

MEI A Level Mathematics Exam Practice

Worked solutions and mark schemes

1 Proof

1 (i) For example, $a = -2, b = 1$

(ii) For example, $a = 1, b = -1$

2 (i) $A \Leftrightarrow B, |x|$ is always positive or zero, so can only equal x when x is positive or zero; thus, the two statements are equivalent.

(ii) $A \Rightarrow B$, John being a pilot implies he has good eyesight, whereas good eyesight does not imply you are a pilot.

(iii) $A \Rightarrow B$, as the two distinct roots come from a positive discriminant; when the discriminant is zero, there is only one distinct root, so the reverse implication is not true.

3 $x - 2 = 0 \Rightarrow x(x - 2) = 0$ is correct

$x(x - 2) = 0 \Rightarrow x = 0$ or $x = 2$ is correct

There is no contradiction: the fact that $x = 0$ or 2 includes the possibility that $x = 2$ only.

4 $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = \frac{1}{2} \times 9 \times 10 = 45$, which is a multiple of 3.

Hence, 9876543210 is a multiple of 3.

As it ends in 0, it is also a multiple of 2; it is, therefore, a multiple of 6.

5 The equation is equivalent to $n^3 - n = 571423$

Factorising: $n(n - 1)(n + 1) = 571423$

$(n - 1)n(n + 1)$ is a product of three consecutive integers and is, therefore, a multiple of 3.

The sum of the digits of 571 423 is 22, which is not a multiple of 3.

Hence, 571 423 itself is not a multiple of 3.

So, the two sides of the equation cannot be equal, and it is not possible for a positive integer n to satisfy

$$n^3 = n + 571\,423$$

6 Rewrite the quadratic by completing the square:

$$ax^2 + bx + c = 0 \Leftrightarrow a\left(x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a}\right) = 0$$

$$\Leftrightarrow x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

Now 'solve' the equation:

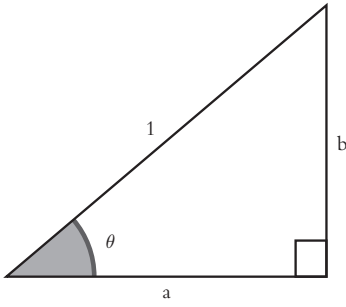
$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$\Leftrightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Leftrightarrow 2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$\Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- 7 Create a right-angled triangle as shown.



Then, $\cos \theta = \frac{a}{1}$ and $\sin \theta = \frac{b}{1}$

By Pythagoras' theorem, $a^2 + b^2 = 1$, so that $\cos^2 \theta + \sin^2 \theta = 1$

- 8 Let $n = N + 1$

Then $N(N + 2) = (n - 1)(n + 1) = n^2 - 1$, which is never a perfect square.

- 9 Test all pairs of integers with a product of 6.

Starting with 1×6 :

$$x - 2 = 1, \quad y + 3 = 6 \Rightarrow x = 3, \quad y = 3$$

$$x - 2 = -1, \quad y + 3 = -6 \Rightarrow x = 1, \quad y = -9 \text{ (not valid)}$$

$$x - 2 = 2, \quad y + 3 = 3 \Rightarrow x = 4, \quad y = 0 \text{ (not valid)}$$

$$x - 2 = -2, \quad y + 3 = -3 \Rightarrow x = 0, \quad y = -6 \text{ (not valid)}$$

$$x - 2 = 3, \quad y + 3 = 2 \Rightarrow x = 5, \quad y = -1 \text{ (not valid)}$$

$$x - 2 = -3, \quad y + 3 = -2 \Rightarrow x = -1, \quad y = -5 \text{ (not valid)}$$

$$x - 2 = 6, \quad y + 3 = 1 \Rightarrow x = 8, \quad y = -2 \text{ (not valid)}$$

$$x - 2 = -6, \quad y + 3 = -1 \Rightarrow x = -4, \quad y = -4 \text{ (not valid)}$$

So, $x = 3, y = 3$ is the only pair that satisfies the equation.

- 10 (i) Assume that $A \Rightarrow B$ and that B is not true.

If A is true then, because $A \Rightarrow B$, we have a contradiction, as we are assuming that B is not true. Hence, A is not true.

So, $B' \Rightarrow A'$

- (ii) Assume that $B' \Rightarrow A'$ and that A is true.

If B is not true then, because $B' \Rightarrow A'$, we have a contradiction, as we are assuming that A is true. Hence, B is true.

