$$

$$3 \text{ (i)} \quad LHS = \sin^2 2\theta (\operatorname{cosec}^2 \theta - \sec^2 \theta)$$

$$= \sin^2 2\theta \left(\frac{1}{\sin^2 \theta} - \frac{1}{\cos^2 \theta} \right)$$

$$= \sin^2 2\theta \left(\frac{\cos^2 \theta - \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} \right)$$

$$= \sin^2 2\theta \left(\frac{4(\cos^2 \theta - \sin^2 \theta)}{4\sin^2 \theta \cos^2 \theta} \right)$$

$$= \sin^2 2\theta \left(\frac{4\cos 2\theta}{\sin^2 2\theta} \right)$$

$$= 4\cos 2\theta$$

$$= RHS$$

$$(ii) \text{ (a)} \quad 4\cos 2\theta = 3$$

$$\cos 2\theta = \frac{3}{4}$$

$$2\theta = \cos^{-1}\left(\frac{3}{4}\right)$$

$$2\theta = 41.4^\circ \text{ or } 318.6^\circ$$

$$\theta = 20.7^\circ \text{ or } 159.3^\circ$$

$$(b) \quad \sin^2 2\theta (\operatorname{cosec}^2 \theta - \sec^2 \theta) = 4\cos 2\theta$$

$$\operatorname{cosec}^2 \theta - \sec^2 \theta = \frac{4\cos 2\theta}{\sin^2 2\theta}$$

$$\operatorname{cosec}^2 15^\circ - \sec^2 15^\circ = \frac{4\cos 30^\circ}{(\sin 30^\circ)^2}$$

$$= \frac{4 \times \frac{\sqrt{3}}{2}}{\left(\frac{1}{2}\right)^2}$$

$$= \frac{2\sqrt{3}}{\frac{1}{4}} = 8\sqrt{3}$$

$$4 \quad \cot 2\theta = 1 + \tan \theta$$

$$\frac{1}{\tan 2\theta} = 1 + \tan \theta$$

$$\frac{1 - \tan^2 \theta}{2\tan \theta} = 1 + \tan \theta$$

$$1 - \tan^2 \theta = 2\tan \theta + 2\tan^2 \theta$$

$$3\tan^2 \theta + 2\tan \theta - 1 = 0$$

$$(3\tan \theta - 1)(\tan \theta + 1) = 0$$

$$\tan \theta = \frac{1}{3} \text{ or } \tan \theta = -1$$

$$\theta = 18.4^\circ, 135^\circ$$

$$5 \text{ (i)} \quad \sin 2\theta (3\sec \theta + 4\operatorname{cosec} \theta)$$

$$= 2\sin \theta \cos \theta \left(\frac{3}{\cos \theta} + \frac{4}{\sin \theta} \right)$$

$$= 6\sin \theta + 8\cos \theta$$

$$(ii) \quad 6\sin \theta + 8\cos \theta \equiv R(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$$

$$6 \equiv R\cos \alpha$$

$$8 \equiv R\sin \alpha$$

$$\text{Dividing gives } \tan \alpha = \frac{8}{6} \Rightarrow \alpha = 53.1^\circ$$

$$R^2 = 6^2 + 8^2 \Rightarrow R = 10$$

$$\text{So } 6\sin \theta + 8\cos \theta \equiv 10\sin(\theta + 53.1^\circ)$$

$$(iii) \quad 10\sin(\theta + 53.1^\circ) = 7$$

$$\sin(\theta + 53.1^\circ) = \frac{7}{10}$$

$$(\theta + 53.1^\circ) = 44.43^\circ, 135.57^\circ, 404.43^\circ, \dots$$

$$\theta = 82.4^\circ, 351.3^\circ$$

$$6 \text{ (i)} \quad \cot \theta - 2\tan \theta = \sin 2\theta$$

$$\frac{\cos \theta}{\sin \theta} - \frac{2\sin \theta}{\cos \theta} = 2\sin \theta \cos \theta$$

$$\cos^2 \theta - 2\sin^2 \theta = 2\sin^2 \theta \cos^2 \theta$$

$$\cos^2 \theta - 2(1 - \cos^2 \theta) = 2(1 - \cos^2 \theta)\cos^2 \theta$$

$$\cos^2 \theta - 2 + 2\cos^2 \theta = 2\cos^2 \theta - 2\cos^4 \theta$$

$$2\cos^4 \theta + \cos^2 \theta - 2 = 0$$

$$a = 2, b = 1, c = -2$$

$$(ii) \quad 2\cos^4 \theta + \cos^2 \theta - 2 = 0$$

Using the quadratic formula,

$$\cos^2 \theta = -1.28 \text{ (no solutions)}$$

$$\cos^2 \theta = 0.781$$

$$\cos \theta = \pm\sqrt{0.781}$$

$$\theta = 152.1^\circ$$

Stretch and challenge (Page 38)

$$1 \text{ (i)} \quad R = \frac{2V^2 \sin B}{g \cos^2 A} \cos(A + B)$$

$$= \frac{2V^2 \sin B}{10 \cos^2 \frac{\pi}{4}} \cos\left(B + \frac{\pi}{4}\right)$$

$$= \frac{2V^2}{5} \sin B \cos\left(B + \frac{\pi}{4}\right)$$

$$\begin{aligned}
 \text{(iii)} \quad 150 &= \frac{2 \times 40^2}{5} \sin B \cos \left(B + \frac{\pi}{4} \right) \\
 \sin B \cos \left(B + \frac{\pi}{4} \right) &= \frac{15}{64} \\
 \sin B \left(\cos B \cos \frac{\pi}{4} - \sin B \sin \frac{\pi}{4} \right) &= \frac{15}{64} \\
 \frac{\sqrt{2}}{2} \sin B (\cos B - \sin B) &= \frac{15}{64} \\
 \sin B (\cos B - \sin B) &= \frac{15\sqrt{2}}{64} \\
 \sin B \cos B - \sin^2 B &= \frac{15\sqrt{2}}{64} \\
 \frac{1}{2} \sin 2B - \left(\frac{\cos 2B - 1}{2} \right) &= \frac{15\sqrt{2}}{64} \\
 \sin 2B - \cos 2B &= \frac{15\sqrt{2}}{32} - 1 \\
 \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin 2B - \frac{1}{\sqrt{2}} \cos 2B \right) &= \frac{15\sqrt{2}}{32} - 1 \\
 \sqrt{2} \sin \left(2B - \frac{\pi}{4} \right) &= \frac{15\sqrt{2}}{32} - 1 \\
 \sin \left(2B - \frac{\pi}{4} \right) &= \frac{15}{32} - \frac{1}{\sqrt{2}} \\
 2B - \frac{\pi}{4} &= -0.241, 3.38 \\
 B &= 0.272, 2.08
 \end{aligned}$$

Since $2.08 > \frac{\pi}{2}$ then $B = 0.272 = 15.6^\circ$

$$\begin{aligned}
 2 \quad \tan x - \tan \frac{1}{8}x &= \frac{\sin kx}{\cos x \cos \frac{1}{8}x} \\
 \left(\tan x - \tan \frac{1}{8}x \right) \left(\cos x \cos \frac{1}{8}x \right) &= \sin kx \\
 \tan x \cos x \cos \frac{1}{8}x - \tan \frac{1}{8}x \cos x \cos \frac{1}{8}x &= \sin kx \\
 \sin x \cos \frac{1}{8}x - \cos x \sin \frac{1}{8}x &= \sin kx \\
 \sin \left(x - \frac{1}{8}x \right) &= \sin kx \\
 \sin \frac{7}{8}x &= \sin kx \\
 k &= \frac{7}{8}
 \end{aligned}$$

$$\begin{aligned}
 3 \quad (\sin A + \cos A)^2 &= \sin^2 A + 2 \sin A \cos A + \cos^2 A \\
 &= \sin^2 A + \cos^2 A + 2 \sin A \cos A \\
 \text{So} \quad 1.5^2 &= 1 + 2 \sin A \cos A \\
 \sin A \cos A &= \frac{5}{8} \\
 (\sin A + \cos A)^3 &= \sin^3 A + 3 \sin^2 A \cos A + 3 \sin A \cos^2 A + \cos^3 A \\
 &= \sin^3 A + \cos^3 A + 3 \sin A \cos A (\sin A + \cos A) \\
 1.5^3 &= \sin^3 A + \cos^3 A + 3 \times \frac{5}{8} \times 1.5 \\
 \sin^3 A + \cos^3 A &= \frac{9}{16}
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \log_{10}(\tan \theta) - \log_{10}(\sin \theta) &= \log_{10} \left(\frac{\tan \theta}{\sin \theta} \right) \\
 &= \log_{10} \left(\frac{\sin \theta}{\cos \theta} \right) \\
 &= \log_{10} \left(\frac{1}{\cos \theta} \right) \\
 &= \log_{10} \left(\frac{1}{0.1} \right) \\
 &= \log_{10}(10)
 \end{aligned}$$

$$\begin{aligned}
 5 \quad 1 - \frac{\sin^2 \theta}{1 + \cot \theta} - \frac{\cos^2 \theta}{1 + \tan \theta} &\equiv \frac{1}{2} \sin 2\theta \\
 LHS &= 1 - \left(\frac{\sin^2 \theta}{1 + \cot \theta} + \frac{\cos^2 \theta}{1 + \tan \theta} \right) \\
 &= 1 - \left(\frac{\sin^2 \theta}{1 + \cot \theta} + \frac{\cos^2 \theta}{1 + \tan \theta} \right) \\
 &= 1 - \left(\frac{\sin^2 \theta}{1 + \frac{\cos \theta}{\sin \theta}} + \frac{\cos^2 \theta}{1 + \frac{\sin \theta}{\cos \theta}} \right) \\
 &= 1 - \left(\frac{\sin^2 \theta}{\frac{\sin \theta + \cos \theta}{\sin \theta}} + \frac{\cos^2 \theta}{\frac{\cos \theta + \sin \theta}{\cos \theta}} \right) \\
 &= 1 - \left(\frac{\sin^3 \theta}{\sin \theta + \cos \theta} + \frac{\cos^3 \theta}{\sin \theta + \cos \theta} \right) \\
 &= 1 - \left(\frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} \right) \\
 &= \frac{\sin \theta + \cos \theta - (\sin^3 \theta + \cos^3 \theta)}{\sin \theta + \cos \theta} \\
 &= \frac{\sin \theta + \cos \theta - ((\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta))}{\sin \theta + \cos \theta} \\
 &= \frac{\sin \theta + \cos \theta - ((\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta))}{\sin \theta + \cos \theta} \\
 &= \frac{\sin \theta + \cos \theta - (\sin \theta + \cos \theta)}{\sin \theta + \cos \theta} \\
 &= \frac{\sin^2 \theta \cos \theta + \sin \theta \cos^2 \theta}{\sin \theta + \cos \theta} \\
 &= \frac{\sin \theta \cos \theta (\sin \theta + \cos \theta)}{\sin \theta + \cos \theta} \\
 &= \sin \theta \cos \theta \\
 &= \frac{1}{2} (2 \sin \theta \cos \theta) \\
 &= \frac{1}{2} \sin 2\theta \\
 &= RHS
 \end{aligned}$$

6 Using ratios:

Arm: 4 min: 2π so 1 min: $\frac{2\pi}{4}$ so t min: $\frac{\pi t}{2}$

Wheel: 3 min: 2π so 1 min: $\frac{2\pi}{3}$ so t min: $\frac{2\pi t}{3}$

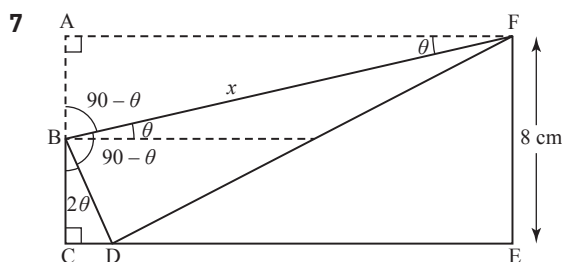
OR using calculus:

$$\frac{d\theta}{dt} = \frac{\pi}{2} \text{ radians/min} \Rightarrow \theta = \frac{\pi}{2}t$$

$$\frac{d\alpha}{dt} = \frac{2\pi}{3} \text{ radians/min} \Rightarrow \alpha = \frac{2\pi}{3}t$$

So the required function is the sum of the vertical (sine) functions of the motions of the arm and the wheel together, added to the starting height of 35 m:

$$h(t) = 35 + 20\sin\left(\frac{\pi}{2}t\right) + 15\sin\left(\frac{2\pi}{3}t\right)$$



$$\sin\theta = \frac{AB}{x} \Rightarrow AB = x\sin\theta \text{ and } BD = x\sin\theta$$

$$\angle ABF = \angle DBF = 90^\circ - \theta$$

$$\angle CBD = 180^\circ - 2(90^\circ - \theta) = 2\theta$$

$$\cos 2\theta = \frac{BC}{BD} = \frac{BC}{x\sin\theta} \Rightarrow BC = x\sin\theta\cos 2\theta$$

$$AC = AB + BC$$

$$8 = x\sin\theta + x\sin\theta\cos 2\theta$$

$$8 = x\sin\theta(1 + \cos 2\theta)$$

$$8 = x\sin\theta(1 + 2\cos^2\theta - 1)$$

$$8 = x\sin\theta \times 2\cos^2\theta$$

$$4 = x\sin\theta\cos^2\theta$$

$$x = \frac{4}{\sin\theta\cos^2\theta}$$

$$8 \text{ (i)} \quad \cos 4\theta = 1 - 2\sin^2 2\theta$$

$$= 1 - 2(2\sin\theta\cos\theta)^2$$

$$= 1 - 2(4\sin^2\theta\cos^2\theta)$$

$$= 1 - 8\sin^2\theta\cos^2\theta$$

$$\text{(ii)} \quad \text{Let } \theta = \frac{1}{24}\pi$$

$$\cos 4\left(\frac{1}{24}\pi\right) = 1 - 8\sin^2\left(\frac{1}{24}\pi\right)\cos^2\left(\frac{1}{24}\pi\right)$$

$$\cos\left(\frac{1}{6}\pi\right) = 1 - 8\sin^2\left(\frac{1}{24}\pi\right)\cos^2\left(\frac{1}{24}\pi\right)$$

$$\sin^2\left(\frac{1}{24}\pi\right)\cos^2\left(\frac{1}{24}\pi\right) = \frac{1 - \cos\left(\frac{1}{6}\pi\right)}{8}$$

$$= \frac{1 - \frac{\sqrt{3}}{2}}{8}$$

$$= \frac{1}{8} - \frac{\sqrt{3}}{16}$$

$$= \frac{2 - \sqrt{3}}{16}$$

$$\text{(iii)} \quad \cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 4\theta = 2\cos^2 2\theta - 1 \Rightarrow \cos^2 2\theta = \frac{\cos 4\theta + 1}{2}$$

$$2\cos^2 2\theta - \frac{8}{3}\sin^2\theta\cos^2\theta$$

$$= 2\left(\frac{\cos 4\theta + 1}{2}\right) - \frac{8}{3}\left(\frac{1 - \cos 4\theta}{8}\right)$$

$$= \cos 4\theta + 1 - \frac{1}{3} + \frac{1}{3}\cos 4\theta$$

$$= \frac{4}{3}\cos 4\theta + \frac{2}{3}$$

Since $-1 \leq \cos 4\theta \leq 1$,

greatest value is 2, least value is $-\frac{2}{3}$

4 Differentiation

4.1 The product rule (Page 40)

$$\begin{aligned}
 1 \quad (i) \quad y &= (x-2)(x+3)^2 \\
 \frac{dy}{dx} &= 1(x+3)^2 + 2(x+3)^1 \times (x-2) \\
 &= (x^2 + 6x + 9) + 2(x^2 + x - 6) \\
 &= 3x^2 + 8x - 3 \\
 (ii) \quad y &= x^3(1-2x)^4 \\
 \frac{dy}{dx} &= 3x^2(1-2x)^4 + x^3 \times 4(1-2x)^3 \times -2 \\
 &= 3x^2(1-2x)^4 - 8x^3(1-2x)^3 \\
 &= x^2(1-2x)^3 [3(1-2x) - 8x] \\
 &= x^2(1-2x)^3 [3 - 6x - 8x] \\
 &= x^2(1-2x)^3 (3 - 14x)
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad y &= 3x^2\sqrt{1+4x} \\
 \frac{dy}{dx} &= 6x\sqrt{1+4x} + \frac{1}{2}(1+4x)^{-\frac{1}{2}} \times 4 \times 3x^2 \\
 &= 6x\sqrt{1+4x} + \frac{6x^2}{\sqrt{1+4x}} \\
 &= \frac{6x(1+4x) + 6x^2}{\sqrt{1+4x}} \\
 &= \frac{6x + 30x^2}{\sqrt{1+4x}} \\
 &= \frac{6x(1+5x)}{\sqrt{1+4x}}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad f(x) &= 4x(x-3)^5 \\
 f'(x) &= 4(x-3)^5 + 5(x-3)^4 \times 4x \\
 &= 4(x-3)^4((x-3) + 5x) \\
 &= 4(x-3)^4(6x-3) \\
 f'(4) &= 4(4-3)^4(6 \times 4 - 3) = 84 \\
 y &= mx + c \Rightarrow 16 = 84 \times 4 + c \Rightarrow c = -320 \\
 y &= 84x - 320
 \end{aligned}$$

$$\begin{aligned}
 3 \quad h(x) &= x^2(x+3)^3 \\
 h'(x) &= 2x(x+3)^3 + 3(x+3)^2 \times x^2 \\
 &= x(x+3)^2 [2(x+3) + 3x] \\
 &= x(x+3)^2 [5x+6]
 \end{aligned}$$

$$\begin{aligned}
 \text{Stationary points are where } h'(x) &= 0 \\
 \Rightarrow x &= 0, -3, -\frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 4 \quad f(x) &= (x-1)^k(x+2)^{k+1} \\
 (i) \quad f'(x) &= k(x-1)^{k-1}(x+2)^{k+1} \\
 &\quad + (k+1)(x+2)^k(x-1)^k \\
 &= (x-1)^{k-1}(x+2)^k [k(x+2) + (k+1)(x-1)] \\
 &= (x-1)^{k-1}(x+2)^k [kx + 2k + kx - k + x - 1] \\
 &= (x-1)^{k-1}(x+2)^k [(2k+1)x + k - 1] \\
 (2k+1)x + k - 1 &= 0 \\
 (2k+1)x &= 1 - k \\
 x &= \frac{1-k}{2k+1}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \frac{1-k}{2k+1} &= -\frac{1}{3} \Rightarrow -3(1-k) = 2k+1 \\
 -3 + 3k &= 2k+1 \\
 k &= 4
 \end{aligned}$$

4.2 The quotient rule (Page 42)

$$\begin{aligned}
 1 \quad (i) \quad y &= \frac{x^3}{x-1} \\
 \frac{dy}{dx} &= \frac{3x^2(x-1) - 1 \times x^3}{(x-1)^2} \\
 &= \frac{3x^3 - 3x^2 - x^3}{(x-1)^2} \\
 &= \frac{2x^3 - 3x^2}{(x-1)^2} \\
 &= \frac{x^2(2x-3)}{(x-1)^2}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad y &= \frac{x+2}{3x^2} \\
 \frac{dy}{dx} &= \frac{1 \times 3x^2 - 6x(x+2)}{(3x^2)^2} \\
 &= \frac{3x^2 - 6x^2 - 12x}{(3x^2)^2} \\
 &= \frac{-3x^2 - 12x}{9x^4} \\
 &= \frac{-3x(x+4)}{9x^4} \\
 &= -\frac{(x+4)}{3x^3}
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad y &= \frac{2x}{\sqrt{6x-1}} \\
 \frac{dy}{dx} &= \frac{2\sqrt{6x-1} - \frac{1}{2}(6x-1)^{-\frac{1}{2}} \times 6 \times 2x}{6x-1} \\
 &= \frac{2\sqrt{6x-1} - \frac{6x}{\sqrt{6x-1}}}{6x-1} \\
 &= \frac{\frac{2(6x-1) - 6x}{\sqrt{6x-1}}}{6x-1} \\
 &= \frac{\frac{6x-2}{\sqrt{6x-1}}}{6x-1} \\
 &= \frac{6x-2}{\sqrt{(6x-1)^3}}
 \end{aligned}$$

$$2 \quad g(x) = \frac{2x-1}{1-x^2}$$

$$\begin{aligned} g'(x) &= \frac{2(1-x^2) - (-2x)(2x-1)}{(1-x^2)^2} \\ &= \frac{2-2x^2+4x^2-2x}{(1-x^2)^2} \\ &= \frac{2x^2-2x+2}{(1-x^2)^2} \end{aligned}$$

$$g'(2) = \frac{2(2)^2 - 2(2) + 2}{(1-(2)^2)^2} = \frac{6}{9} = \frac{2}{3}$$

$$\text{Gradient of the normal} = -\frac{3}{2}$$

$$y = mx + c \Rightarrow -1 = -\frac{3}{2} \times 2 + c \Rightarrow c = 2$$

$$y = -\frac{3}{2}x + 2$$

$$3 \quad (i) \quad y = \frac{x^2 - x + 2}{x + 1}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x-1)(x+1) - 1(x^2-x+2)}{(x+1)^2} \\ &= \frac{2x^2+x-1-x^2+x-2}{(x+1)^2} \\ &= \frac{x^2+2x-3}{(x+1)^2} \\ &= \frac{(x-1)(x+3)}{(x+1)^2} \end{aligned}$$

$$(ii) \quad \text{Stationary points when } \frac{(x-1)(x+3)}{(x+1)^2} = 0$$

$$\Rightarrow x = -3 \text{ or } 1$$

$$y = -7 \text{ or } 1$$

$$(-3, -7) \text{ and } (1, 1)$$

4.3 Differentiating natural logarithms and exponentials (Page 44)

$$1 \quad (i) \quad y = \ln(x+4)$$

$$\frac{dy}{dx} = \frac{1}{x+4}$$

$$(ii) \quad y = \ln(3x^2)$$

$$\frac{dy}{dx} = \frac{6x}{3x^2} = \frac{2}{x}$$

$$(iii) \quad y = e^{1-x}$$

$$\frac{dy}{dx} = -e^{1-x}$$

$$(iv) \quad y = 3e^{x^3+1}$$

$$\frac{dy}{dx} = 3e^{x^3+1} \times 3x^2 = 9x^2e^{x^3+1}$$

$$(v) \quad y = 3\ln\left(\frac{x}{x-1}\right) = 3(\ln x - \ln(x-1))$$

$$\frac{dy}{dx} = 3\left(\frac{1}{x} - \frac{1}{x-1}\right) = \frac{3}{x} - \frac{3}{x-1} = -\frac{3}{x(x-1)}$$

$$(vi) \quad y = 5\ln(1+\sqrt{x})^2 = 10\ln(1+\sqrt{x})$$

$$\begin{aligned} \frac{dy}{dx} &= 10\left(\frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+\sqrt{x}}\right) = 10\left(\frac{1}{2\sqrt{x}(1+\sqrt{x})}\right) \\ &= \frac{10}{2\sqrt{x}+2x} \end{aligned}$$

$$(vii) \quad y = xe^{x-2}$$

$$\frac{dy}{dx} = 1 \times e^{x-2} + e^{x-2} \times x = e^{x-2}(1+x)$$

$$(viii) \quad y = \frac{x^3}{e^x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2 \times e^x - e^x \times x^3}{(e^x)^2} = \frac{x^2e^x(3-x)}{e^{2x}} \\ &= \frac{x^2(3-x)}{e^x} \end{aligned}$$

$$(ix) \quad y = \frac{1+e^x}{1-e^{-x}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x(1-e^{-x}) - e^{-x}(1+e^x)}{(1-e^{-x})^2} \\ &= \frac{e^x - e^0 - e^{-x} - e^0}{(1-e^{-x})^2} \\ &= \frac{e^x - e^{-x} - 2}{(1-e^{-x})^2} \\ &= \frac{e^{3x} - 2e^{2x} - e^x}{e^{2x} - 2e^x + 1} \end{aligned}$$

$$(x) \quad y = \sqrt{e^x} \ln x^2 = (e^x)^{\frac{1}{2}} \times 2\ln x = e^{\frac{1}{2}x} \times 2\ln x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}e^{\frac{1}{2}x} \times 2\ln x + \frac{2}{x} \times e^{\frac{1}{2}x} \\ &= e^{\frac{1}{2}x} \ln x + \frac{2e^{\frac{1}{2}x}}{x} \\ &= e^{\frac{1}{2}x} \left(\ln x + \frac{2}{x} \right) \end{aligned}$$

$$(xi) \quad y = \frac{e^{3x}-1}{\ln(3x-1)}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3e^{3x} \ln(3x-1) - \frac{3}{3x-1}(e^{3x}-1)}{[\ln(3x-1)]^2} \\ &= \frac{3e^{3x} \ln(3x-1) - \frac{3(e^{3x}-1)}{3x-1}}{[\ln(3x-1)]^2} \end{aligned}$$

(xiii) $y = \ln(1 + e^{2x})e^{5x}$

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{2e^{2x}}{1 + e^{2x}} \right) e^{5x} + 5e^{5x} \ln(1 + e^{2x}) \\ &= \frac{2e^{7x}}{1 + e^{2x}} + 5e^{5x} \ln(1 + e^{2x})\end{aligned}$$

2 (i) $\frac{dV}{dt} = 24\,000 \times -0.04e^{-0.04t} = -960e^{-0.04t}$

At $t = 6$, $\frac{dV}{dt} = -960e^{-0.04 \times 6} = -755.16$

Decreasing in value at a rate of \$755/year.

(ii) $-960e^{-0.04t} = -594$

$$e^{-0.04t} = \frac{-594}{-960}$$

$$-0.04t = \ln\left(\frac{594}{960}\right)$$

$$t = \frac{\ln\left(\frac{594}{960}\right)}{-0.04} = 12 \text{ years}$$

3 (i) $\frac{dy}{dx} = \frac{1 \times \ln x - \frac{1}{x} \times x}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{x} \times (\ln x)^2 - 2(\ln x)^1 \times \frac{1}{x} \times (\ln x - 1)}{(\ln x)^4}$$

$$= \frac{\frac{1}{x}(\ln x)^2 - \frac{2}{x}(\ln x)^2 + \frac{2\ln x}{x}}{(\ln x)^4}$$

$$= \frac{\frac{1}{x}\ln x(\ln x - 2\ln x + 2)}{(\ln x)^4}$$

$$= \frac{2 - \ln x}{x(\ln x)^3}$$

(ii) (a) Stationary point is where $\frac{dy}{dx} = 0$

$$\frac{\ln x - 1}{(\ln x)^2} = 0 \Rightarrow \ln x - 1 = 0$$

$$\Rightarrow \ln x = 1 \Rightarrow x = e$$

$$y = \frac{e}{\ln e} = e$$

Point is (e, e)

(b) Gradient is a maximum where $\frac{d^2y}{dx^2} = 0$

$$\frac{2 - \ln x}{x(\ln x)^3} = 0 \Rightarrow 2 - \ln x = 0 \Rightarrow \ln x = 2$$

$$\Rightarrow x = e^2$$

$$y = \frac{e^2}{\ln e^2} = \frac{1}{2}e^2$$

$$\text{Point is } \left(e^2, \frac{1}{2}e^2\right)$$

4 (i) $\frac{dy}{dx} = 2xe^{-x} + -e^{-x} \times x^2 = xe^{-x}(2 - x)$

$$\frac{d^2y}{dx^2} = (2e^{-x} + -e^{-x} \times 2x) - (2xe^{-x} - x^2e^{-x})$$

$$= 2e^{-x} - 4xe^{-x} + x^2e^{-x}$$

$$= e^{-x}(x^2 - 4x + 2)$$

(ii) Stationary point is where $\frac{dy}{dx} = 0$

$$xe^{-x}(2 - x) = 0$$

$$x = 0 \text{ or } 2$$

Clearly the x coordinate of $A > 0$, so

$$A \text{ is } \left(2, \frac{4}{e^2}\right).$$

(iii) $e^{-x}(x^2 - 4x + 2) = 0$

$$x^2 - 4x + 2 = 0$$

$$x = 2 \pm \sqrt{2}$$

These points represent the x values where the gradient is a maximum or a minimum.

5 $y = x^3 \ln x$

$$\frac{dy}{dx} = 3x^2 \ln x + \frac{1}{x} \times x^3$$

$$= 3x^2 \ln x + x^2$$

$$= x^2(3\ln x + 1)$$

Stationary point is where $\frac{dy}{dx} = 0$

$$x^2(3\ln x + 1) = 0$$

$$3\ln x + 1 = 0$$

$$\ln x = -\frac{1}{3}$$

$$x = e^{-\frac{1}{3}}$$

4.4 Differentiating trigonometrical functions (Page 47)

1 (i) $y = 3\sin 2x$

$$\frac{dy}{dx} = 3\cos 2x \times 2 = 6\cos 2x$$

(ii) $y = \cos(1 + 3x)$

$$\frac{dy}{dx} = -3\sin(1 + 3x)$$

(iii) $y = \tan(x^2)$

$$\frac{dy}{dx} = \sec^2(x^2) \times 2x = 2x \sec^2(x^2)$$

(iv) $y = x^3 \sin 2x$

$$\frac{dy}{dx} = 3x^2 \sin 2x + 2\cos 2x \times x^3$$

$$= x^2(3\sin 2x + 2x \cos 2x)$$

$$(v) \quad y = \frac{\cos 5x}{x^2}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-5 \sin 5x \times x^2 - 2x \cos 5x}{(x^2)^2} \\ &= \frac{-5x^2 \sin 5x - 2x \cos 5x}{x^4} \\ &= \frac{-5x \sin 5x - 2 \cos 5x}{x^3} \end{aligned}$$

$$(vi) \quad y = \frac{\sin^3 x}{x^2}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(3(\sin x)^2 \times \cos x) \times x^2 - 2x \sin^3 x}{(x^2)^2} \\ &= \frac{3x^2 \sin^2 x \cos x - 2x \sin^3 x}{x^4} \\ &= \frac{x \sin^2 x (3 \cos x - 2 \sin x)}{x^4} \\ &= \frac{\sin^2 x (3 \cos x - 2 \sin x)}{x^3} \end{aligned}$$

$$(vii) \quad y = e^{\sin x + 1}$$

$$\frac{dy}{dx} = \cos x e^{\sin x + 1}$$

$$(viii) \quad y = \ln(\tan 2x)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2 \sec^2 2x}{\tan 2x} = \frac{\cos^2 2x}{\frac{\sin 2x}{\cos 2x}} = \frac{2}{\sin 2x \cos 2x} \\ &= \frac{4}{\sin 4x} \end{aligned}$$

$$(ix) \quad y = \sin(\ln 3x)$$

$$\frac{dy}{dx} = \cos(\ln 3x) \times \frac{3}{3x} = \frac{\cos(\ln 3x)}{x}$$

$$(x) \quad y = \tan(e^{x^2})$$

$$\frac{dy}{dx} = 2x e^{x^2} \sec^2(e^{x^2})$$

$$(xi) \quad y = \sqrt{\cos 2x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(\cos 2x)^{-\frac{1}{2}} \times -2 \sin 2x \\ &= \frac{-2 \sin 2x}{2\sqrt{\cos 2x}} \\ &= -\frac{\sin 2x}{\sqrt{\cos 2x}} \end{aligned}$$

$$(xii) \quad y = (\sin(2e^{x-1}))^3$$

$$\begin{aligned} \frac{dy}{dx} &= 3(\sin(2e^{x-1}))^2 \times \cos 2(e^{x-1}) \times 2e^{x-1} \\ &= 6e^{x-1} \sin^2(2e^{x-1}) \cos(2e^{x-1}) \end{aligned}$$

$$(xiii) \quad y = \cos^4[\ln(\sin e^x)]$$

$$\begin{aligned} \frac{dy}{dx} &= 4(\cos[\ln(\sin e^x)])^3 \times -\sin[\ln(\sin e^x)] \\ &\quad \times \frac{\cos e^x \times e^x}{\sin e^x} \\ &= -\frac{4e^x \cos e^x}{\sin e^x} \cos^3[\ln(\sin e^x)] \sin[\ln(\sin e^x)] \\ &= -4e^x \cot e^x \cos^3[\ln(\sin e^x)] \sin[\ln(\sin e^x)] \end{aligned}$$

$$(xiv) \quad y = e^{\sin^2(\ln x)}$$

$$\begin{aligned} \frac{dy}{dx} &= e^{\sin^2(\ln x)} \times 2(\sin(\ln x))^1 \times \cos(\ln x) \times \frac{1}{x} \\ &= \frac{2 \sin(\ln x) \cos(\ln x) e^{\sin^2(\ln x)}}{x} \end{aligned}$$

$$2 \quad (i) \quad y = x^2 \sin x$$

$$\frac{dy}{dx} = 2x \sin x + x^2 \cos x = x(2 \sin x + x \cos x)$$

$$(ii) \quad \text{At } x = \pi,$$

$$\frac{dy}{dx} = \pi(2 \sin \pi + \pi \cos \pi) = -\pi^2$$

$$y = mx + c \Rightarrow 0 = -\pi^2 \times \pi + c \Rightarrow c = \pi^3$$

$$y = -\pi^2 x + \pi^3$$

$$3 \quad f(x) = 2 \sin^2 3x = 2(\sin 3x)^2$$

$$\begin{aligned} f'(x) &= 4(\sin 3x)^1 \times 3 \cos 3x \\ &= 12 \sin 3x \cos 3x \\ &= 6 \sin 6x \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{18}\right) &= 6 \sin\left(6 \times \frac{\pi}{18}\right) \\ &= 6 \sin\left(\frac{\pi}{3}\right) \\ &= 6 \times \frac{\sqrt{3}}{2} \\ &= 3\sqrt{3} \end{aligned}$$

$$4 \quad (i) \quad y = \frac{e^{2x}}{\cos x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2e^{2x} \cos x - (-\sin x)e^{2x}}{\cos^2 x} \\ &= \frac{e^{2x}(2 \cos x + \sin x)}{\cos^2 x} \end{aligned}$$

$$(ii) \quad \frac{dy}{dx} = 0 \quad \text{when} \quad \frac{e^{2x}(2 \cos x + \sin x)}{\cos^2 x} = 0$$

$$2 \cos x + \sin x = 0$$

$$2 + \tan x = 0$$

$$\tan x = -2$$

$$x = \tan^{-1}(-2)$$

$$x = -1.107, 2.034$$

$$\text{Since } 0 \leq x \leq \pi, x = 2.03 \text{ (3 s.f.)}$$

$$\begin{aligned}
 5 \quad y &= 3 \sin^2 x \cos^3 x = 3(\sin x)^2 (\cos x)^3 \\
 \frac{dy}{dx} &= 6(\sin x)^1 \times \cos x \times (\cos x)^3 \\
 &\quad + 3(\cos x)^2 \times -\sin x \times 3(\sin x)^2 \\
 &= 6 \sin x \cos^4 x - 9 \sin^3 x \cos^2 x \\
 &= 3 \sin x \cos^2 x (2 \cos^2 x - 3 \sin^2 x) \\
 \frac{dy}{dx} &= 0 \text{ when } 3 \sin x \cos^2 x (2 \cos^2 x - 3 \sin^2 x) = 0
 \end{aligned}$$

$$2 \cos^2 x - 3 \sin^2 x = 0$$

$$2 - 3 \tan^2 x = 0$$

$$\tan^2 x = \frac{2}{3}$$

$$\tan x = \pm \sqrt{\frac{2}{3}}$$

$$\text{Since } 0 \leq x \leq \frac{\pi}{2}, \quad x = 0.685$$

$$\begin{aligned}
 6 \quad y &= \frac{\cos 2x}{e^{2x}} \\
 \frac{dy}{dx} &= \frac{-2 \sin 2x \times e^{2x} - 2e^{2x} \cos 2x}{(e^{2x})^2} \\
 &= \frac{-2e^{2x} (\sin 2x + \cos 2x)}{e^{4x}}
 \end{aligned}$$

$$\frac{dy}{dx} = 0 \text{ when } \frac{-2e^{2x} (\sin 2x + \cos 2x)}{e^{4x}} = 0$$

$$\sin 2x + \cos 2x = 0$$

$$\tan 2x + 1 = 0$$

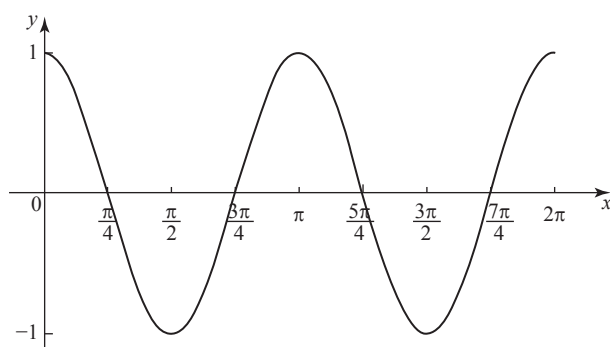
$$\tan 2x = -1$$

$$2x = \tan^{-1}(-1)$$

$$2x = -\frac{1}{4}\pi, \frac{3}{4}\pi, \frac{7}{4}\pi, \frac{11}{4}\pi, \dots$$

$$x = \frac{3}{8}\pi, \frac{7}{8}\pi$$

$$7 \quad \text{(i)} \quad y = \ln(\cos 2x) + 2x \text{ is undefined when } \cos 2x \leq 0.$$



From the graph of $\cos 2x$,

$$\frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \text{ and } \frac{5\pi}{4} \leq x \leq \frac{7\pi}{4}$$

$$\text{(ii)} \quad \frac{dy}{dx} = \frac{-2 \sin 2x}{\cos 2x} + 2$$

$$\frac{-2 \sin 2x}{\cos 2x} + 2 = 0$$

$$-2 \tan 2x = -2$$

$$\tan 2x = 1$$

$$2x = \tan^{-1}(1)$$

$$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

Since $\ln(\cos 2x)$ is only defined when $\cos 2x > 0$,

$$2x \neq \frac{5\pi}{4} \text{ or } \frac{13\pi}{4}$$

$$\text{So } x = \frac{\pi}{8}, \frac{9\pi}{8}$$

$$\text{(iii)} \quad \frac{dy}{dx} = \frac{-2 \sin 2x}{\cos 2x} + 2 = -2 \tan 2x + 2$$

$$\frac{d^2 y}{dx^2} = -4 \sec^2 2x$$

Since $\sec^2 \theta \geq 0$, $-4 \sec^2 2x \leq 0$ so all of the points are maximum points.