So, $A \Rightarrow B$

Alternatively, we can replace A with B' and B with A' in part (i).

- (iii) To prove that $A \Leftrightarrow B$, we can prove that $A \Rightarrow B$ and $B \Rightarrow A$.

From parts (i) and (ii), $A \Rightarrow B$ and $B' \Rightarrow A'$ are equivalent.

Also, $B \Rightarrow A$ and $A' \Rightarrow B'$ are equivalent.

So, we can prove instead that $A \Rightarrow B$ and $A' \Rightarrow B'$

Mark scheme

	Answer	Marks	Guidance
1	(i) For example, $a = -2$, $b = 1$	B1	Or other correct example
	(ii) For example, $a = 1$, $b = -1$	B1	Or other correct example
2	(i) $A \Leftrightarrow B$ $ x $ is always positive or zero so can only equal x when x is positive or zero; thus the two statements are equivalent	B1 E1	Correct choice Valid explanation
	(ii) $A \Rightarrow B$ John being a pilot implies he has good eyesight, whereas good eyesight does not imply you are a pilot	B1 E1	Correct choice Valid explanation
	(iii) $A \Rightarrow B$ The two distinct roots come from a positive discriminant; when the discriminant is zero, there is only one distinct root	B1 E1	Correct choice Valid explanation
3	$x - 2 = 0 \Rightarrow x(x - 2) = 0$ is correct $x(x - 2) = 0 \Rightarrow x = 0$ or $x = 2$ is correct There is no contradiction The fact that $x = 0$ or 2 includes the possibility that $x = 2$ only	E1 E1 B1 E1	Confirms correct statement Confirms correct statement Denies contradiction Valid explanation
4	$9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 45$ 9876543210 is a multiple of 3 As it ends in 0, it is also a multiple of 2 It is, therefore, a multiple of 6	B1 B1 B1	Proves multiple of 3 Proves multiple of 2 States conclusion
5	$n(n - 1)(n + 1) = 571\,423$ $(n - 1)n(n + 1) \dots$ $\dots$ is a product of three consecutive integers... $\dots$ and is, therefore, a multiple of 3 Sum of the digits of $571\,423$ is 22, which is not a multiple of 3 $571\,423$ is not a multiple of 3 So, result is not true	M1 A1 B1 B1 M1 A1 B1	Reformulates original problem Correct factorisation Interprets product as consecutive integers Identifies multiple of 3 Tests for divisibility by 3 Correct conclusion that $571\,423$ is not a multiple of 3 Correct conclusion
6	$ax^2 + bx + c = 0$ $\Leftrightarrow a\left(x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a}\right) = 0$ $\Leftrightarrow x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$ $\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$	B1 B1	Rearranges to enable completing of the square Completes the square (any method)

	Answer	Marks	Guidance
	$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ $\Leftrightarrow (2ax + b)^2 = b^2 - 4ac$ $\Leftrightarrow 2ax + b = \pm\sqrt{b^2 - 4ac}$ $\Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	B1 B1	Rearranges to enable square root to be taken Final step correct
7	Create a right-angled triangle with correct labels $a^2 + b^2 = 1$ $\Rightarrow \cos^2 \theta + \sin^2 \theta = 1$	B1 B1 B1 E1	With appropriate labelling (hypotenuse need not be 1) Hypotenuse of 1 (oe method) Uses Pythagoras' theorem Completely convincing
8	Let $n = N + 1$ $N(N + 2) = n^2 - 1$ Which is never a perfect square	B1 B1 E1	Reformulates problem Substitutes and simplifies Completely convincing
9	Proof by exhaustion attempted Correct answers $x = 3, y = 3$ is the only pair that satisfies the equation	B1 B1 B1 B1 B1	At least four possibilities considered All eight possibilities considered At least four correct At least six correct Completely correct solution (conclusion need not be stated explicitly)
10 (i)	Assume that $A \Rightarrow B$ and that B is not true If A is true, then, because $A \Rightarrow B$, we have a contradiction, as we are assuming that B is not true; hence, A is not true So, $B' \Rightarrow A'$	B1 B1 B1	Makes assumption Reaches contradiction Conclusion stated
(ii)	Assume that $B' \Rightarrow A'$ and that A is true If B is not true, then, because $B' \Rightarrow A'$, we have a contradiction, as we are assuming that A is true; hence, B is true So, $A \Rightarrow B$	B1 B1 B1	Makes assumption Reaches contradiction Conclusion stated
(iii)	To prove $A \Leftrightarrow B$, we can prove that $A \Rightarrow B$ and $B \Rightarrow A$ From parts (i) and (ii), $A \Rightarrow B$ and $B' \Rightarrow A'$ are equivalent Also, $B \Rightarrow A$ and $A' \Rightarrow B'$ are equivalent So, we can prove instead that $A \Rightarrow B$ and $A' \Rightarrow B'$	B1 B1 B1 B1	States equivalent implications Uses previous parts States another equivalence Conclusion stated