4.5 Differentiating functions defined implicitly (Page 52)

$$1 \quad \text{(i)} \quad \frac{d}{dx} 2y^3 = 6y^2 \frac{dy}{dx}$$

$$\text{(ii)} \quad \frac{d}{dx} (3x^2 - 5y^4 - 8) = 6x - 20y^3 \frac{dy}{dx}$$

$$\text{(iii)} \quad \frac{d}{dx} (\sin 2x + \cos 2y) = 2 \cos 2x - 2 \sin 2y \frac{dy}{dx}$$

$$\text{(iv)} \quad \frac{d}{dx} e^{3y} = 3e^{3y} \frac{dy}{dx}$$

$$\text{(v)} \quad \frac{d}{dx} (4x^2 y) = 8xy + 4x^2 \frac{dy}{dx}$$

$$\text{(vi)} \quad \ln(xy) = \ln x + \ln y$$

$$\frac{d}{dx} (\ln x + \ln y) = \frac{1}{x} + \frac{1}{y} \frac{dy}{dx}$$

$$\text{(vii)} \quad \frac{d}{dx} (\tan xy^2 - e^y)$$

$$= \sec^2(xy^2) \times \left(y^2 + 2xy \frac{dy}{dx} \right) - e^y \frac{dy}{dx}$$

$$\text{(viii)} \quad (e^{\sin y})^x = e^{x \sin y}$$

$$\frac{d}{dx} (e^{x \sin y}) = e^{x \sin y} \times \left(\sin y + \cos y \frac{dy}{dx} \times x \right)$$

$$\text{(ix)} \quad \frac{d}{dx} \left(\frac{x \sin y}{y \sin x} \right)$$

$$\left(\sin y + \cos y \frac{dy}{dx} \times x \right) y \sin x$$

$$= \frac{-\left(\frac{dy}{dx} \sin x + y \cos x \right) x \sin y}{(y \sin x)^2}$$

$$\text{(x)} \quad \frac{d}{dx} (y^3 \ln(\sin xy))$$

$$= 3y^2 \frac{dy}{dx} \ln(\sin xy) + \frac{\cos xy \times \left(y + x \frac{dy}{dx} \right)}{\sin xy} y^3$$

2 (i) $y^2 - 2x^3 = 5$

$$2y \frac{dy}{dx} - 6x^2 = 0$$

$$\frac{dy}{dx} = \frac{6x^2}{2y} = \frac{3x^2}{y}$$

(ii) $x^2y = \sin y$

$$2xy + x^2 \frac{dy}{dx} = \cos y \frac{dy}{dx}$$

$$x^2 \frac{dy}{dx} - \cos y \frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx}(x^2 - \cos y) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - \cos y} = \frac{2xy}{\cos y - x^2}$$

(iii) $2y^3 + x = 4xy$

$$6y^2 \frac{dy}{dx} + 1 = 4y + 4x \frac{dy}{dx}$$

$$(6y^2 - 4x) \frac{dy}{dx} = 4y - 1$$

$$\frac{dy}{dx} = \frac{4y - 1}{6y^2 - 4x}$$

(iv) $e^{xy} - 2x = 12$

$$e^{xy} \times \left(y + x \frac{dy}{dx} \right) - 2 = 0$$

$$y + x \frac{dy}{dx} = \frac{2}{e^{xy}}$$

$$x \frac{dy}{dx} = \frac{2}{e^{xy}} - y$$

$$\frac{dy}{dx} = \frac{2}{xe^{xy}} - \frac{y}{x}$$

3 $x^2 = 2\cos y$

$$2x = -2\sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{-2\sin y} = -\frac{x}{\sin y}$$

At the point $\left(1, \frac{\pi}{3}\right)$,

$$\frac{dy}{dx} = \frac{1}{-\sin \frac{\pi}{3}} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$$

$$y = mx + c \Rightarrow \frac{\pi}{3} = -\frac{2}{\sqrt{3}} \times 1 + c \Rightarrow c = \frac{\pi}{3} + \frac{2}{\sqrt{3}}$$

$$y = -\frac{2}{\sqrt{3}}x + \frac{\pi}{3} + \frac{2}{\sqrt{3}}$$

4 $\ln(2xy) + y^2 = 1$

$$\ln(2x) + \ln y + y^2 = 1$$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

At the point $\left(\frac{1}{2}, 1\right)$,

$$\frac{1}{\frac{1}{2}} + \frac{1}{1} \frac{dy}{dx} + 2 \times 1 \frac{dy}{dx} = 0$$

$$2 + \frac{dy}{dx} + 2 \frac{dy}{dx} = 0$$

$$3 \frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = -\frac{2}{3}$$

$$\Rightarrow m(\text{normal}) = \frac{3}{2}$$

$$y = mx + c \Rightarrow 1 = \frac{3}{2} \times \frac{1}{2} + c \Rightarrow c = \frac{1}{4}$$

$$y = \frac{3}{2}x + \frac{1}{4}$$

5 $4x^2 + y^2 = 8$

$$8x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-8x}{2y} = -\frac{4x}{y}$$

$$-\frac{4x}{y} = 2 \Rightarrow -4x = 2y \Rightarrow y = -2x$$

$$4x^2 + (-2x)^2 = 8$$

$$8x^2 = 8$$

$$x^2 = 1$$

$$x = \pm 1$$

$$y = \mp 2$$

Points are $(1, -2)$ and $(-1, 2)$

6 $x^2 + 6y^2 - 2x + 8y = 39$

$$2x + 12y \frac{dy}{dx} - 2 + 8 \frac{dy}{dx} = 0$$

$$(12y + 8) \frac{dy}{dx} = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{12y + 8} = \frac{1 - x}{6y + 4}$$

Parallel to the x -axis when $\frac{dy}{dx} = 0$

$$\frac{1 - x}{6y + 4} = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$$

$$1^2 + 6y^2 - 2 \times 1 + 8y = 39$$

$$6y^2 + 8y - 40 = 0$$

$$3y^2 + 4y - 20 = 0$$

$$(3y + 10)(y - 2) = 0$$

$$y = -\frac{10}{3} \text{ or } 2$$

Points are $\left(1, -\frac{10}{3}\right)$ and $(1, 2)$

7 (i) $5x^2 - 2xy + 3y^2 - 70 = 0$

$$10x - \left(2y + 2x \frac{dy}{dx}\right) + 6y \frac{dy}{dx} = 0$$

$$10x - 2y - 2x \frac{dy}{dx} + 6y \frac{dy}{dx} = 0$$

$$(6y - 2x) \frac{dy}{dx} = 2y - 10x$$

$$\frac{dy}{dx} = \frac{2y - 10x}{6y - 2x} = \frac{y - 5x}{3y - x}$$

(ii) Parallel to the x -axis when $\frac{dy}{dx} = 0$

$$\Rightarrow y - 5x = 0 \Rightarrow y = 5x$$

$$5x^2 - 2x(5x) + 3(5x)^2 - 70 = 0$$

$$5x^2 - 10x^2 + 75x^2 - 70 = 0$$

$$70x^2 = 70$$

$$x^2 = 1$$

$$x = 1 \text{ or } -1$$

$$y = 5 \text{ or } -5$$

Points are (1, 5) and (-1, -5)

(iii) Parallel to the y -axis $\Rightarrow \frac{dy}{dx}$ is undefined

$$\Rightarrow 3y - x = 0 \Rightarrow x = 3y$$

$$5(3y)^2 - 2(3y)y + 3y^2 - 70 = 0$$

$$45y^2 - 6y^2 + 3y^2 - 70 = 0$$

$$42y^2 = 70$$

$$y^2 = \frac{70}{42}$$

$$y = \pm \frac{\sqrt{15}}{3}$$

$$x = \pm \sqrt{15}$$

Points are $\left(\sqrt{15}, \frac{\sqrt{15}}{3}\right)$ and $\left(-\sqrt{15}, -\frac{\sqrt{15}}{3}\right)$

8 $y^2 + y = x^3 + 2x$

(i) $y^2 + y = 2^3 + 2 \times 2$

$$y^2 + y = 12$$

$$y^2 + y - 12 = 0$$

$$(y + 4)(y - 3) = 0$$

$$y = -4 \text{ or } 3$$

Points are (2, -4) and (2, 3)

(ii) $2y \frac{dy}{dx} + \frac{dy}{dx} = 3x^2 + 2$

$$(2y + 1) \frac{dy}{dx} = 3x^2 + 2$$

$$\frac{dy}{dx} = \frac{3x^2 + 2}{2y + 1}$$

At (2, -4), $\frac{dy}{dx} = \frac{3 \times 2^2 + 2}{2 \times -4 + 1} = -2$

At (2, 3), $\frac{dy}{dx} = \frac{3 \times 2^2 + 2}{2 \times 3 + 1} = 2$

9 $x^3 + 4x^2y + y^3 = 6$

$$3x^2 + (8xy + 4x^2 \frac{dy}{dx}) + 3y^2 \frac{dy}{dx} = 0$$

$$(4x^2 + 3y^2) \frac{dy}{dx} = -3x^2 - 8xy$$

$$\frac{dy}{dx} = \frac{-3x^2 - 8xy}{4x^2 + 3y^2}$$

At (1, 1), $\frac{dy}{dx} = \frac{-3(1)^2 - 8(1)(1)}{4(1)^2 + 3(1)^2} = -\frac{11}{7}$

Gradient of normal = $\frac{7}{11}$

$$y = mx + c \Rightarrow 1 = \frac{7}{11} \times 1 + c \Rightarrow c = \frac{4}{11}$$

$$y = \frac{7}{11}x + \frac{4}{11}$$

$$11y = 7x + 4$$

$$7x - 11y + 4 = 0$$

4.6 Parametric equations and parametric differentiation (Page 57)

1 (i) $\frac{dy}{dt} = 2t, \frac{dx}{dt} = 2$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{2} = t$$

(ii) $\frac{dy}{d\theta} = 3\cos\theta, \frac{dx}{d\theta} = -2\sin\theta$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= 3\cos\theta \times \frac{1}{-2\sin\theta} \\ &= \frac{3\cos\theta}{-2\sin\theta} \\ &= -\frac{3}{2}\cot\theta \end{aligned}$$

(iii) $\frac{dy}{dt} = 1 - 2e^{2t}, \frac{dx}{dt} = 3e^t$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= (1 - 2e^{2t}) \times \frac{1}{3e^t} \\ &= \frac{1 - 2e^{2t}}{3e^t} \end{aligned}$$

(iv) $\frac{dy}{d\theta} = 5\sec^2\theta, \frac{dx}{d\theta} = 2\sec\theta \tan\theta$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= 5\sec^2\theta \times \frac{1}{2\sec\theta \tan\theta} \\ &= \frac{5\sec\theta}{2\tan\theta} \\ &= \frac{5 \frac{1}{\cos\theta}}{2 \frac{\sin\theta}{\cos\theta}} = \frac{5}{2} \operatorname{cosec}\theta \end{aligned}$$

2 (i) $x = \frac{t}{1-t}, y = \frac{t^2}{1+t}$

$$\frac{dx}{dt} = \frac{(1-t) - (-1)t}{(1-t)^2} = \frac{1}{(1-t)^2}$$

$$\frac{dy}{dt} = \frac{2t(1+t) - t^2}{(1+t)^2} = \frac{2t+t^2}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t+t^2}{(1+t)^2} \times (1-t)^2$$

(ii) Stationary points when $\frac{dy}{dx} = 0$

$$(1-t)^2 = 0 \Rightarrow t = 1$$

But the curve is undefined for $t = 1$

$$2t + t^2 = 0$$

$$t(2+t) = 0$$

$$t = 0 \text{ or } t = -2$$

$$t = 0 \Rightarrow (0, 0), t = -2 \Rightarrow \left(-\frac{2}{3}, -4\right)$$

3 (i) $x = 3e^{4t}, y = 6te^{2t}$

$$\frac{dx}{dt} = 12e^{4t}, \frac{dy}{dt} = 6e^{2t} + 2e^{2t} \times 6t = 6e^{2t}(1+2t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 6e^{2t}(1+2t) \times \frac{1}{12e^{4t}} = \frac{1+2t}{2e^{2t}}$$

(ii) When $t = 0, x = 3$ and $y = 0$

$$\frac{dy}{dx} = \frac{1+2 \times 0}{2e^{2 \times 0}} = \frac{1}{2}$$

Tangent is $y = mx + c$

$$0 = \frac{1}{2} \times 3 + c \Rightarrow c = -\frac{3}{2}$$

$$\text{Tangent is } y = \frac{1}{2}x - \frac{3}{2} \text{ or } x - 2y - 3 = 0$$

4 $x = 2\sin\theta + \cos\theta$

$$y = \sin\theta + 2\cos\theta$$

(i) $\frac{dy}{d\theta} = \cos\theta - 2\sin\theta, \frac{dx}{d\theta} = 2\cos\theta - \sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= (\cos\theta - 2\sin\theta) \times \frac{1}{2\cos\theta - \sin\theta}$$

$$= \frac{\cos\theta - 2\sin\theta}{2\cos\theta - \sin\theta}$$

$$\text{When } \theta = \frac{\pi}{2}, \frac{dy}{dx} = \frac{\cos\frac{\pi}{2} - 2\sin\frac{\pi}{2}}{2\cos\frac{\pi}{2} - \sin\frac{\pi}{2}}$$

$$= \frac{0-2}{0-1} = 2$$

$$\text{When } \theta = \frac{\pi}{2}, x = 2\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 2$$

$$y = \sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} = 1$$

$$y = mx + c \Rightarrow 1 = 2 \times 2 + c \Rightarrow c = -3$$

Equation of the tangent is $y = 2x - 3$

(ii) (a) $x^2 = (2\sin\theta + \cos\theta)^2$
 $= 4\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta$

$$y^2 = (\sin\theta + 2\cos\theta)^2$$

$$= \sin^2\theta + 4\sin\theta\cos\theta + 4\cos^2\theta$$

$$x^2 + y^2 = 4\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta$$

$$+ \sin^2\theta + 4\sin\theta\cos\theta + 4\cos^2\theta$$

$$= 5\sin^2\theta + 8\sin\theta\cos\theta + 5\cos^2\theta$$

$$= 5(\sin^2\theta + \cos^2\theta) + 8\sin\theta\cos\theta$$

$$= 5 + 4(2\sin\theta\cos\theta)$$

$$= 5 + 4\sin 2\theta$$

(b) Since $-1 \leq \sin 2\theta \leq 1, 1 \leq 5 + 4\sin 2\theta \leq 9$

Distance of a point from the origin is $\sqrt{x^2 + y^2}$

So least value is 1, greatest value is 3.

5 $x = 2t - \ln 2t, y = t^2 - \ln t^2, t > 0$

(i) $\frac{dx}{dt} = 2 - \frac{1}{t} = \frac{2t-1}{t}, \frac{dy}{dt} = 2t - \frac{2}{t} = \frac{2t^2-2}{t}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t^2-2}{t} \times \frac{t}{2t-1} = \frac{2t^2-2}{2t-1}$$

Stationary point is where $\frac{dy}{dx} = 0$

$$\frac{2t^2-2}{2t-1} = 0$$

$$2t^2 - 2 = 0$$

$$2(t^2 - 1) = 0$$

$$t^2 = 1$$

$$t = \pm 1$$

Since $t > 0, t = 1$

When $t = 1, x = 2 - \ln 2, y = 1 - \ln 1 = 1$

Point is $(2 - \ln 2, 1)$

(ii) $\frac{2t^2-2}{2t-1} = 2$

$$2t^2 - 2 = 2(2t - 1)$$

$$2t^2 - 2 = 4t - 2$$

$$2t^2 - 4t = 0$$

$$2t(t - 2) = 0$$

$$t = 0 \text{ or } 2$$

Since $t > 0, t = 2$

When $t = 2, x = 4 - \ln 4, y = 4 - \ln 4$

Point is $(4 - \ln 4, 4 - \ln 4)$

6 $x = 2\theta + \sin 2\theta, y = 4\sin\theta$

(i) Maximum y value is when $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$x = 2 \times \frac{\pi}{2} + \sin 2 \times \frac{\pi}{2} = \pi \text{ so A is } (\pi, 4).$$

B is when $y = 0$ so $4\sin\theta = 0$

$$\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \dots$$

Since B is the third time the curve crosses the x axis for $\theta \geq 0$, the value of θ at B must be 2π .

B is $(4\pi, 0)$

$$(ii) \quad \frac{dx}{d\theta} = 2 + 2\cos 2\theta, \quad \frac{dy}{d\theta} = 4\cos \theta$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= 4\cos \theta \times \frac{1}{2 + 2\cos 2\theta} \\ &= \frac{4\cos \theta}{2 + 2\cos 2\theta} \\ &= \frac{4\cos \theta}{2(1 + \cos 2\theta)} \\ &= \frac{2\cos \theta}{1 + (2\cos^2 \theta - 1)} \\ &= \frac{2\cos \theta}{2\cos^2 \theta} \\ &= \frac{1}{\cos \theta} \\ &= \sec \theta \end{aligned}$$

$$(iii) \quad C \text{ is where } \frac{dy}{dx} = 2$$

$$\sec \theta = 2 \Rightarrow \frac{1}{\cos \theta} = 2 \Rightarrow \cos \theta = \frac{1}{2}$$

$$\theta = -\frac{\pi}{3}$$

$$\text{When } \theta = -\frac{\pi}{3},$$

$$x = 2 \times -\frac{\pi}{3} + \sin\left(2 \times -\frac{\pi}{3}\right) = -\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$y = 4\sin\left(-\frac{\pi}{3}\right) = 4 \times -\frac{\sqrt{3}}{2} = -2\sqrt{3}$$

$$\text{So C is } \left(-\frac{2\pi}{3} - \frac{\sqrt{3}}{2}, -2\sqrt{3}\right)$$

$$7 \quad x = 10\cos \theta + 5\cos 2\theta, \quad y = 10\sin \theta + 5\sin 2\theta$$

$$(i) \quad \frac{dx}{d\theta} = -10\sin \theta - 10\sin 2\theta$$

$$\frac{dy}{d\theta} = 10\cos \theta + 10\cos 2\theta$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= (10\cos \theta + 10\cos 2\theta) \times \frac{1}{-10\sin \theta - 10\sin 2\theta} \\ &= \frac{10\cos \theta + 10\cos 2\theta}{-10\sin \theta - 10\sin 2\theta} \\ &= \frac{\cos \theta + \cos 2\theta}{-\sin \theta - \sin 2\theta} \\ &= \frac{\cos \theta + \cos 2\theta}{-(\sin \theta + \sin 2\theta)} \\ &= -\frac{\cos \theta + \cos 2\theta}{\sin \theta + \sin 2\theta} \end{aligned}$$

$$(ii) \quad \text{Highest point is where } \frac{dy}{dx} = 0$$

$$-\frac{\cos \theta + \cos 2\theta}{\sin \theta + \sin 2\theta} = 0 \Rightarrow \cos \theta + \cos 2\theta = 0$$

$$\cos \theta + (2\cos^2 \theta - 1) = 0$$

$$2\cos^2 \theta + \cos \theta - 1 = 0$$

$$(2\cos \theta - 1)(\cos \theta + 1) = 0$$

$$\cos \theta = \frac{1}{2} \text{ or } -1$$

$$\text{From the graph, } \theta = \frac{\pi}{3}. \text{ When } \theta = \frac{\pi}{3},$$

$$x = 10\cos \frac{\pi}{3} + 5\cos \frac{2\pi}{3} = \frac{5}{2}$$

$$y = 10\sin \frac{\pi}{3} + 5\sin \frac{2\pi}{3} = \frac{15\sqrt{3}}{2}$$

$$\text{Hence A is } \left(\frac{5}{2}, \frac{15\sqrt{3}}{2}\right)$$

$$(iii) \quad x^2 = (10\cos \theta + 5\cos 2\theta)^2$$

$$= 100\cos^2 \theta + 100\cos \theta \cos 2\theta + 25\cos^2 2\theta$$

$$y^2 = (10\sin \theta + 5\sin 2\theta)^2$$

$$= 100\sin^2 \theta + 100\sin \theta \sin 2\theta + 25\sin^2 2\theta$$

$$\begin{aligned} x^2 + y^2 &= 100\cos^2 \theta + 100\cos \theta \cos 2\theta + 25\cos^2 2\theta \\ &\quad + 100\sin^2 \theta + 100\sin \theta \sin 2\theta + 25\sin^2 2\theta \\ &= 100(\cos^2 \theta + \sin^2 \theta) + (25\cos^2 2\theta + \sin^2 2\theta) \\ &\quad + 100(\cos \theta \cos 2\theta + \sin \theta \sin 2\theta) \\ &= 125 + 100(\cos(2\theta - \theta)) \\ &= 125 + 100\cos \theta \end{aligned}$$

$$(iv) \quad \text{Since } -1 \leq \cos \theta \leq 1, \text{ least distance is 5 m,}$$

$$\text{greatest distance is 15 m.}$$

Further practice (Page 62)

$$1 \quad y = e^{-\frac{1}{4}x} \sqrt{3+x^2}$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{4}e^{-\frac{1}{4}x} \sqrt{3+x^2} + \frac{1}{2}(3+x^2)^{-\frac{1}{2}} \times 2x \times e^{-\frac{1}{4}x} \\ &= -\frac{1}{4}e^{-\frac{1}{4}x} \sqrt{3+x^2} + \frac{xe^{-\frac{1}{4}x}}{\sqrt{3+x^2}} \\ &= \frac{-\frac{1}{4}e^{-\frac{1}{4}x}(3+x^2) + xe^{-\frac{1}{4}x}}{\sqrt{3+x^2}} \\ &= \frac{e^{-\frac{1}{4}x} \left(-\frac{3}{4} - \frac{1}{4}x^2 + x\right)}{\sqrt{3+x^2}} \end{aligned}$$

Stationary points are where $\frac{dy}{dx} = 0$

$$-\frac{3}{4} - \frac{1}{4}x^2 + x = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

$$x = 1 \text{ or } 3$$

2 $y = \frac{(\ln x)^2}{x}$

$$\frac{dy}{dx} = \frac{2(\ln x)^1 \times \frac{1}{x} \times x - 1 \times (\ln x)^2}{x^2}$$

$$= \frac{2\ln x - (\ln x)^2}{x^2}$$

$$= \frac{\ln x(2 - \ln x)}{x^2}$$

Stationary point is where $\frac{dy}{dx} = 0$

$$\frac{\ln x(2 - \ln x)}{x^2} = 0$$

$$\ln x = 0 \text{ or } 2 - \ln x = 0$$

$$x = 1 \text{ or } e^2$$

$$y = 0 \text{ or } \frac{(\ln e^2)^2}{e^2} = \frac{4}{e^2}$$

$$\text{So M is } \left(e^2, \frac{4}{e^2}\right)$$

3 $y = \sin 2x + x$

$$\frac{dy}{dx} = 2\cos 2x + 1$$

Stationary points are where $\frac{dy}{dx} = 0$

$$2\cos 2x + 1 = 0$$

$$2\cos 2x = -1$$

$$\cos 2x = -\frac{1}{2}$$

$$2x = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\frac{d^2y}{dx^2} = -4\sin 2x$$

$$\text{When } x = \frac{\pi}{3}, \frac{d^2y}{dx^2} = -4\sin 2 \times \frac{\pi}{3} < 0$$

$$\Rightarrow x = \frac{\pi}{3} \text{ is a maximum.}$$

$$\text{When } x = \frac{2\pi}{3}, \frac{d^2y}{dx^2} = -4\sin 2 \times \frac{2\pi}{3} > 0$$

$$\Rightarrow x = \frac{2\pi}{3} \text{ is a minimum.}$$

4 $x^2 + y^2 - xy - 48 = 0$

(i) $2x + 2y\frac{dy}{dx} - \left(y + x\frac{dy}{dx}\right) = 0$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x} = \frac{2x - y}{x - 2y}$$

(ii) Points where the tangent is parallel to the x -axis are where $\frac{dy}{dx} = 0$

$$\frac{2x - y}{x - 2y} = 0 \Rightarrow 2x - y = 0 \Rightarrow y = 2x$$

$$x^2 + (2x)^2 - x(2x) - 48 = 0$$

$$x^2 + 4x^2 - 2x^2 - 48 = 0$$

$$3x^2 = 48$$

$$x^2 = 16$$

$$x = \pm 4$$

$$y = \pm 8$$

Points are (4, 8) and (-4, -8)

(iii) Points where the tangent is parallel to the y -axis are where $\frac{dy}{dx}$ is undefined.

$$\frac{2x - y}{x - 2y} \text{ is undefined when } x - 2y = 0 \Rightarrow x = 2y$$

$$(2y)^2 + y^2 - (2y)y - 48 = 0$$

$$4y^2 + y^2 - 2y^2 - 48 = 0$$

$$3y^2 = 48$$

$$y^2 = 16$$

$$y = \pm 4$$

$$x = \pm 8$$

Points are (8, 4) and (-8, -4)

5 $2x^2 + xy + y^2 = 14$

$$4x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -4x - y$$

$$\frac{dy}{dx} = \frac{-4x - y}{x + 2y}$$

Stationary points are where $\frac{dy}{dx} = 0$

$$-4x - y = 0 \Rightarrow y = -4x$$

$$2x^2 + x(-4x) + (-4x)^2 = 14$$

$$2x^2 - 4x^2 + 16x^2 = 14$$

$$14x^2 = 14$$

$$x^2 = 1$$

$$x = \pm 1$$

$$y = \mp 4$$

Points are (1, -4) and (-1, 4)

6 $x = \frac{2t}{3t+4}, y = 3\ln(3t+4)$

$$\begin{aligned} \text{(i)} \quad \frac{dx}{dt} &= \frac{2(3t+4) - 3(2t)}{(3t+4)^2} = \frac{8}{(3t+4)^2}, \frac{dy}{dt} = \frac{9}{3t+4} \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = \frac{9}{3t+4} \times \frac{(3t+4)^2}{8} \\ &= \frac{9(3t+4)}{8} \\ &= \frac{27t+36}{8} \end{aligned}$$

ii) When $x = 2, 2 = \frac{2t}{3t+4} \Rightarrow 2(3t+4) = 2t$

$$6t + 8 = 2t$$

$$4t = -8$$

$$t = -2$$

$$\text{When } t = -2, \frac{dy}{dx} = \frac{27 \times -2 + 36}{8} = -\frac{9}{4}$$

Past exam questions (Page 63)

1 $x^2 + 2xy - y^2 + 8 = 0$

ii) $2x + 2y + 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$

At $(-2, 2),$

$$2(-2) + 2(2) + 2(-2) \frac{dy}{dx} - 2(2) \frac{dy}{dx} = 0$$

$$-4 + 4 - 4 \frac{dy}{dx} - 4 \frac{dy}{dx} = 0$$

$$-8 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0$$

So tangent is parallel to the x -axis.

ii) When $x = -2,$

$$(-2)^2 + 2(-2)(y) - y^2 + 8 = 0$$

$$4 - 4y - y^2 + 8 = 0$$

$$y^2 + 4y - 12 = 0$$

$$(y-2)(y+6) = 0$$

So $y = 2$ or $y = -6$

The other point is $(-2, -6)$

$$2x + 2y + (2x - 2y) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x-y}{x-y}$$

$$= \frac{x+y}{y-x}$$

$$\text{At } (-2, -6), \frac{dy}{dx} = \frac{(-2) + (-6)}{(-6) - (-2)} = \frac{-8}{-4} = 2$$

Equation of the tangent is $y = mx + c$

$$-6 = 2(-2) + c \Rightarrow c = -2$$

So $y = 2x - 2$

2 $x = \ln(\tan t), y = \sin^2 t$

ii) $\frac{dx}{dt} = \frac{\sec^2 t}{\tan t}, \frac{dy}{dt} = 2 \sin t \cos t$

$$= \frac{1}{\frac{\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t}$$

$$= \frac{1}{\sin t \cos t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2 \sin t \cos t \times \sin t \cos t$$

$$= 2 \sin^2 t \cos^2 t$$

ii) When $x = 0,$

$$\ln(\tan t) = 0$$

$$\tan t = 1$$

$$t = \frac{\pi}{4}$$

$$\text{When } t = \frac{\pi}{4}, \frac{dy}{dx} = 2 \sin^2 \frac{\pi}{4} \cos^2 \frac{\pi}{4}$$

$$= 2 \left(\frac{1}{\sqrt{2}} \right)^2 \left(\frac{1}{\sqrt{2}} \right)^2$$

$$= \frac{1}{2}$$

$$y = \sin^2 \frac{\pi}{4} = \frac{1}{2}$$

Equation of the tangent is $y = mx + c$

$$\frac{1}{2} = \frac{1}{2}(0) + c \Rightarrow c = \frac{1}{2}$$

$$\text{So } y = \frac{1}{2}x + \frac{1}{2}$$

3 $x = 1 + 2 \sin^2 \theta, y = 4 \tan \theta$

ii) $\frac{dx}{d\theta} = 4 \sin \theta \cos \theta, \frac{dy}{d\theta} = 4 \sec^2 \theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 4 \sec^2 \theta \times \frac{1}{4 \sin \theta \cos \theta}$$

$$= \frac{4}{\cos^2 \theta} \times \frac{1}{4 \sin \theta \cos \theta}$$

$$= \frac{1}{\sin \theta \cos^3 \theta}$$

ii) When $\theta = \frac{\pi}{4},$

$$\frac{dy}{dx} = \frac{1}{\sin \frac{\pi}{4} \cos^3 \frac{\pi}{4}}$$

$$= \frac{1}{\frac{1}{\sqrt{2}} \times \left(\frac{1}{\sqrt{2}} \right)^3}$$

$$= \frac{1}{\frac{1}{4}} = 4$$

$$\text{When } \theta = \frac{\pi}{4}, x = \sin^2 \frac{\pi}{4} = 2$$

$$y = 4 \tan \frac{\pi}{4} = 4$$

Equation of tangent is $y = mx + c$

$$4 = 4(2) + c \Rightarrow c = -4$$

$$\text{So } y = 4x - 4$$

$$\begin{aligned} 4 \quad y &= \frac{e^{2x}}{1 + e^{2x}} \\ \frac{dy}{dx} &= \frac{2e^{2x}(1 + e^{2x}) - 2e^{2x}(e^{2x})}{(1 + e^{2x})^2} \\ &= \frac{2e^{2x}}{(1 + e^{2x})^2} \end{aligned}$$

When $x = \ln 3$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{2e^{2\ln 3}}{(1 + e^{2\ln 3})^2} \\ &= \frac{2e^{\ln 3^2}}{(1 + e^{\ln 3^2})^2} \\ &= \frac{2(9)}{(1 + 9)^2} \\ &= \frac{18}{100} \\ &= \frac{9}{50} \end{aligned}$$

$$\begin{aligned} 5 \quad y &= \frac{\sin x}{1 + \cos x} \\ \frac{dy}{dx} &= \frac{\cos x(1 + \cos x) - (-\sin x)\sin x}{(1 + \cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} \\ &= \frac{\cos x + 1}{(1 + \cos x)^2} \\ &= \frac{1}{1 + \cos x} \end{aligned}$$

Since $-1 < \cos x \leq 1$ on the interval $-\pi < x < \pi$,

$$\text{then } \frac{1}{1 + \cos x} > 0$$

$$6 \quad x = t^2 + 1, y = 4t + \ln(2t - 1)$$

$$\begin{aligned} \text{(ii)} \quad \frac{dx}{dt} &= 2t, \quad \frac{dy}{dt} = 4 + \frac{2}{2t - 1} \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= \left(4 + \frac{2}{2t - 1}\right) \left(\frac{1}{2t}\right) \\ &= \frac{2}{t} + \frac{1}{t(2t - 1)} \\ &= \frac{4t - 1}{t(2t - 1)} \end{aligned}$$

$$\text{(ii)} \quad \text{When } t = 1, \frac{dy}{dx} = \frac{4(1) - 1}{1(2(1) - 1)} = 3$$

$$\text{Gradient of normal is } -\frac{1}{3}$$

When $t = 1$,

$$x = 1^2 + 1 = 2$$

$$y = 4(1) + \ln(2(1) - 1) = 4$$

Equation of normal is $y = mx + c$

$$4 = -\frac{1}{3} \times 2 + c \Rightarrow c = \frac{14}{3}$$

$$\text{So } y = -\frac{1}{3}x + \frac{14}{3}$$

OR

$$3y = -x + 14$$

$$x + 3y - 14 = 0$$

Stretch and challenge (Page 64)

- 1 (i) The x -coordinate of any point on the curve

$$= OD - AB$$

$$= r\theta - r \sin \theta$$

$$= r(\theta - \sin \theta)$$

The y -coordinate of any point on the curve

$$= CD - CB$$

$$= r - r \cos \theta$$

$$= r(1 - \cos \theta)$$

$$\text{(ii)} \quad \frac{dx}{d\theta} = r - r \cos \theta = r(1 - \cos \theta), \quad \frac{dy}{d\theta} = r \sin \theta$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= r \sin \theta \times \frac{1}{r(1 - \cos \theta)} \end{aligned}$$

$$= \frac{r \sin \theta}{r(1 - \cos \theta)}$$

$$= \frac{\sin \theta}{1 - \cos \theta}$$

$$= \frac{\sin \theta}{1 - \cos \theta} \times \frac{1 + \cos \theta}{1 + \cos \theta}$$

$$= \frac{\sin \theta(1 + \cos \theta)}{1 - \cos^2 \theta}$$

$$= \frac{\sin \theta(1 + \cos \theta)}{\sin^2 \theta}$$

$$= \frac{1 + \cos \theta}{\sin \theta}$$

$$\cos 2\theta = 2\cos^2 \theta - 1 \Rightarrow \cos \theta = 2\cos^2 \left(\frac{\theta}{2}\right) - 1$$

$$\cos \theta + 1 = 2\cos^2 \left(\frac{\theta}{2}\right)$$

$$\sin 2\theta = 2\sin \theta \cos \theta \Rightarrow \sin \theta = 2\sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)$$

$$\begin{aligned}\Rightarrow \frac{1 + \cos \theta}{\sin \theta} &= \frac{2 \cos^2 \left(\frac{\theta}{2} \right)}{2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right)} \\ &= \frac{\cos \left(\frac{\theta}{2} \right)}{\sin \left(\frac{\theta}{2} \right)} \\ &= \cot \left(\frac{\theta}{2} \right)\end{aligned}$$

(iii) When $y = \frac{1}{2}r$,

$$r(1 - \cos \theta) = \frac{1}{2}r$$

$$1 - \cos \theta = \frac{1}{2}$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

$$\frac{dy}{dx} = \cot \left(\frac{\theta}{2} \right) = \cot \left(\frac{\pi}{6} \right) = \sqrt{3}$$

When $y = \frac{1}{2}r$,

$$x = \frac{1}{2}r \left(\frac{\pi}{3} - \sin \frac{\pi}{3} \right) = \frac{1}{2}r \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) = \frac{r\pi}{6} - \frac{\sqrt{3}r}{4}$$

Equation of the tangent is $y = mx + c$

$$\frac{1}{2}r = \sqrt{3} \left(\frac{r\pi}{6} - \frac{\sqrt{3}r}{4} \right) + c$$

$$c = \frac{1}{2}r - \left(\frac{\sqrt{3}r\pi}{6} - \frac{3r}{4} \right)$$

$$= \frac{1}{2}r \left(1 - \frac{\sqrt{3}\pi}{3} + \frac{3}{2} \right)$$

$$= \frac{1}{2}r \left(\frac{5}{2} - \frac{\sqrt{3}\pi}{3} \right)$$

$$\text{So } y = \sqrt{3}x + \frac{1}{2}r \left(\frac{5}{2} - \frac{\sqrt{3}\pi}{3} \right)$$

(iv) $s = \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}$

$$s = \sqrt{(r - r \cos \theta)^2 + (r \sin \theta)^2}$$

$$= \sqrt{r^2 - 2r^2 \cos \theta + r^2 \cos^2 \theta + r^2 \sin^2 \theta}$$

$$= \sqrt{r^2 - 2r^2 \cos \theta + r^2 (\cos^2 \theta + \sin^2 \theta)}$$

$$= \sqrt{2r^2 - 2r^2 \cos \theta}$$

$$= \sqrt{2r^2 (1 - \cos \theta)}$$

$$= r \sqrt{2(1 - \cos \theta)}$$

(v) $a = \frac{ds}{d\theta}$

$$s = \sqrt{2r^2 (1 - \cos \theta)} = (2r^2 - 2r^2 \cos \theta)^{\frac{1}{2}}$$

$$a = \frac{ds}{d\theta} = \frac{1}{2} (2r^2 - 2r^2 \cos \theta)^{-\frac{1}{2}} \times 2r^2 \sin \theta$$

$$= \frac{r^2 \sin \theta}{\sqrt{2r^2 (1 - \cos \theta)}}$$

$$= \frac{r^2 \sin \theta}{r \sqrt{2(1 - \cos \theta)}}$$

$$= \frac{r \sin \theta}{\sqrt{2(1 - \cos \theta)}}$$

2 $x = t^3, y = t^2$

(i) $\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{3t^2} = \frac{2}{3t}$$

Equation of the tangent at $t = p$:

$$y = mx + c \Rightarrow p^2 = \frac{2}{3p} \times p^3 + c$$

$$c = p^2 - \frac{2}{3}p^2 = \frac{1}{3}p^2$$

$$y = \frac{2}{3p}x + \frac{1}{3}p^2$$

$$3py = 2x + p^3$$

$$3py - 2x = p^3$$

(ii) $3p(7) - 2(-10) = p^3$

$$p^3 - 21p - 20 = 0$$

By inspection, $p + 1$ is a factor:

$$p^3 - 21p - 20 = (p + 1)(p^2 - p - 20)$$

$$= (p + 1)(p + 4)(p - 5)$$

$$p = -4, -1 \text{ or } 5$$

$$t = -4, -1 \text{ or } 5$$

$$t = -4 \Rightarrow (-64, 16)$$

$$t = -1 \Rightarrow (-1, 1)$$

$$t = 5 \Rightarrow (125, 25)$$

3 (i) The height of the gutter is given by $10 \sin \theta$.