2 Trigonometry

- 1 Using inverse sine button on your calculator:

$$\sin^{-1} 0.25 = 14.5^\circ.$$

- 2 (i) To convert from degrees to radians, divide by 180 and multiply by π :

$$\begin{aligned}\Rightarrow 315 \div 180 \times \pi &= \frac{315}{180} \pi \\ &= \frac{7}{4} \pi\end{aligned}$$

- (ii) To convert from radians to degrees, divide by π and multiply by 180:

$$\begin{aligned}\Rightarrow 2\frac{4}{9} \pi \div \pi \times 180 &= \frac{22}{9} \times 180 \\ &= 440^\circ\end{aligned}$$

- 3 Using the formula for the length of an arc, $l = r\theta$:

$$20 = r\theta$$

And the area of a sector, $\text{area} = \frac{1}{2}r^2\theta$:

$$160 = \frac{1}{2}r^2\theta$$

$$\Rightarrow 160 = \frac{1}{2}r \times 20 \Rightarrow r = 16 \text{ cm}$$

- 4 Using exact values for the trigonometric functions:

$$\begin{aligned}\sqrt{8} \times \sin 315^\circ \times \cos 30^\circ &= \sqrt{8} \times -\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} \\ &= -\sqrt{3}\end{aligned}$$

- 5 Using the formula for the length of an arc, $l = r\theta$:

$$6 = 8\theta \Rightarrow \theta = \frac{3}{4}$$

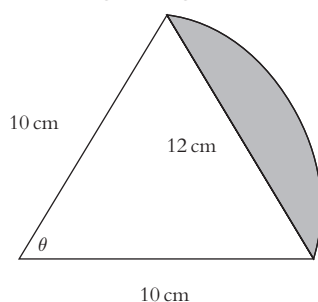
Using the formula for the area of a sector, $\text{area} = \frac{1}{2}r^2\theta$:

$$\begin{aligned}\text{area} &= \frac{1}{2} \times 8^2 \times \frac{3}{4} \\ &= 24 \text{ cm}^2\end{aligned}$$

- 6 Using formula for the area of a triangle, $\text{area} = \frac{1}{2}ab \sin \theta$:

$$\begin{aligned}\text{area} &= \frac{1}{2} \times 14 \times 10 \times \sin 0.5 \\ &= 33.6 \text{ cm}^2\end{aligned}$$

- 7 Drawing a diagram makes this clearer:



The segment is shaded in the diagram.

The first step is to determine the angle, θ , using trigonometry in the right-angled triangle that is half the isosceles triangle in the diagram.

$$\sin \frac{\theta}{2} = \frac{6}{10}$$

$$= 0.6$$

$$\Rightarrow \theta = 2 \times 0.6435$$

$$= 1.287 \text{ radians}$$

The area of the segment is the difference between the area of the sector and the area of the triangle:

$$\text{area} = \frac{1}{2} \times 10^2 \times 1.287 - \frac{1}{2} \times 10 \times 10 \times \sin 1.287$$

$$= 16.4 \text{ cm}^2$$

- 8 $\cos \theta = 0.43$, so principal solution from the calculator is:

$$\cos^{-1}(0.43) = 64.5^\circ \Rightarrow \theta = 64.5^\circ$$

$$y = \cos \theta \text{ is symmetric about } \theta = 0, \text{ so } 0.43 = \cos(-64.5^\circ) \Rightarrow \theta = -64.5^\circ$$

$$y = \cos \theta \text{ is periodic, with period } 360^\circ, \text{ so } 0.43 = \cos(-64.5^\circ + 360^\circ) = \cos(295.5^\circ) \Rightarrow \theta = 295.5^\circ$$

- 9 (i) Substituting the exact values into the equation gives:

$$\sin \frac{\pi}{4} \times \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{1}{2}$$