The area, A , is given by

$$10 \times 10 \sin \theta + \frac{1}{2} \times 10 \cos \theta \times 10 \sin \theta$$

$$= 100 \sin \theta + 50 \sin \theta \cos \theta$$

$$= 100 \sin \theta + 25(2 \sin \theta \cos \theta)$$

$$= 100 \sin \theta + 25 \sin 2\theta$$

- (ii) Maximum area is when $\frac{dA}{d\theta} = 0$

$$\frac{dA}{d\theta} = 100\cos\theta + 50\cos 2\theta$$

$$100\cos\theta + 50\cos 2\theta = 0$$

$$2\cos\theta + \cos 2\theta = 0$$

$$2\cos\theta + (2\cos^2\theta - 1) = 0$$

$$2\cos^2\theta + 2\cos\theta - 1 = 0$$

Using the quadratic formula,

$$\cos\theta = 0.366 \text{ (or } -1.366)$$

$$\theta = 1.20 \text{ or } 68.5^\circ$$

$$A = 100\sin 1.20 + 25\sin(2 \times 1.20)$$

$$= 110.1 \text{ cm}^2$$

$$\frac{d^2A}{d\theta^2} = -100\sin\theta - 100\sin 2\theta$$

$$\text{When } \theta = 1.20, \frac{d^2A}{d\theta^2} < 0$$

so this value of θ is a maximum.

4 (i) $l = \frac{2}{\cos\theta} + \frac{3}{\sin\theta}$

- (ii) Maximum length is when $\frac{dl}{d\theta} = 0$

$$l = \frac{2}{\cos\theta} + \frac{3}{\sin\theta} = 2(\cos\theta)^{-1} + 3(\sin\theta)^{-1}$$

$$\frac{dl}{d\theta} = -2(\cos\theta)^{-2} \times -\sin\theta - 3(\sin\theta)^{-2} \times \cos\theta$$

$$= \frac{2\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta}$$

$$= \frac{2\sin^3\theta - 3\cos^3\theta}{\cos^2\theta \sin^2\theta}$$

$$\frac{2\sin^3\theta - 3\cos^3\theta}{\cos^2\theta \sin^2\theta} = 0$$

$$2\sin^3\theta - 3\cos^3\theta = 0$$

$$\frac{2\sin^3\theta}{\cos^3\theta} - \frac{3\cos^3\theta}{\cos^3\theta} = \frac{0}{\cos^3\theta}$$

$$2\tan^3\theta - 3 = 0$$

$$\tan^3\theta = \frac{3}{2}$$

$$\tan\theta = \sqrt[3]{\frac{3}{2}}$$

$$\theta = \tan^{-1} \sqrt[3]{\frac{3}{2}} = 0.853 \text{ (or } 48.9^\circ)$$

When $\theta = 0.853$,

$$l = \frac{2}{\cos 0.853} + \frac{3}{\sin 0.853} = 7.02 \text{ m}$$

5 $x = t + \frac{1}{t}, y = t - \frac{1}{t}$

$$x^2 = \left(t + \frac{1}{t}\right)^2 = t^2 + 2 + \frac{1}{t^2}$$

$$y^2 = \left(t - \frac{1}{t}\right)^2 = t^2 - 2 + \frac{1}{t^2}$$

$$x^2 - y^2 = t^2 + 2 + \frac{1}{t^2} - \left(t^2 - 2 + \frac{1}{t^2}\right)$$

$$x^2 - y^2 = 4$$

6 $\tan BKC = \frac{30}{x}, \tan AKC = \frac{35.6}{x}$

$$\tan\theta = \tan(AKC - BKC)$$

$$= \frac{\tan AKC - \tan BKC}{1 + \tan AKC \times \tan BKC}$$

$$= \frac{\frac{35.6}{x} - \frac{30}{x}}{1 + \frac{35.6}{x} \times \frac{30}{x}}$$

$$= \frac{\frac{5.6}{x}}{1 + \frac{1068}{x^2}}$$

$$= \frac{\frac{5.6}{x}}{\frac{x^2 + 1068}{x^2}}$$

$$= \frac{5.6}{x} \times \frac{x^2}{x^2 + 1068}$$

$$= \frac{5.6x}{x^2 + 1068}$$

Maximum value of θ corresponds to the maximum value of $\tan\theta$.

$$\frac{d}{dx} \left(\frac{5.6x}{x^2 + 1068} \right)$$

$$= \frac{5.6(x^2 + 1068) - 2x \times 5.6x}{(x^2 + 1068)^2}$$

$$= \frac{5.6x^2 + 5980.8 - 11.2x^2}{(x^2 + 1068)^2}$$

$$= \frac{5980.8 - 5.6x^2}{(x^2 + 1068)^2}$$

$$\text{Maximum is when } 5980.8 - 5.6x^2 = 0$$

$$5.6x^2 = 5980.6$$

$$x^2 = \frac{5980.6}{5.6}$$

$$x = 32.7 \text{ m}$$

7 (i) $x = r \cos \theta = Ae^{k\theta} \cos \theta$

$$\frac{dx}{d\theta} = \cos \theta A k e^{k\theta} - \sin \theta \times A e^{k\theta}$$

$$= Ae^{k\theta} (k \cos \theta - \sin \theta)$$

$$y = r \sin \theta = Ae^{k\theta} \sin \theta$$

$$\frac{dy}{d\theta} = A k e^{k\theta} \sin \theta + A e^{k\theta} \cos \theta$$

$$= Ae^{k\theta} (k \sin \theta + \cos \theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{Ae^{k\theta} (k \sin \theta + \cos \theta)}{Ae^{k\theta} (k \cos \theta - \sin \theta)}$$

$$= \frac{k \sin \theta + \cos \theta}{k \cos \theta - \sin \theta}$$

$$= \frac{k \tan \theta + 1}{k - \tan \theta}$$

(ii) $\frac{dy}{dx} = \tan(\alpha + \theta)$

(gradient of the tangent using exterior angle of a triangle)

$$\text{So } \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} = \frac{k \tan \theta + 1}{k - \tan \theta}$$

$$= \frac{\tan \theta + \frac{1}{k}}{1 - \frac{1}{k} \tan \theta} \Rightarrow \tan \alpha = \frac{1}{k}$$

$$\alpha = \tan^{-1}\left(\frac{1}{k}\right)$$

OR use $(\tan \alpha + \tan \theta)(k - \tan \theta)$

$$= (1 - \tan \alpha \tan \theta)(k \tan \theta + 1):$$

$$(k \tan \alpha - 1)(\tan^2 \theta + 1) = 0$$

$$\tan \alpha = \frac{1}{k} \quad (\text{since } \tan^2 \theta + 1 \neq 0)$$

OR $\alpha = \tan^{-1}\left(\frac{1}{k}\right)$

$$\frac{dy}{dx} = \tan(\alpha + \theta) \text{ and from (i)}$$

$$\frac{dy}{dx} = \frac{k \sin \theta + \cos \theta}{k \cos \theta - \sin \theta}$$

$$\text{hence } \tan(\alpha + \theta) = \frac{k \sin \theta + \cos \theta}{k \cos \theta - \sin \theta}$$

and this is true for all values of θ . In particular it is true for $\theta = 0$, and then

$$\tan(\alpha + 0) = \frac{k \sin 0 + \cos 0}{k \cos 0 - \sin 0} = \frac{1}{k}$$

$$\text{so } \tan \alpha = \frac{1}{k} \text{ and } \alpha = \tan^{-1}\left(\frac{1}{k}\right).$$

5 Integration

5.1 Integrals involving the exponential function (Page 66)

1 (i) $\int e^{2x} dx = \frac{1}{2}e^{2x} + c$

(ii) $\int e^{1-3x} dx = -\frac{1}{3}e^{1-3x} + c$

(iii) $\int \frac{4}{e^{2x+3}} dx = \int 4e^{-2x-3} dx$
 $= 4 \times \frac{1}{-2} e^{-2x-3} + c$
 $= -2e^{-2x-3} + c$
 $= -\frac{2}{e^{2x+3}} + c$

(iv) $\int \frac{e^{4x} + 1}{e^x} dx = \int \left(\frac{e^{4x}}{e^x} + \frac{1}{e^x} \right) dx$
 $= \int (e^{3x} + e^{-x}) dx$
 $= \frac{1}{3}e^{3x} + (-e^{-x}) + c$
 $= \frac{1}{3}e^{3x} - \frac{1}{e^x} + c$

(v) $\int 9e^{\frac{1}{3}x} dx = 9 \times 3e^{\frac{1}{3}x} + c = 27e^{\frac{1}{3}x} + c$

(vi) $\int (e^x - 1)^2 dx = \int ((e^x)^2 - 2e^x + 1) dx$
 $= \int (e^{2x} - 2e^x + 1) dx$
 $= \frac{1}{2}e^{2x} - 2e^x + x + c$

2 $\int_1^2 3e^{4x} dx = \left[\frac{3}{4}e^{4x} \right]_1^2 = \frac{3}{4}e^{4 \times 2} - \frac{3}{4}e^{4 \times 1}$
 $= \frac{3}{4}e^8 - \frac{3}{4}e^4$
 $= \frac{3}{4}e^4(e^4 - 1)$

3 $\int_1^\infty \frac{e^x + 2}{e^{2x}} dx = \int_1^\infty \left(\frac{e^x}{e^{2x}} + \frac{2}{e^{2x}} \right) dx$
 $= \int_1^\infty (e^{-x} + 2e^{-2x}) dx$
 $= [-e^{-x} - e^{-2x}]_1^\infty$
 $= \left[-\frac{1}{e^x} - \frac{1}{e^{2x}} \right]_1^\infty$
 $= \left(-\frac{1}{e^\infty} - \frac{1}{e^{2\infty}} \right) - \left(-\frac{1}{e^1} - \frac{1}{e^2} \right)$
 $= (0) - \left(-\frac{1}{e^1} - \frac{1}{e^2} \right)$
 $= \frac{1}{e} + \frac{1}{e^2}$
 $= \frac{e+1}{e^2}$

$$\begin{aligned}
 4 \quad \int \frac{(e^x + 1)^2}{e^{2x}} dx &= \int \frac{(e^{2x} + 2e^x + 1)}{e^{2x}} dx \\
 &= \int \left(\frac{e^{2x}}{e^{2x}} + \frac{2e^x}{e^{2x}} + \frac{1}{e^{2x}} \right) dx \\
 &= \int (1 + 2e^{-x} + e^{-2x}) dx \\
 &= x - 2e^{-x} - \frac{1}{2}e^{-2x} + c \\
 &= x - \frac{2}{e^x} - \frac{1}{2e^{2x}} + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad (i) \quad \frac{dy}{dx} &= e^{\frac{1}{3}x} - 3e^{-x} \\
 y &= \int (e^{\frac{1}{3}x} - 3e^{-x}) dx \\
 &= 3e^{\frac{1}{3}x} + 3e^{-x} + c
 \end{aligned}$$

Substituting $(0, -2)$,

$$-2 = 3e^{\frac{1}{3} \times 0} + 3e^{-0} + c \Rightarrow c = -8$$

$$\text{Equation is } y = 3e^{\frac{1}{3}x} + 3e^{-x} - 8$$

$$(ii) \quad \text{You want } x \text{ so that } \frac{dy}{dx} = e^{\frac{1}{3}x} - 3e^{-x} = 0$$

$$e^{\frac{1}{3}x} - \frac{3}{e^x} = 0$$

$$e^{\frac{1}{3}x} \times e^x - \frac{3}{e^x} \times e^x = 0 \times e^x$$

$$e^{\frac{4}{3}x} - 3 = 0$$

$$e^{\frac{4}{3}x} = 3$$

$$\frac{4}{3}x = \ln 3$$

$$x = \frac{3}{4} \ln 3 \approx 0.824$$

$$\frac{d^2y}{dx^2} = \frac{1}{3}e^{\frac{1}{3}x} + 3e^{-x}$$

$$\text{When } x = \frac{3}{4} \ln 3, \frac{d^2y}{dx^2} = \frac{1}{3}e^{\frac{1}{3} \times \frac{3}{4} \ln 3} + 3e^{-\frac{3}{4} \ln 3} > 0$$

So $x = \frac{3}{4} \ln 3$ is a minimum.

$$\begin{aligned}
 6 \quad \int_0^{\ln 4} (ke^{3x} + (k-2)e^{-\frac{1}{2}x}) dx &= 185 \\
 \left[\frac{k}{3}e^{3x} - 2(k-2)e^{-\frac{1}{2}x} \right]_0^{\ln 4} &= 185 \\
 \left(\frac{k}{3}e^{3 \ln 4} - 2(k-2)e^{-\frac{1}{2} \ln 4} \right) - \left(\frac{k}{3}e^{3 \times 0} - 2(k-2)e^{-\frac{1}{2} \times 0} \right) &= 185 \\
 \left(\frac{k}{3} \times 64 - 2(k-2) \times \frac{1}{2} \right) - \left(\frac{k}{3} - 2(k-2) \right) &= 185 \\
 \left(\frac{64k}{3} - (k-2) \right) - \left(\frac{k}{3} - 2(k-2) \right) &= 185 \\
 \left(\frac{61k}{3} + 2 \right) - \left(-\frac{5k}{3} + 4 \right) &= 185 \\
 22k - 2 &= 185 \\
 22k &= 187 \\
 k &= \frac{17}{2}
 \end{aligned}$$

5.2 Integrals involving the natural logarithm function (Page 68)

$$1 \quad (i) \quad \int \frac{1}{2x} dx = \frac{1}{2} \ln|x| + c$$

$$(ii) \quad \int \frac{2}{x} dx = 2 \ln|x| + c$$

$$\begin{aligned}
 (iii) \quad \int \frac{3}{2x+1} dx &= 3 \times \frac{1}{2} \ln|2x+1| + c \\
 &= \frac{3}{2} \ln|2x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 (iv) \quad \int \left(\frac{1}{3x} + \frac{1}{1-2x} \right) dx &= \frac{1}{3} \ln|x| + \frac{1}{-2} \ln|1-2x| + c \\
 &= \frac{1}{3} \ln|x| - \frac{1}{2} \ln|1-2x| + c \\
 &= \ln|x|^{\frac{1}{3}} - \ln|1-2x|^{\frac{1}{2}} + c \\
 &= \ln \left| \frac{x^{\frac{1}{3}}}{(1-2x)^{\frac{1}{2}}} \right| + c
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \int_2^4 \frac{2}{x+1} dx &= [2 \ln|x+1|]_2^4 \\
 &= 2 \ln|4+1| - 2 \ln|2+1| \\
 &= 2 \ln|5| - 2 \ln|3| \\
 &= 2 \ln \frac{5}{3} \\
 &= \ln \left(\frac{5}{3} \right)^2 \\
 &= \ln \frac{25}{9}
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \int_{-1}^1 \frac{9}{1-3x} dx &= [-3\ln|1-3x|]_{-1}^1 \\
 &= (-3\ln|1-3 \times 1|) - (-3\ln|1-3 \times -1|) \\
 &= (-3\ln|-2|) - (-3\ln|4|) \\
 &= -3\ln 2 + 3\ln 4 \\
 &= \ln 2^{-3} + \ln 4^3 \\
 &= \ln \frac{1}{8} + \ln 64 \\
 &= \ln \left(\frac{1}{8} \times 64 \right) \\
 &= \ln 8
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \int \frac{x-1}{x^2-1} dx &= \int \frac{x-1}{(x-1)(x+1)} dx \\
 &= \int \frac{1}{x+1} dx \\
 &= \ln|x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \int \frac{3x^2-7x}{3x+2} dx \\
 (3x^2-7x) \div (3x+2) &= x-3 + \frac{6}{3x+2} \\
 \Rightarrow \int \frac{3x^2-7x}{3x+2} dx &= \int \left(x-3 + \frac{6}{3x+2} \right) dx \\
 &= \frac{x^2}{2} - 3x + 2\ln|3x+2| + c
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \int_1^k \frac{3}{3x-1} dx &= 1 \\
 [\ln|3x-1|]_1^k &= 1 \\
 \ln|3k-1| - \ln|3-1| &= 1 \\
 \ln|3k-1| - \ln|2| &= 1 \\
 \ln\left(\frac{3k-1}{2}\right) &= 1 \\
 \frac{3k-1}{2} &= e \\
 3k-1 &= 2e \\
 k &= \frac{2e+1}{3}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad (i) \quad \int_0^{\frac{\pi}{4}} \sin 2x dx &= \left[-\frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{4}} \\
 &= \left(-\frac{1}{2} \cos 2 \times \frac{\pi}{4} \right) - \left(-\frac{1}{2} \cos 2 \times 0 \right) \\
 &= (0) - \left(-\frac{1}{2} \right) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sec^2 \frac{1}{2} x dx &= \left[2 \tan \frac{1}{2} x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left(2 \tan \frac{1}{2} \times \frac{\pi}{2} \right) - \left(2 \tan \frac{1}{2} \times \frac{\pi}{3} \right) \\
 &= \left(2 \tan \frac{\pi}{4} \right) - \left(2 \tan \frac{\pi}{6} \right) \\
 &= (2 \times 1) - \left(2 \times \frac{1}{\sqrt{3}} \right) \\
 &= 2 - \frac{2}{\sqrt{3}}
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \cos x = \frac{1}{2} &\Rightarrow x = \frac{\pi}{3} \\
 A &= \int_0^{\frac{\pi}{3}} \cos x dx - \frac{\pi}{3} \times \frac{1}{2} \\
 &= [\sin x]_0^{\frac{\pi}{3}} - \frac{\pi}{6} \\
 &= \left(\sin \frac{\pi}{3} - \sin 0 \right) - \frac{\pi}{6} \\
 &= \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.342
 \end{aligned}$$

$$\begin{aligned}
 4 \quad (i) \quad \int \sin^2 x dx &= \int \frac{1}{2} (1 - \cos 2x) dx \\
 &= \frac{1}{2} x - \frac{1}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \int 4 \sin x \cos x dx &= 2 \int 2 \sin x \cos x dx \\
 &= 2 \int \sin 2x dx \\
 &= 2 \left(-\frac{1}{2} \cos 2x \right) + c \\
 &= -\cos 2x + c
 \end{aligned}$$