- (ii) Another solution will require the same signs for sine and cosine, so the third quadrant is indicated:

$$\sin\left(-\frac{3\pi}{4}\right) = \cos\left(-\frac{3\pi}{4}\right)$$

$$= -\frac{\sqrt{2}}{2}$$

$$\text{So, } \sin\left(-\frac{3\pi}{4}\right) \times \cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \times -\frac{\sqrt{2}}{2}$$

$$= \frac{1}{2}$$

$$\text{So, } x = -\frac{3\pi}{4}$$

- 10 (i) perimeter = $2r + \text{arc length}$, so arc length = $12 - 2r$

$$\text{arc length} = r\theta, \text{ so } \theta = \frac{(12 - 2r)}{r}$$

$$= \frac{12}{r} - 2$$

$$\text{sector area} = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} r^2 \left(\frac{12}{r} - 2 \right)$$

$$= 6r - r^2$$

Or

$$\text{sector area} = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} r(r\theta)$$

$$= \frac{1}{2} r(\text{arc length})$$

$$= \frac{1}{2} r(12 - 2r)$$

$$= 6r - r^2$$

- (ii) area = $6r - r^2 = r(6 - r) = 9 - (r - 3)^2$, which is a maximum when $r = 3$
(By symmetry or from graph.)

Or

$$\frac{d}{dr}(6r - r^2) = 6 - 2r, \text{ which is zero when } r = 3$$

when $r = 3$,

$$\theta = \frac{12}{r} - 2$$

$$= 4 - 2$$

$$= 2 \text{ rad}$$

Or

when $r = 3$,

$$\text{arc length} = 12 - 2r$$

$$= 6$$

$$6 = r\theta, \text{ so } \theta = \frac{6}{3}$$

$$= 2$$

- 11 (i) Using the small angle approximations:

$$\cos \theta \approx 1 - \frac{\theta^2}{2} \text{ so } 1 - \cos \theta \approx \frac{\theta^2}{2}$$

$$\sin 2\theta \approx 2\theta, \text{ and } \theta \sin 2\theta \approx 2\theta^2$$

$$\text{So, } \frac{1 - \cos \theta}{\theta \sin 2\theta} \approx \frac{\frac{\theta^2}{2}}{2\theta^2} = \frac{1}{4}$$

- (ii) For small positive angles, θ radians:

$$\begin{aligned} \lim_{\theta \rightarrow 0} \left(\frac{1 - \cos 3\theta}{\theta \tan 3\theta} \right) &= \lim_{\theta \rightarrow 0} \left(\frac{1 - 1 + 9\frac{\theta^2}{2}}{\theta 3\theta} \right) \\ &= \lim_{\theta \rightarrow 0} \left(\frac{9\theta^2}{6\theta^2} \right) \\ &= \frac{3}{2} \end{aligned}$$

- 12 (i) hypotenuse of triangle = 5 cm = radius of circle
So, chord at horizon coincides with base of triangle.

$$\text{angle at vertex of triangle} = 2 \arctan \left(\frac{3}{4} \right)$$

$$= 1.287 \text{ rad (or } 73.7^\circ)$$

$$\text{sector of circle contains angle } 2\pi - 1.287 = 4.996 \text{ rad (286.3}^\circ)$$

$$\text{So, shaded area} = \frac{1}{2} \times 5^2 \times 4.996$$

$$= 62.45$$

$$= 62.5 \text{ cm}^2 \text{ (3 s.f.)}$$

- (ii) area of semicircle = $\frac{1}{2} \times \pi \times 5^2$
 $= 39.27 \text{ cm}^2$

$$\text{area of triangle} = \frac{1}{2} \times 6 \times 4$$

$$= 12 \text{ cm}^2$$

$$\begin{aligned}
 \text{So, shaded area} &= 39.27 - 12 \\
 &= 27.27 \\
 &= 27.3 \text{ cm}^2 \text{ (3 s.f.)}
 \end{aligned}$$

(iii) distance of centre of circle below horizon = 1 cm

$$\begin{aligned}
 \text{angle at centre of circle} &= 2 \arccos\left(\frac{1}{5}\right) \\
 &= 2.739 \text{ rad}
 \end{aligned}$$

$$\begin{aligned}
 \text{area of segment above horizon} &= \frac{1}{2} \times 5^2 \times (2.739 - \sin 2.739) \\
 &= 29.34 \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{So, shaded area} &= 29.34 - 12 \\
 &= 17.3 \text{ cm}^2 \text{ (3 s.f.)}
 \end{aligned}$$

13 (i) Each hexagon is made of six equilateral triangles.