5.3 Integrals involving trigonometrical functions (Page 70)

$$\begin{aligned}
 1 \quad (i) \quad \int \sin 4x dx &= -\frac{1}{4} \cos 4x + c \\
 (ii) \quad \int \cos(3x-1) dx &= \frac{1}{3} \sin(3x-1) + c \\
 (iii) \quad \int \left(4 \cos \frac{1}{2} x + 1 \right) dx &= 8 \sin \frac{1}{2} x + x + c \\
 (iv) \quad \int (\cos 2x - \sec^2 3x + 4) dx \\
 &= \frac{1}{2} \sin 2x - \frac{1}{3} \tan 3x + 4x + c
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad \cos 2x = 1 - 2 \sin^2 x &\Rightarrow \sin^2 x = \frac{1}{2} (1 - \cos 2x) \\
 \sin^4 x &= \left(\frac{1}{2} (1 - \cos 2x) \right)^2 \\
 &= \frac{1}{4} (1 - 2 \cos 2x + \cos^2 2x) \\
 \cos 4x &= 2 \cos^2 2x - 1 \Rightarrow \cos^2 2x = \frac{1}{2} (\cos 4x + 1) \\
 \Rightarrow \int \sin^4 x dx \\
 &= \int \frac{1}{4} (1 - 2 \cos 2x + \cos^2 2x) dx \\
 &= \int \frac{1}{4} \left(1 - 2 \cos 2x + \frac{1}{2} (\cos 4x + 1) \right) dx \\
 &= \int \left(\frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x + \frac{1}{8} \right) dx
 \end{aligned}$$

$$= \int \left(\frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) dx$$

$$= \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c$$

(iv) $\int (\sin 4x \cos x - \cos 4x \sin x) dx$

$$= \int \sin(4x - x) dx$$

$$= \int \sin 3x dx$$

$$= -\frac{1}{3} \cos 3x + c$$

5 $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + (2\cos^2 2x - 1)} dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x} dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2} \cos 2x dx$$

$$= \sqrt{2} \times \left[\frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2} \times \left(\frac{1}{2} \sin 2 \times \frac{\pi}{4} - \frac{1}{2} \sin 0 \right)$$

$$= \sqrt{2} \times \left(\frac{1}{2} - 0 \right)$$

$$= \frac{\sqrt{2}}{2}$$

6 (i) Points of intersection are when

$$\sin 2x = \cos x$$

$$2 \sin x \cos x = \cos x$$

$$2 \sin x \cos x - \cos x = 0$$

$$\cos x (2 \sin x - 1) = 0$$

$$\cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \text{ and } \frac{\pi}{2} \text{ and } y = \cos x$$

$$A = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x) dx$$

$$= \left[-\frac{1}{2} \cos 2x - \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{2} \cos 2 \times \frac{\pi}{2} - \sin \frac{\pi}{2} \right) - \left(-\frac{1}{2} \cos 2 \times \frac{\pi}{6} - \sin \frac{\pi}{6} \right)$$

$$= \left(\frac{1}{2} - 1 \right) - \left(-\frac{1}{2} \times \frac{1}{2} - \frac{1}{2} \right)$$

$$= \left(-\frac{1}{2} \right) - \left(-\frac{3}{4} \right)$$

$$= \frac{1}{4}$$

7 (i) $2 \sin x = 1 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2 \sin x - 1) dx$$

$$= \left[-2 \cos x - x \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \left(-2 \cos \frac{5\pi}{6} - \frac{5\pi}{6} \right) - \left(-2 \cos \frac{\pi}{6} - \frac{\pi}{6} \right)$$

$$= \left(-2 \times -\frac{\sqrt{3}}{2} - \frac{5\pi}{6} \right) - \left(-2 \times \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right)$$

$$= \left(\sqrt{3} - \frac{5\pi}{6} \right) - \left(-\sqrt{3} - \frac{\pi}{6} \right)$$

$$= 2\sqrt{3} - \frac{2\pi}{3}$$

(ii) $V = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \pi (2 \sin x)^2 dx - \pi \times 1^2 \times \frac{2\pi}{3}$

$$= 4\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x dx - \frac{2\pi^2}{3}$$

$$= 4\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2} (1 - \cos 2x) dx - \frac{2\pi^2}{3}$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 - \cos 2x) dx - \frac{2\pi^2}{3}$$

$$= 2\pi \left[x - \frac{1}{2} \sin 2x \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{2\pi^2}{3}$$

$$= 2\pi \left[\left(\frac{5\pi}{6} - \frac{1}{2} \sin 2 \times \frac{5\pi}{6} \right) - \left(\frac{\pi}{6} - \frac{1}{2} \sin 2 \times \frac{\pi}{6} \right) \right] - \frac{2\pi^2}{3}$$

$$= 2\pi \left[\left(\frac{5\pi}{6} - \frac{1}{2} \times -\frac{\sqrt{3}}{2} \right) - \left(\frac{\pi}{6} - \frac{1}{2} \times \frac{\sqrt{3}}{2} \right) \right] - \frac{2\pi^2}{3}$$

$$= 2\pi \left[\left(\frac{5\pi}{6} + \frac{\sqrt{3}}{4} \right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) \right] - \frac{2\pi^2}{3}$$

$$= 2\pi \left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2} \right] - \frac{2\pi^2}{3}$$

$$= \frac{4\pi^2}{3} + \sqrt{3}\pi - \frac{2\pi^2}{3}$$

$$= \frac{2\pi^2}{3} + \sqrt{3}\pi$$

$$\approx 12.0$$

8 (i) $LHS = \cos 3x$

$$= \cos(2x + x)$$

$$= \cos 2x \cos x - \sin 2x \sin x$$

$$= (2 \cos^2 x - 1) \cos x - 2 \sin x \cos x \sin x$$

$$= 2 \cos^3 x - \cos x - 2 \sin^2 x \cos x$$

$$= 2 \cos^3 x - \cos x - 2(1 - \cos^2 x) \cos x$$

$$= 2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x$$

$$= 4 \cos^3 x - 3 \cos x$$

$$= RHS$$

(ii) $\cos 3x = 4\cos^3 x - 3\cos x$

$$\Rightarrow \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$$

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^3 x \, dx &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{4}(\cos 3x + 3\cos x) \, dx \\&= \frac{1}{4} \left[\frac{1}{3} \sin 3x + 3 \sin x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\&= \frac{1}{4} \left[\left(\frac{1}{3} \sin \frac{3\pi}{2} + 3 \sin \frac{\pi}{2} \right) - \left(\frac{1}{3} \sin \pi + 3 \sin \frac{\pi}{3} \right) \right] \\&= \frac{1}{4} \left[\left(-\frac{1}{3} + 3 \right) - \left(0 + \frac{3\sqrt{3}}{2} \right) \right] \\&= \frac{1}{4} \left[\frac{8}{3} - \frac{3\sqrt{3}}{2} \right] \\&= \frac{2}{3} - \frac{3\sqrt{3}}{8}\end{aligned}$$

9 (i) $5\cos x - 3\sin x$
 $= A(\cos x + \sin x) + B(\cos x - \sin x)$
 $= (A+B)\cos x + (A-B)\sin x$
 $A+B=5$
 $A-B=-3$
 $A=1, B=4$

(ii) $\int_0^{\frac{\pi}{2}} \frac{5\cos x - 3\sin x}{\cos x + \sin x} \, dx$
 $= \int_0^{\frac{\pi}{2}} \frac{(\cos x + \sin x) + 4(\cos x - \sin x)}{\cos x + \sin x} \, dx$
 $= \int_0^{\frac{\pi}{2}} \left(\frac{(\cos x + \sin x)}{\cos x + \sin x} + \frac{4(\cos x - \sin x)}{\cos x + \sin x} \right) \, dx$
 $= \int_0^{\frac{\pi}{2}} \left(1 + \frac{4(\cos x - \sin x)}{\cos x + \sin x} \right) \, dx$
 $= [x + 4\ln(\cos x + \sin x)]_0^{\frac{\pi}{2}}$
 $= \left(\frac{\pi}{2} + 4\ln\left(\cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right) \right)$
 $\quad - (0 + 4\ln(\cos 0 + \sin 0))$
 $= \left(\frac{\pi}{2} + 4\ln(0+1) \right) - (0 + 4\ln(1+0))$
 $= \left(\frac{\pi}{2} + 0 \right) - (0+0)$
 $= \frac{\pi}{2}$

10 $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos 2x}{\cos^2 x} \, dx$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\cos^2 x - 1}{\cos^2 x} \, dx$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{2\cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) \, dx$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2 - \sec^2 x) \, dx$
 $= [2x - \tan x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$
 $= \left(\frac{2\pi}{3} - \tan \frac{\pi}{3} \right) - \left(\frac{\pi}{3} - \tan \frac{\pi}{6} \right)$
 $= \left[\left(\frac{2\pi}{3} - \sqrt{3} \right) - \left(\frac{\pi}{3} - \frac{1}{\sqrt{3}} \right) \right]$
 $= \frac{\pi}{3} - \sqrt{3} + \frac{1}{\sqrt{3}}$
 $= \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$

5.4 Numerical integration (Page 74)

1 $\int_0^2 f(x) \, dx$
 $= \frac{1}{2} \times \frac{2-0}{4} [2.1 + 2(0.8 + 1.5 + 1.9) + 3.8]$
 $= 3.575$

2 (i)

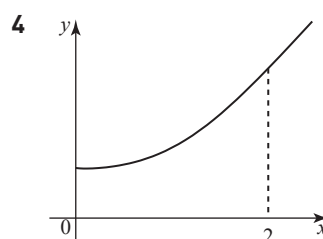
x	0	0.25	0.5	0.75	1
$f(x)$	1	0.9846	0.8888	0.7032	0.5

(ii) $\int_0^1 f(x) \, dx$
 $= \frac{1}{2} \times \frac{1-0}{4} [1 + 2(0.9846 + 0.8888 + 0.7032) + 0.5]$
 $= 0.832 \text{ (3 s.f.)}$

(iii) Underestimate of the true area under the curve

3 (i) Estimate of area
 $= \frac{1}{2} \times \frac{4-0}{8} [0 + 2(2.1 + 2.2 + 2.3 + 1.8 + 1.2 + 1.2 + 1.0) + 0]$
 $= 5.9 \text{ m}^2$

(ii) $3 \text{ km/h} = 3000 \text{ m/h} = 50 \text{ m/min}$
 $\text{Volume} = 50 \times 5.9 = 295 \text{ m}^3/\text{min}$



$$\begin{aligned}
 \text{(i)} \quad & \int_0^2 \sqrt{1+x^3} \, dx \\
 & \approx \frac{1}{2} \times \frac{2-0}{4} [1 + 2(1.0606 + 1.4142 + 2.0916) + 3] \\
 & = 3.28 \text{ (3 s.f.)}
 \end{aligned}$$

- (ii) Chris is correct. The answer from part (i) is an overestimate of the true area. Increasing the number of strips would decrease the value found.

Further practice (Page 77)

$$\begin{aligned}
 1 \text{ (i)} \quad & \int_{-2}^2 \frac{4}{2x+1} \, dx = [2\ln|2x+1|]_{-2}^2 \\
 & = 2\ln|2 \times 2 + 1| - 2\ln|2 \times -2 + 1| \\
 & = 2\ln|5| - 2\ln|-3| \\
 & = 2\ln 5 - 2\ln 3 \\
 & = \ln 25 - \ln 9 \\
 & = \ln \frac{25}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \int_k^2 \frac{4}{2x+1} \, dx = [2\ln|2x+1|]_k^2 \\
 & = (2\ln|2 \times 2 + 1|) - (2\ln|2k+1|) \\
 & = (2\ln|5|) - (2\ln|2k+1|) \\
 & = \ln 25 - \ln(2k+1)^2 \\
 & = \ln \frac{25}{(2k+1)^2} \\
 & \ln \frac{25}{(2k+1)^2} = 0 \\
 & \frac{25}{(2k+1)^2} = 1 \\
 & 25 = (2k+1)^2 \\
 & 2k+1 = \pm 5 \\
 & k = -3 \text{ or } 2 \\
 & \text{Since } k < 0, k = -3
 \end{aligned}$$

$$\begin{aligned}
 2 \text{ (i)} \quad & (2\cos x + \sin x)^2 \\
 & = 4\cos^2 x + 4\sin x \cos x + \sin^2 x \\
 & = 3\cos^2 x + 4\sin x \cos x + \sin^2 x + \cos^2 x \\
 & = 3\cos^2 x + 4\sin x \cos x + 1 \\
 & = 3\left(\frac{1}{2}(\cos 2x + 1)\right) + 2(2\sin x \cos x) + 1 \\
 & = \frac{3}{2}\cos 2x + \frac{3}{2} + 2\sin 2x + 1 \\
 & = 2\sin 2x + \frac{3}{2}\cos 2x + \frac{5}{2} \\
 & a = 2, b = \frac{3}{2}, c = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \int_0^{\frac{\pi}{2}} (2\cos x + \sin x)^2 \, dx \\
 & = \int_0^{\frac{\pi}{2}} \left(2\sin 2x + \frac{3}{2}\cos 2x + \frac{5}{2}\right) \, dx \\
 & = \left[-\cos 2x + \frac{3}{4}\sin 2x + \frac{5}{2}x\right]_0^{\frac{\pi}{2}} \\
 & = \left[\left(-\cos 2 \times \frac{\pi}{2} + \frac{3}{4}\sin 2 \times \frac{\pi}{2} + \frac{5}{2} \times \frac{\pi}{2}\right) - \left(-\cos 2 \times 0 + \frac{3}{4}\sin 2 \times 0 + \frac{5}{2} \times 0\right)\right] \\
 & = \left(1 + 0 + \frac{5\pi}{4}\right) - (-1 + 0 + 0) \\
 & = \frac{5\pi}{4} + 2
 \end{aligned}$$

$$\begin{aligned}
 3 \text{ LHS} &= \tan^2 x + \sin^2 x \\
 &= \sec^2 x - 1 + \frac{1}{2}(1 - \cos 2x) \\
 &= \sec^2 x - 1 + \frac{1}{2} - \frac{1}{2}\cos 2x \\
 &= \sec^2 x - \frac{1}{2}\cos 2x - \frac{1}{2} \\
 &= \text{RHS} \\
 & \int_0^{\frac{\pi}{6}} (\tan^2 x + \sin^2 x) \, dx \\
 &= \int_0^{\frac{\pi}{6}} \left(\sec^2 x - \frac{1}{2}\cos 2x - \frac{1}{2}\right) \, dx \\
 &= \left[\tan x - \frac{1}{4}\sin 2x - \frac{1}{2}x\right]_0^{\frac{\pi}{6}} \\
 &= \left[\left(\tan \frac{\pi}{6} - \frac{1}{4}\sin 2 \times \frac{\pi}{6} - \frac{1}{2} \times \frac{\pi}{6}\right) - \left(\tan 0 - \frac{1}{4}\sin 2 \times 0 - \frac{1}{2} \times 0\right)\right] \\
 &= \left(\frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right) - (0) \\
 &= \frac{5}{8\sqrt{3}} - \frac{\pi}{12} \\
 &= \frac{5\sqrt{3}}{24} - \frac{\pi}{12}
 \end{aligned}$$

$$\begin{aligned}
 4 \text{ (i)} \quad & \int_0^{\pi} \sqrt{4 - \cos x} \, dx \\
 & \approx \frac{1}{2} \times \frac{\pi - 0}{2} [\sqrt{3} + 2(2) + \sqrt{5}] \\
 & = 6.26 \text{ (3 s.f.)}
 \end{aligned}$$

- (ii) The first trapezium will overestimate the area from $0 \leq x \leq \frac{\pi}{2}$ but the second trapezium will underestimate the area from $\frac{\pi}{2} < x \leq \pi$, so taken together the trapezium rule will give a good estimate of the true value of the integral.

$$\begin{aligned}
 5 \quad (i) \quad & \frac{d}{dx}(e^x(\sin 2x - 2\cos 2x)) \\
 &= e^x(\sin 2x - 2\cos 2x) + (2\cos 2x + 4\sin 2x) \times e^x \\
 &= e^x(\sin 2x - 2\cos 2x + 2\cos 2x + 4\sin 2x) \\
 &= e^x(5\sin 2x)
 \end{aligned}$$

$$(ii) \quad \text{Since } \frac{d}{dx}(e^x(\sin 2x - 2\cos 2x)) = 5e^x \sin 2x$$

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx \\
 &= \frac{1}{5} \left[e^x (\sin 2x - 2\cos 2x) \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{5} \left[e^{\frac{\pi}{4}} \left(\sin 2 \times \frac{\pi}{4} - 2\cos 2 \times \frac{\pi}{4} \right) - \right. \\
 &\quad \left. e^0 (\sin 0 - 2\cos 0) \right] \\
 &= \frac{1}{5} \left[e^{\frac{\pi}{4}} (1 - 0) - (-2) \right] \\
 &= \frac{1}{5} \left[e^{\frac{\pi}{4}} + 2 \right] \\
 &= \frac{1}{5} e^{\frac{\pi}{4}} + \frac{2}{5}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad (i) \quad & R\cos(\theta - \alpha) \equiv R(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\
 & \cos\theta + \sqrt{3}\sin\theta \equiv R(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\
 & 1 = R\cos\alpha \\
 & \sqrt{3} = R\sin\alpha \\
 & \sqrt{3} = \tan\alpha \Rightarrow \alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3} \\
 & R = \sqrt{1^2 + \sqrt{3}^2} = 2 \\
 & \text{So } \cos\theta + \sqrt{3}\sin\theta = 2\cos\left(\theta - \frac{\pi}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad & \int_0^{\frac{\pi}{3}} \frac{1}{(\cos\theta + \sqrt{3}\sin\theta)^2} d\theta \\
 &= \int_0^{\frac{\pi}{3}} \frac{1}{\left(2\cos\left(\theta - \frac{\pi}{3}\right)\right)^2} d\theta \\
 &= \int_0^{\frac{\pi}{3}} \frac{1}{4\cos^2\left(\theta - \frac{\pi}{3}\right)} d\theta \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{3}} \sec^2\left(\theta - \frac{\pi}{3}\right) d\theta \\
 &= \frac{1}{4} \left[\tan\left(\theta - \frac{\pi}{3}\right) \right]_0^{\frac{\pi}{3}} \\
 &= \frac{1}{4} \left[\tan\left(\frac{\pi}{3} - \frac{\pi}{3}\right) - \tan\left(0 - \frac{\pi}{3}\right) \right] \\
 &= \frac{1}{4} \left[\tan 0 - \tan\left(-\frac{\pi}{3}\right) \right] \\
 &= \frac{1}{4} [0 - (-\sqrt{3})] \\
 &= \frac{\sqrt{3}}{4}
 \end{aligned}$$

Past exam questions (Page 78)

$$\begin{aligned}
 1 \quad (i) \quad & y = \cot x = \frac{\cos x}{\sin x} \\
 & \frac{dy}{dx} = \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x} \\
 &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\
 &= -\frac{1}{\sin^2 x} \\
 &= -\operatorname{cosec}^2 x
 \end{aligned}$$

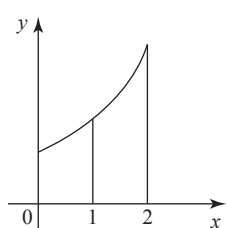
$$\begin{aligned}
 (ii) \quad & \cot^2 x = \operatorname{cosec}^2 x - 1 \\
 & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 x \, dx \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\operatorname{cosec}^2 x - 1) \, dx \\
 &= \left[-\cot x - x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \left(-\cot \frac{\pi}{2} - \frac{\pi}{2} \right) - \left(-\cot \frac{\pi}{4} - \frac{\pi}{4} \right) \\
 &= \left(0 - \frac{\pi}{2} \right) - \left(-1 - \frac{\pi}{4} \right) \\
 &= 1 - \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad & \cos 2x = 1 - 2\sin^2 x \\
 & \Rightarrow \frac{1}{1 - \cos 2x} = \frac{1}{1 - (1 - 2\sin^2 x)} \\
 &= \frac{1}{2\sin^2 x} = \frac{1}{2} \operatorname{cosec}^2 x \\
 & \int \frac{1}{1 - \cos 2x} \, dx = \int \frac{1}{2} \operatorname{cosec}^2 x \, dx \\
 &= -\frac{1}{2} \cot x + c
 \end{aligned}$$

$$\begin{aligned}
 2 \quad (i) \quad & \cos(3x - x) = \cos 3x \cos x + \sin 3x \sin x \\
 & \cos(3x + x) = \cos 3x \cos x - \sin 3x \sin x \\
 & LHS = \frac{1}{2}(\cos 2x - \cos 4x) \\
 &= \frac{1}{2}[\cos(3x - x) - \cos(3x + x)] \\
 &= \frac{1}{2}[(\cos 3x \cos x + \sin 3x \sin x) - \\
 &\quad (\cos 3x \cos x - \sin 3x \sin x)] \\
 &= \frac{1}{2}[2\sin 3x \sin x] \\
 &= \sin 3x \sin x \\
 &= RHS
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 3x \sin x \, dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} (\cos 2x - \cos 4x) \, dx \\
 &= \frac{1}{2} \left[\frac{1}{2} \sin 2x - \frac{1}{4} \sin 4x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= \left(\frac{1}{4} \sin \frac{2}{3} \pi - \frac{1}{8} \sin \frac{4}{3} \pi \right) \\
 &\quad - \left(\frac{1}{4} \sin \frac{\pi}{3} - \frac{1}{8} \sin \frac{2}{3} \pi \right) \\
 &= \left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{16} \right) - \left(\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{16} \right) \\
 &= \frac{\sqrt{3}}{8}
 \end{aligned}$$

3 (i)



x	0	1	2
y	1	$\sqrt{2}$	3

$$\begin{aligned}
 \text{Area } A &\approx \frac{1}{2}(1 + 2\sqrt{2} + 3) \\
 &= 3.41 \text{ (2 d.p.)}
 \end{aligned}$$

(ii) Area $(A + B) = 6$

$$\Rightarrow \text{Area } B \approx 6 - 3.41 = 2.59 \text{ (2 d.p.)}$$

Since the calculation of Area A overestimates the true value of A , this approximation will be an underestimate of the true value of B .

$$\begin{aligned}
 4 \quad & \int_0^k e^{4x} \, dx = \left[\frac{1}{4} e^{4x} \right]_0^k \\
 &= \frac{1}{4} e^{4k} - \frac{1}{4} e^0 \\
 &= \frac{1}{4} (e^{4k} - 1) \\
 & \int_0^{2k} e^x \, dx = \left[e^x \right]_0^{2k} \\
 &= e^{2k} - e^0 \\
 &= e^{2k} - 1
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \frac{1}{4} (e^{4k} - 1) &= e^{2k} - 1 \\
 e^{4k} - 1 &= 4e^{2k} - 4
 \end{aligned}$$

$$e^{4k} - 4e^{2k} + 3 = 0$$

$$\text{Let } m = e^{2k} \Rightarrow$$

$$m^2 - 4m + 3 = 0$$

$$(m - 3)(m - 1) = 0$$

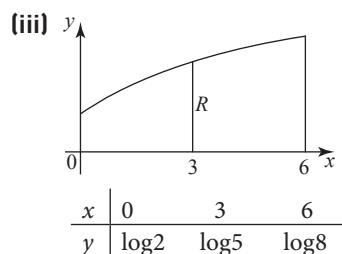
$$m = 1 \text{ or } 3$$

$$2^{2k} = 1 \Rightarrow k = 0$$

$$e^{2k} = 3 \Rightarrow k = \frac{\ln 3}{2}$$

$$\begin{aligned}
 5 \quad \text{(i)} \quad & \int \frac{4 + e^x}{2e^{2x}} \, dx \\
 &= \int \left(\frac{4}{2e^{2x}} + \frac{e^x}{2e^{2x}} \right) \, dx \\
 &= \int \left(\frac{2}{e^{2x}} + \frac{1}{2e^x} \right) \, dx \\
 &= \int \left(2e^{-2x} + \frac{1}{2}e^{-x} \right) \, dx \\
 &= \frac{2e^{-2x}}{-2} - \frac{1}{2}e^{-x} + c \\
 &= -e^{-2x} - \frac{1}{2}e^{-x} + c \\
 &= -\frac{1}{e^{2x}} - \frac{1}{2e^x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \int_2^{10} \frac{1}{2x + 5} \, dx \\
 &= \left[\frac{1}{2} \ln |2x + 5| \right]_2^{10} \\
 &= \left(\frac{1}{2} \ln |2(10) + 5| \right) - \left(\frac{1}{2} \ln |2(2) + 5| \right) \\
 &= \frac{1}{2} \ln 25 - \frac{1}{2} \ln 9 \\
 &= \ln 25^{\frac{1}{2}} - \ln 9^{\frac{1}{2}} \\
 &= \ln 5 - \ln 3 \\
 &= \ln \left(\frac{5}{3} \right)
 \end{aligned}$$



$$\begin{aligned}
 \text{Area } R &\approx \frac{3}{2} (\log 2 + 2 \log 5 + \log 8) \\
 &= 3.9 \text{ (1 d.p.)}
 \end{aligned}$$

$$\begin{aligned}
 6 \text{ (i)} \quad \text{Let } I &= \int (2\cos\theta - 3)(\cos\theta + 1) d\theta \\
 &= \int (2\cos^2\theta - \cos\theta - 3) d\theta \\
 \cos 2\theta &= 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1) \\
 \text{So } I &= \int (\cos 2\theta + 1 - \cos\theta - 3) d\theta \\
 &= \int (\cos 2\theta - \cos\theta - 2) d\theta \\
 &= \frac{1}{2}\sin 2\theta - \sin\theta - 2\theta + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) (a)} \quad \int \left(\frac{4}{2x+1} + \frac{1}{2x} \right) dx \\
 = 4 \times \frac{1}{2} \ln|2x+1| + \frac{1}{2} \ln x + c \\
 = 2\ln|2x+1| + \frac{1}{2} \ln x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_1^4 \left(\frac{4}{2x+1} + \frac{1}{2x} \right) dx \\
 = \left[2\ln|2x+1| + \frac{1}{2} \ln x \right]_1^4 \\
 = \left(2\ln|2(4)+1| + \frac{1}{2} \ln(4) \right) \\
 \quad - \left(2\ln|2(1)+1| + \frac{1}{2} \ln(1) \right) \\
 = \left(2\ln 9 + \frac{1}{2} \ln 4 \right) - \left(2\ln 3 + \frac{1}{2} \ln(1) \right) \\
 = \left(\ln 9^2 + \ln 4^{\frac{1}{2}} \right) - \left(\ln 3^2 + \frac{1}{2} (0) \right) \\
 = \ln 81 + \ln 2 - \ln 9 \\
 = \ln \frac{81 \times 2}{9} \\
 = \ln 18
 \end{aligned}$$

Stretch and challenge (Page 79)

$$1 \text{ If } y = \frac{x^3}{6} + \frac{1}{2x}, \quad \frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$\begin{aligned}
 L &= \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2} \right)^2} dx \\
 &= \int_1^2 \sqrt{1 + \left(\frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4} \right)} dx \\
 &= \int_1^2 \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4}} dx \\
 &= \int_1^2 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2} dx \\
 &= \int_1^2 \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx \\
 &= \left[\frac{x^3}{6} - \frac{1}{2x} \right]_1^2
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{2^3}{6} - \frac{1}{2 \times 2} \right) - \left(\frac{1^3}{6} - \frac{1}{2 \times 1} \right) \\
 &= \left(\frac{8}{6} - \frac{1}{4} \right) - \left(\frac{1}{6} - \frac{1}{2} \right) \\
 &= \frac{13}{12} - \left(-\frac{1}{3} \right) \\
 &= \frac{17}{12}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \int_0^{\frac{\pi}{4}} \sin^2 x \cos^2 x dx &= \int_0^{\frac{\pi}{4}} \sin^2 x (1 - \sin^2 x) dx \\
 &= \int_0^{\frac{\pi}{4}} (\sin^2 x - \sin^4 x) dx
 \end{aligned}$$

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\begin{aligned}
 \Rightarrow \sin^4 x &= \frac{1}{4}(1 - \cos 2x)^2 \\
 &= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \\
 &= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1)\right) \\
 &= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{1}{8} \\
 &= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x
 \end{aligned}$$

$$\text{So } \int_0^{\frac{\pi}{4}} (\sin^2 x - \sin^4 x) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}(1 - \cos 2x) - \left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \right) \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{8} - \frac{1}{8}\cos 4x \right) dx$$

$$= \frac{1}{8} \int_0^{\frac{\pi}{4}} (1 - \cos 4x) dx$$

$$= \frac{1}{8} \left[x - \frac{1}{4}\sin 4x \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8} \left[\left(\frac{\pi}{4} - \frac{1}{4}\sin 4 \times \frac{\pi}{4} \right) - 0 \right]$$

$$= \frac{1}{8} \left[\left(\frac{\pi}{4} \right) - 0 \right]$$

$$= \frac{\pi}{32}$$

$$3 \text{ (i)} \quad m = \frac{3\pi}{4}$$

$$r = k\theta \Rightarrow \frac{3\pi}{2} = k \times \frac{3\pi}{4} \Rightarrow k = 2$$

$$\left[t, \frac{\pi}{2} \right] \text{ on curve } \Rightarrow r = 2 \times \frac{\pi}{2} = \pi \Rightarrow t = \pi$$

(ii) $r(\theta) = 1 + \cos\theta$, $r'(\theta) = -\sin\theta$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2} d\theta \\ &= 2 \int_0^\pi \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2} d\theta \\ &= 2 \int_0^\pi \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} d\theta \\ &= 2 \int_0^\pi \sqrt{2 + 2\cos\theta} d\theta \\ &= 2 \int_0^\pi \sqrt{2(1 + \cos\theta)} d\theta \\ &= 2\sqrt{2} \int_0^\pi \sqrt{1 + \cos\theta} d\theta \end{aligned}$$

$$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

so $\cos\theta + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$

$$\begin{aligned} \text{and } L &= 2\sqrt{2} \int_0^\pi \sqrt{2\cos^2\left(\frac{\theta}{2}\right)} d\theta \\ &= 4 \int_0^\pi \sqrt{\cos^2\left(\frac{\theta}{2}\right)} d\theta \\ &= 4 \int_0^\pi \cos\left(\frac{\theta}{2}\right) d\theta \\ &= 4 \left[2\sin\left(\frac{\theta}{2}\right) \right]_0^\pi \\ &= 4 \left[2\sin\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{0}{2}\right) \right] \\ &= 4[2 - 0] \\ &= 8 \end{aligned}$$

(iii) $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$

$$\frac{dx}{d\theta} = r - r\cos\theta = r(1 - \cos\theta), \quad \frac{dy}{d\theta} = r\sin\theta$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{(r(1 - \cos\theta))^2 + (r\sin\theta)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta) + r^2\sin^2\theta} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + 1)} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(2 - 2\cos\theta)} d\theta \\ &= \int_0^{2\pi} \sqrt{2r^2(1 - \cos\theta)} d\theta \\ &= \sqrt{2}r \int_0^{2\pi} \sqrt{1 - \cos\theta} d\theta \\ &= \sqrt{2}r \int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}} d\theta \\ &= 2r \int_0^{2\pi} \sqrt{\sin^2\frac{\theta}{2}} d\theta \end{aligned}$$

$$\begin{aligned} &= 2r \int_0^{2\pi} \sin\frac{\theta}{2} d\theta \\ &= 2r \left[-2\cos\frac{\theta}{2} \right]_0^{2\pi} \\ &= -4r \left(\cos\frac{2\pi}{2} - \cos\frac{0}{2} \right) \\ &= -4r(-1 - 1) \\ &= -4r(-2) \\ &= 8r \end{aligned}$$

- 4 Consider the boulder as a sphere of radius R centred at the origin. The entire boulder is formed as a volume of revolution, rotating the curve $y^2 = R^2 - x^2$ about the x -axis. The volume underwater is found by integrating from $-R$ to $-R\cos\phi$.

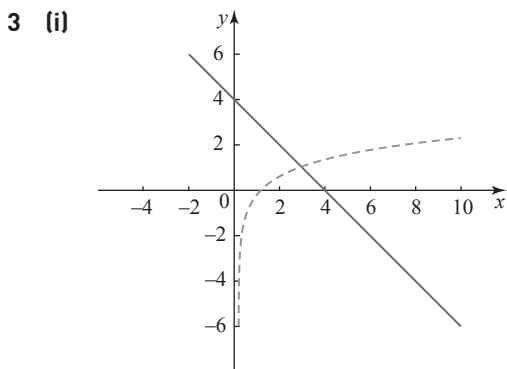
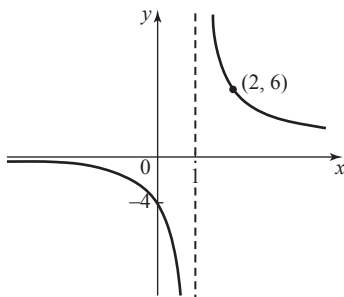
We need the proportion that this is of the total volume, which is $\frac{4}{3}\pi R^3$.

$$\begin{aligned} P(\phi) &= \frac{\int_{-R}^{-R\cos\phi} \pi(R^2 - x^2) dx}{\frac{4}{3}\pi R^3} \\ &= \frac{3}{4R^3} \left[R^2x - \frac{x^3}{3} \right]_{-R}^{-R\cos\phi} \\ &= \frac{3}{4R^3} \left[\left(R^2(-R\cos\phi) - \frac{(-R\cos\phi)^3}{3} \right) - \left(R^2(-R) - \frac{(-R)^3}{3} \right) \right] \\ &= \frac{3}{4R^3} \left[\left(-R^3\cos\phi + \frac{R^3\cos^3\phi}{3} \right) - \left(-R^3 + \frac{R^3}{3} \right) \right] \\ &= \frac{3}{4} \left[\left(-\cos\phi + \frac{\cos^3\phi}{3} \right) - \left(-1 + \frac{1}{3} \right) \right] \\ &= \frac{3}{4} \left[\left(-\cos\phi + \frac{\cos^3\phi}{3} \right) - \left(-\frac{2}{3} \right) \right] \\ &= \frac{1}{4} [-3\cos\phi + \cos^3\phi + 2] \\ &= \frac{1}{4} (\cos^3\phi - 3\cos\phi + 2) \end{aligned}$$

6 Numerical solution of equations

6.1 Interval estimation – change-of-sign methods (Page 81)

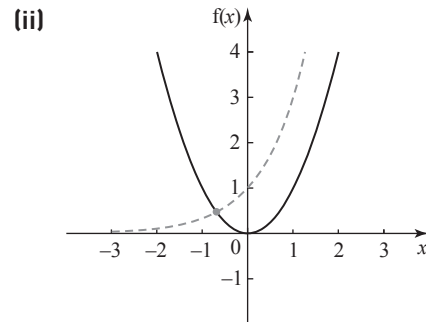
- 1 (i) $e^{x-3} = x^3 \Rightarrow e^{x-3} - x^3 = 0$
 $f(x) = e^{x-3} - x^3$
- (ii) $f(0) = e^{0-3} - 0^3 = e^{-3} > 0$
 $f(1) = e^{1-3} - 1^3 = e^{-2} - 1 < 0$
 Since there is a change in sign, the root must lie between 0 and 1.
- (iii) $f(0) = e^{0-3} - 0^3 = e^{-3} > 0$
 $f(0.5) = e^{0.5-3} - 0.5^3 = -0.0429... < 0$
 Since there is a change in sign, the root must lie between 0 and 0.5.
- (iv) $f(0.4) = e^{0.4-3} - 0.4^3 = 0.0103... > 0$
 $f(0.5) = e^{0.5-3} - 0.5^3 = -0.0429... < 0$
 So the root lies between $x = 0.4$ and $x = 0.5$
- 2 The statement is false because the curve is not continuous between $x = 0$ and $x = 2$.



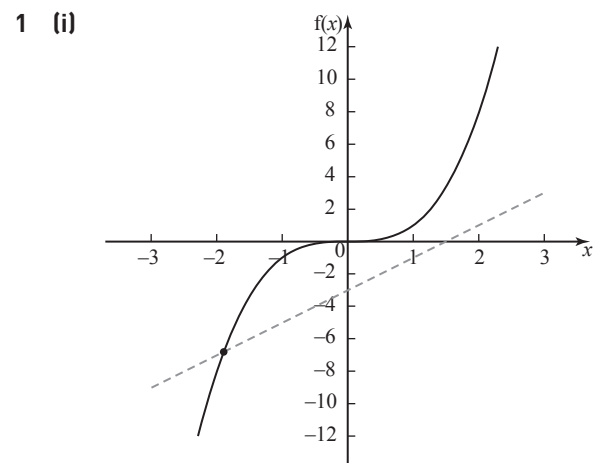
- (ii) Either $4 - x - \ln x = 0$ or $\ln x + x - 4 = 0$
- (iii) $f(x) = \ln x + x - 4$
 $f(2) = \ln 2 + 2 - 4 < 0$
 $f(3) = \ln 3 + 3 - 4 > 0$
 The integer bounds are 2 and 3.

- (iv) $f(2.9) = \ln 2.9 + 2.9 - 4 = -0.0353...$
 $f(3.0) = \ln 3 + 3 - 4 = 0.0986...$
 Change in sign $\Rightarrow$ root lies between 2.9 and 3.0

- 4 (i) $x^2 = 3^x \Rightarrow x^2 - 3^x = 0$
 $f(x) = x^2 - 3^x$
 $f(-0.7) = (-0.7)^2 - 3^{-0.7} = 0.0265...$
 $f(-0.6) = (-0.6)^2 - 3^{-0.6} = -0.157...$
 Sign change $\Rightarrow$ root in the interval $[-0.7, -0.6]$



6.2 Fixed-point iteration and problems with the method (Page 84)



- (ii) The root lies in the interval $[-2, -1]$.