The radius of the circle is the same as the edge of the triangles that make up the smaller hexagon and the same as the perpendicular height of the triangles that make up the larger hexagon.

$$\text{The area of each equilateral triangle of edge } r \text{ cm is } \frac{1}{2} r^2 \sin \frac{\pi}{3} = \frac{\sqrt{3}}{4} r^2$$

$$\text{So, the area of the smaller hexagon is } \frac{3\sqrt{3}}{2} r^2$$

$$\text{The hypotenuse of each equilateral triangle of perpendicular height } r \text{ cm is } r \div \frac{\sqrt{3}}{2} = \frac{2\sqrt{3}}{3} r$$

$$\text{So, the area of each of these triangles is } \frac{1}{2} \left(\frac{2\sqrt{3}}{3} r \right)^2 \sin \frac{\pi}{3} = \frac{\sqrt{3}}{3} r^2$$

$$\text{So, the area of the larger hexagon is } 2\sqrt{3} r^2$$

$$\text{The shaded area is } 2\sqrt{3} r^2 - \frac{3\sqrt{3}}{2} r^2 = \frac{\sqrt{3}}{2} r^2$$

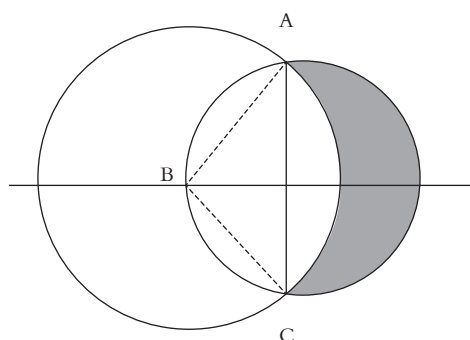
(ii) area of circle = πr^2

area of smaller hexagon < area of circle < area of larger hexagon

$$\text{So, } \frac{3\sqrt{3}}{2} r^2 < \pi r^2 < 2\sqrt{3} r^2$$

$$\Rightarrow \frac{3\sqrt{3}}{2} < \pi < 2\sqrt{3}$$

14 (i)



For the larger circle, the arc length $AC = 15$ cm and $r = 10$ cm.

So, angle $ABC = 1.5$ rad.

The straight line distance $AC = 2 \times 10 \times \sin 0.75 = 13.63$ cm.

The smaller circle passes through the centre of the larger circle, so the diameter of the smaller circle is $10 + 4$, so the radius is 7 cm.

The angle subtended at the centre of the smaller circle by the chord AC is 2θ , where $2 \times 7 \times \sin \theta = 13.63$, so $2\theta = 2.681$ rad, and the angle subtended at the centre of the smaller circle by the major arc is $2\pi - 2\theta = 3.60$ rad.

So, the length of the major arc AC is $(7 \times 3.60) = 25.2$ cm and the perimeter is 40.2 cm.

- (ii) area of crescent = (area of segment of smaller circle subtended by angle 3.60 rad)
 – (area of segment of larger circle subtended by angle 1.5 rad)

$$\begin{aligned}\text{area of segment of smaller circle} &= \frac{1}{2} \times 7^2 (3.60 - \sin 3.60) \\ &= 99.04 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{area of segment of larger circle} &= \frac{1}{2} \times 10^2 (1.5 - \sin 1.5) \\ &= 25.13 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{So, area of crescent shape} &= 99.07 - 25.13 \\ &= 73.9 \text{ cm}^2\end{aligned}$$

Mark scheme

	Answer	Marks	Guidance
1	14.5°	B1	cao
2 (i)	$\frac{7}{4}\pi$	B1	cao
(ii)	440°	B1	cao
3	$20 = r\theta$	B1	Uses arc length formula
	$160 = \frac{1}{2}r^2\theta = \frac{1}{2} \times r \times 20$	M1	Solves equation
	16 cm	A1	cao
4	$\sqrt{8} \times -\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}$	M1	Substitutes exact values correctly
	$-\sqrt{3}$	A1	cao
5	$6 = 8\theta \Rightarrow \theta = \frac{3}{4}$	B1	Correct angle
	$\text{area} = \frac{1}{2} \times 8^2 \times \frac{3}{4}$	M1	Uses area formula
	24 cm^2	A1	cao
6	$\frac{1}{2} \times 14 \times 10 \times \sin 0.5$	M1	Uses area formula
	33.6 cm^2	A1	awrt 33.6
7	$\sin \frac{\theta}{2} = 0.6 \Rightarrow \theta = 1.287$	M1	Determines the value of the angle
	$\frac{1}{2} \times 10^2 \times 1.287$	M1	Determines the area of the sector
	$-\frac{1}{2} \times 10 \times 10 \times \sin 1.287$	M1	Determines the area of the triangle
	16.4 cm^2	A1	awrt 16.4
8	$\arccos(0.43) = 64.5^\circ$	B1	Principal solution to 3 s.f. or better (1.13 rad)
	-64.5°	M1	Finds negative of their principal solution, must be in range -90° to 360° ($^\circ$ may be implied)
	$-64.5^\circ + 360^\circ = 295.5^\circ$	A1	Adds 360° to their negative angle ($^\circ$ may be implied) Alternatively, from B1 M0 or B0 M0 SC1 for 360° – their 64.5°