- (iii) $x_{n+1} = \sqrt[3]{2x_n - 3}$ $x_0 = -2$
 $x_1 = \sqrt[3]{2 \times -2 - 3} = -1.9129...$
 $x_2 = -1.8969...$
 $x_3 = -1.8940...$
 $x_4 = -1.8934...$
 $\vdots$
 $x_n = -1.8933$ (4 d.p.)

- (iv) $x_{n+1} = \frac{1}{2}(x_n^3 + 3)$ or $\frac{2}{x_n} - \frac{3}{x_n^2}$

$$2 \quad x_{n+1} = 4 - \ln x_n$$

$$x_1 = 4 - \ln 2.9 = 2.9353\dots$$

$$x_2 = 2.9232\dots$$

$$x_3 = 2.9273\dots$$

$$x_4 = 2.9259\dots$$

$$\vdots$$

$$x_n = 2.926 \text{ (3 d.p.)}$$

$$3 \text{ (i)} \quad x^3 - x^2 = 15 \Rightarrow x^3 - x^2 - 15 = 0$$

$$f(x) = x^3 - x^2 - 15$$

$$f(2) = 2^3 - 2^2 - 15 = -11 < 0$$

$$f(3) = 3^3 - 3^2 - 15 = 3 > 0$$

Change in sign $\Rightarrow$ root lies between 2 and 3

$$\text{(ii)} \quad x_{n+1} = \sqrt[3]{15 + x_n^2}$$

$$x_1 = \sqrt[3]{15 + 2.5^2} = 2.7698\dots$$

$$x_2 = 2.8302\dots$$

$$\vdots$$

$$x_n = 2.849 \text{ (3 d.p.)}$$

$$4 \text{ (i)} \quad x_{n+1} = \sqrt{\frac{3 - x_n}{x_n}}$$

$$x = \sqrt{\frac{3 - x}{x}}$$

$$x^2 = \frac{3 - x}{x}$$

$$x^3 = 3 - x$$

$$x^3 + x - 3 = 0$$

$$\text{(ii)} \quad x_1 = \sqrt{\frac{3 - 1.5}{1.5}} = 1$$

$$x_2 = 1.4142\dots$$

$$x_3 = 1.0589\dots$$

$$x_4 = 1.3539\dots$$

$$\vdots$$

$$x_n = 1.21 \text{ (2 d.p.)}$$

$$5 \text{ (i)} \quad \text{Area of sector} = \frac{1}{2}r^2\theta$$

$$\text{Area of triangle OAB} = \frac{1}{2}r^2 \sin \theta$$

$$\text{Area of segment}$$

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = \frac{1}{2}r^2(\theta - \sin \theta)$$

$$\text{(ii)} \quad \text{Area of segment} = \frac{1}{6}\pi r^2$$

$$\frac{1}{2}r^2(\theta - \sin \theta) = \frac{1}{6}\pi r^2$$

$$3r^2(\theta - \sin \theta) = \pi r^2$$

$$3(\theta - \sin \theta) = \pi$$

$$\theta - \sin \theta = \frac{1}{3}\pi$$

$$\theta = \frac{1}{3}\pi + \sin \theta$$

$$\text{(iii)} \quad \theta_{n+1} = \frac{1}{3}\pi + \sin \theta_n$$

$$\theta_1 = \frac{1}{3}\pi + \sin 1 = 1.8887\dots$$

$$\theta_2 = 1.9971\dots$$

$$\theta_3 = 1.9577\dots$$

$$\theta_4 = 1.9733\dots$$

$$\vdots$$

$$\theta_n = 1.97 \text{ (2 d.p.)}$$

6 (i) Iterations diverge.

(ii) Iterations do not converge to a single value.

(iii) Iterations do not converge.

$$7 \text{ (i)} \quad x_1 = 3, x_{n+1} = \sqrt[3]{31 - \frac{5}{2}x_n}$$

$$x_2 = \sqrt[3]{31 - \frac{5}{2}x_1} = \sqrt[3]{31 - \frac{5}{2} \times 3} = 2.8643\dots$$

$$x_3 = 2.8780\dots$$

$$x_4 = 2.8767\dots$$

$$x_5 = 2.8768\dots$$

$$\vdots$$

$$x_n = 2.877 \text{ (3 d.p.)}$$

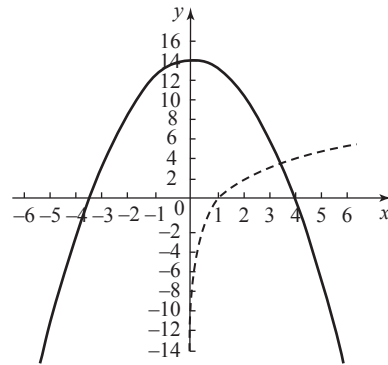
$$\text{(ii)} \quad x = \sqrt[3]{31 - \frac{5}{2}x}$$

$$x^3 = 31 - \frac{5}{2}x$$

$$2x^3 = 62 - 5x$$

$$2x^3 + 5x - 62 = 0$$

8 (i)



$$\text{(ii) (a)} \quad 14 - x^2 = 3 \ln x \Rightarrow 14 - x^2 - 3 \ln x = 0$$

$$f(x) = 14 - x^2 - 3 \ln x$$

$$f(3) = 14 - 3^2 - 3 \ln 3 = 1.704\dots > 0$$

$$f(4) = 14 - 4^2 - 3 \ln 4 = -6.159\dots < 0$$

Change in sign $\Rightarrow \alpha$ lies between 3 and 4

(b)

$$14 - x^2 - 3 \ln x = 0$$

$$14 - 3 \ln x = x^2$$

$$\sqrt{14 - 3 \ln x} = x$$

$$\begin{aligned}
 x_{n+1} &= \sqrt{14 - 3 \ln x_n}, \quad x_0 = 3.5 \\
 x_1 &= \sqrt{14 - 3 \ln 3.5} = 3.2003... \\
 x_2 &= 3.2420... \\
 x_3 &= 3.2360... \\
 x_4 &= 3.2368... \\
 &\vdots \\
 x_n &= 3.24 \text{ (2 d.p.)}
 \end{aligned}$$

Further practice (Page 90)

1 $x_{n+1} = \sqrt{4 - \cos x_n}, \quad x_0 = 5$

$$\begin{aligned}
 x_1 &= \sqrt{4 - \cos 5} = 1.9278... \\
 x_2 &= 2.0855... \\
 x_3 &= 2.1195... \\
 x_4 &= 2.1264... \\
 &\vdots \\
 x_n &= 2.13 \text{ (2 d.p.)}
 \end{aligned}$$

- 2 (i) An equation satisfied by α is

$$\alpha = 2 \ln(48 + 16\alpha^2)$$

$$\frac{\alpha}{2} = \ln(48 + 16\alpha^2)$$

$$e^{\frac{\alpha}{2}} = 48 + 16\alpha^2$$

Starting with the original equation,

$$y = e^{\frac{1}{4}x} \sqrt{3 + x^2}$$

$$\frac{1}{4} = e^{\frac{1}{4}x} \sqrt{3 + x^2}$$

$$\left(\frac{1}{4}\right)^2 = \left(e^{\frac{1}{4}x} \sqrt{3 + x^2}\right)^2$$

$$\frac{1}{16} = \left(e^{\frac{1}{2}x} \sqrt{3 + x^2}\right)$$

$$\frac{1}{e^{\frac{1}{2}x}} = 16(3 + x^2)$$

$$e^{\frac{1}{2}x} = 48 + 16x^2$$

$$\frac{1}{2}x = \ln(48 + 16x^2)$$

$$x = 2 \ln(48 + 16x^2)$$

(ii) $x_{n+1} = 2 \ln(48 + 16x_n^2), \quad x_0 = 14$

$$x_1 = 2 \ln(48 + 16(14)^2) = 16.1318...$$

$$x_2 = 16.6913...$$

$$x_3 = 16.8261...$$

$$x_4 = 16.8580...$$

$\vdots$

$$x_n = 16.87 \text{ (2 d.p.)}$$

3 $f(x) = x^4 - 4x^3 + 4x^2 + 2x - 7$

(i) $f(-1) = (-1)^4 - 4(-1)^3 + 4(-1)^2 + 2(-1) - 7 = 0$
so $\alpha = -1$

(ii) If $\alpha = -1$ is a root of the equation $f(x) = 0$, then $(x + 1)$ is a factor of $f(x)$.

Using polynomial division,

$$\frac{x^4 - 4x^3 + 4x^2 + 2x - 7}{x + 1} = x^3 - 5x^2 + 9x - 7$$

$$\text{So } x^4 - 4x^3 + 4x^2 + 2x - 7$$

$$= (x + 1)(x^3 - 5x^2 + 9x - 7)$$

To find β , solve

$$(x + 1)(x^3 - 5x^2 + 9x - 7) = 0$$

$$x = -1 \text{ or}$$

$$x^3 - 5x^2 + 9x - 7 = 0$$

$$x^3 = 5x^2 - 9x + 7$$

$$x = \sqrt[3]{5x^2 - 9x + 7}$$

(iii) $x_{n+1} = \sqrt[3]{5x_n^2 - 9x_n + 7}, \quad x_0 = 4$

$$x_1 = \sqrt[3]{5 \times 4^2 - 9 \times 4 + 7} = 3.7084...$$

$$x_2 = 3.4867...$$

$$x_3 = 3.3142...$$

$$x_4 = 3.1779...$$

$\vdots$

$$x_n = 2.54 \text{ (2 d.p.)}$$

4 $y = \frac{\cos 2x}{1 - x}$

(i) $\frac{dy}{dx} = \frac{-2 \sin 2x(1 - x) - (-1) \cos 2x}{(1 - x)^2}$

$$= \frac{-2 \sin 2x + 2x \sin 2x + \cos 2x}{(1 - x)^2}$$

Maximum is when $\frac{dy}{dx} = 0$

$$-2 \sin 2x + 2x \sin 2x + \cos 2x = 0$$

$$\sin 2x(2x - 2) + \cos 2x = 0$$

$$\frac{\sin 2x(2x - 2)}{\cos 2x} + \frac{\cos 2x}{\cos 2x} = \frac{0}{\cos 2x}$$

$$\tan 2x(2x - 2) + 1 = 0$$

$$\tan 2x(2x - 2) = -1$$

$$\tan 2x = \frac{-1}{2x - 2} = \frac{1}{2 - 2x}$$

(ii) $\tan 2x = \frac{1}{2 - 2x} \Rightarrow (2 - 2x) \tan 2x - 1 = 0$

$$f(x) = (2 - 2x) \tan 2x - 1$$

$$f(0.3) = (2 - 2 \times 0.3) \tan 2 \times 0.3 - 1 = -0.042...$$

$$f(0.4) = (2 - 2 \times 0.4) \tan 2 \times 0.4 - 1 = 0.236...$$

Change in sign $\Rightarrow$ root lies between 0.3 and 0.4

$$\begin{aligned}
 \text{(iii)} \quad x_{n+1} &= \frac{1}{2} \tan^{-1} \left(\frac{1}{2-2x_n} \right), x_0 = 0.35 \\
 x_1 &= \frac{1}{2} \tan^{-1} \left(\frac{1}{2-2x_0} \right) \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{1}{2-2 \times 0.35} \right) = 0.32785... \\
 x_2 &= 0.31979... \\
 x_3 &= 0.31694... \\
 x_4 &= 0.31594... \\
 &\vdots \\
 x_n &= 0.315 \quad (3 \text{ d.p.})
 \end{aligned}$$

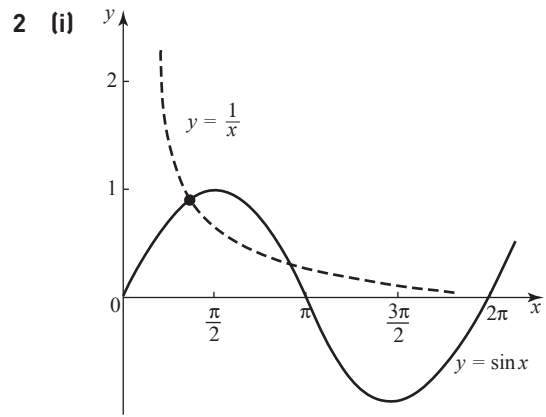
$$\begin{aligned}
 5 \quad \text{(i)} \quad \frac{1}{2} r^2 \theta &= \frac{1}{2} r \times r \tan \theta - \frac{1}{2} r^2 \theta \\
 \frac{1}{2} r^2 \theta &= \frac{1}{2} r^2 \tan \theta - \frac{1}{2} r^2 \theta \\
 \theta &= \tan \theta - \theta \\
 2\theta &= \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \theta_{n+1} &= \tan^{-1}(2\theta_n), \theta_0 = 1 \\
 \theta_1 &= \tan^{-1}(2 \times 1) = 1.1071... \\
 \theta_2 &= 1.1466... \\
 \theta_3 &= 1.1596... \\
 \theta_4 &= 1.1637... \\
 &\vdots \\
 \theta_n &= 1.17 \quad (2 \text{ d.p.})
 \end{aligned}$$

Past exam questions (Page 91)

$$\begin{aligned}
 1 \quad \text{(i)} \quad &\text{Area shaded region} \\
 &= \text{area triangle OCT} - \text{area sector OBC} \\
 &= \frac{1}{2} \times r \times r \tan x - \frac{1}{2} r^2 x \\
 &= \frac{1}{2} r^2 (\tan x - x) \\
 \Rightarrow \frac{1}{2} r^2 (\tan x - x) &= \frac{1}{2} \pi r^2 \\
 \tan x - x &= \pi \\
 \tan x &= x + \pi
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad x_{n+1} &= \tan^{-1}(x_n + \pi) \\
 x_0 &= 1 \\
 x_1 &= 1.3339... \\
 x_2 &= 1.3510... \\
 x_3 &= 1.3518... \\
 x_4 &= 1.3518... \\
 &\vdots \\
 x_n &= 1.35 \quad (2 \text{ d.p.})
 \end{aligned}$$



$$\begin{aligned}
 \text{(ii)} \quad \frac{1}{x} &= \sin x \\
 0 &= \sin x - \frac{1}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } f(x) &= \sin x - \frac{1}{x} \Rightarrow \\
 f(1.1) &= -0.018... \\
 f(1.2) &= 0.0987...
 \end{aligned}$$

Change of sign $\Rightarrow$ root lies between $x = 1.1$ and $x = 1.2$

$$\begin{aligned}
 \text{(iii)} \quad x_{n+1} &= \frac{1}{\sin x_n} \\
 x_0 &= 1.15... \\
 x_1 &= 1.0956... \\
 x_2 &= 1.1246... \\
 x_3 &= 1.1085... \\
 x_4 &= 1.1173... \\
 &\vdots \\
 x_n &= 1.11 \quad (2 \text{ d.p.})
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \text{(i)} \quad \int_0^a (3e^{3x} + 5e^x) dx &= 100 \\
 [e^{3x} + 5e^x]_0^a &= 100 \\
 (e^{3a} + 5e^a) - (e^0 + 5e^0) &= 100 \\
 e^{3a} + 5e^a - 6 &= 100 \\
 e^{3a} + 5e^a &= 106 \\
 e^{3a} &= 106 - 5e^a \\
 \ln(e^{3a}) &= \ln(106 - 5e^a) \\
 3a &= \ln(106 - 5e^a) \\
 a &= \frac{1}{3} \ln(106 - 5e^a)
 \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad a_{n+1} &= \frac{1}{3} \ln(106 - 5e_n^a) \\ a_0 &= 2 \\ a_1 &= 1.41163... \\ a_2 &= 1.48279... \\ a_3 &= 1.47684... \\ a_4 &= 1.47735... \\ &\vdots \\ a_n &= 1.477 \text{ (3 d.p.)} \end{aligned}$$

$$4 \quad x_{n+1} = \frac{4}{x_n^2} + \frac{2x_n}{3}$$

$$\begin{aligned} \text{(i)} \quad x_0 &= 2 \\ x_1 &= 2.33333... \\ x_2 &= 2.29025... \\ x_3 &= 2.28943... \\ x_4 &= 2.28943... \\ &\vdots \\ x_n &= 2.289 \text{ (3 d.p.)} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad 3x^3 &= 12 + 2x^3 \\ x^3 &= 12 \\ x &= \sqrt[3]{12} \\ \text{So } \alpha &= \sqrt[3]{12} \end{aligned}$$

Stretch and challenge (Page 91)

$$1 \quad \text{(i)} \quad \text{The gradient of the line through } x_1 \text{ and } x_2 \text{ is } \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

The equation is

$$\begin{aligned} f(x_1) &= \frac{f(x_1) - f(x_2)}{x_1 - x_2} \times x_1 + c \\ c &= f(x_1) - \frac{[f(x_1) - f(x_2)]x_1}{x_1 - x_2} \end{aligned}$$

The x -intercept (x_3) is

$$\begin{aligned} 0 &= \frac{f(x_1) - f(x_2)}{x_1 - x_2} \times x + f(x_1) - \frac{[f(x_1) - f(x_2)]x_1}{x_1 - x_2} \\ \Rightarrow x_3 &= \frac{\frac{[f(x_1) - f(x_2)]x_1}{x_1 - x_2} - f(x_1)}{\frac{f(x_1) - f(x_2)}{x_1 - x_2}} \\ &= \frac{\frac{[f(x_1) - f(x_2)]x_1 - f(x_1)(x_1 - x_2)}{x_1 - x_2}}{\frac{f(x_1) - f(x_2)}{x_1 - x_2}} \end{aligned}$$

$$\begin{aligned} &= \frac{[f(x_1) - f(x_2)]x_1 - f(x_1)(x_1 - x_2)}{x_1 - x_2} \times \frac{x_1 - x_2}{f(x_1) - f(x_2)} \\ &= \frac{[f(x_1) - f(x_2)]x_1 - f(x_1)(x_1 - x_2)}{f(x_1) - f(x_2)} \\ &= \frac{f(x_1)x_1 - f(x_2)x_1 - f(x_1)x_1 + f(x_1)x_2}{f(x_1) - f(x_2)} \\ &= \frac{f(x_1)x_2 - f(x_2)x_1}{f(x_1) - f(x_2)} \end{aligned}$$

$$\text{So } x_3 = \frac{f(x_1)x_2 - f(x_2)x_1}{f(x_1) - f(x_2)}$$

$$\text{In general, } x_{n+1} = \frac{f(x_{n-1})x_n - f(x_n)x_{n-1}}{f(x_{n-1}) - f(x_n)}$$

It can also be shown that

$$x_{n+1} = x_n - \frac{f(x_n)[x_n - x_{n-1}]}{f(x_n) - f(x_{n-1})}$$

$$\text{(iii)} \quad f(x) = e^{2x} - 3, x_1 = 1 \text{ and } x_2 = 0$$

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)[x_2 - x_1]}{f(x_2) - f(x_1)} \\ &= 0 - \frac{-2[0 - 1]}{-2 - (e^2 - 3)} \\ &= 0 - \frac{2}{1 - e^2} = \frac{2}{e^2 - 1} \end{aligned}$$