	Answer	Marks	Guidance
9 (i)	$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}, \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$	M1	States either in an exact form
	$\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{1}{2}$	A1	Shows that $\sin \frac{\pi}{4} \times \cos \frac{\pi}{4} = \frac{1}{2}$ using exact expressions
(ii)	$-\frac{3\pi}{4}$	B1	cao
10 (i)	arc length = $12 - 2r$	M1	Finds expression for arc length, seen or implied
	sector area = $6r - r^2$	A1	oe
(ii)	Maximum when $r = 3$	B1	Identifies where maximum occurs
	$\theta = \frac{12}{r} - 2$	M1	Uses expression for angle, or uses arc length = $12 - 2 \times 3 = 6$
	$= 2 \text{ rad}$	A1	cao (rad may be implied)
11 (i)	$\cos \theta \approx 1 - \frac{\theta^2}{2}$ so $1 - \cos \theta \approx \frac{\theta^2}{2}$	M1	Uses approximation for $\cos \theta$
	$\sin 2\theta \approx 2\theta$ so $\theta \sin 2\theta \approx 2\theta^2$	M1	Uses approximation for $\sin \theta$
	$\frac{1 - \cos \theta}{\theta \sin 2\theta} \approx \frac{\frac{\theta^2}{2}}{2\theta^2} = \frac{1}{4}$	A1	Deduces given result
(ii)	$1 - \cos 3\theta = 1 - 1 + 9\frac{\theta^2}{2}$	B1	Uses small angle formula for $\cos$
	$\theta \tan 3\theta = \theta 3\theta$	B1	Uses small angle formula for $\tan$
	$\lim_{\theta \rightarrow 0} \left(\frac{9\theta^2}{6\theta^2} \right)$	B1	Simplifies
	$\frac{3}{2}$	B1	cao
12 (i)	$2\pi - 2 \arctan\left(\frac{3}{4}\right) = 4.996... \text{ rad}$	M1	Uses or calculates an appropriate angle (0.64, 1.29, 2.50, 5.00 rad oe)
	$\frac{1}{2} \times 5^2 \times 4.996 = 62.5 \text{ cm}^2$	A1	cao
(ii)	$39.27 - 12$	M1	$\frac{1}{2} \times \pi \times 5^2$ and $\frac{1}{2} \times 6 \times 4$
	27.3 cm^2	A1	cao
(iii)	$2 \arccos\left(\frac{1}{5}\right) = 2.739 \text{ rad}$	B1	Uses or calculates an appropriate angle (1.37, 2.74 rad oe)
	$\frac{1}{2} \times 5^2 \times (2.739 - \sin 2.739) = 29.34$	M1	ft
	$29.34 - 12 = 17.3 \text{ cm}^2$	A1	17.3 (3 s.f.)
13 (i)	area of smaller hexagon $= 6 \times \frac{1}{2} r^2 \sin \frac{\pi}{3}$	M1	Uses $= 6 \times \frac{1}{2} r^2 \sin \frac{\pi}{3}$
	$= \frac{3\sqrt{3}}{2} r^2$	A1	Seen or implied

	Answer	Marks	Guidance
	edge of larger hexagon = $\frac{2\sqrt{3}}{3}r$	M1	Uses edge length $\times \frac{1}{2}r^2 \sin \frac{\pi}{3}$
	area of larger hexagon = $2\sqrt{3}r^2$	A1	Seen or implied
	shaded area = $2\sqrt{3}r^2 - \frac{3\sqrt{3}}{2}r^2$	M1	Finds difference between their hexagon areas, oe
	$= \frac{\sqrt{3}}{2}r^2$	A1	cao in surd form
13 (ii)	area of smaller hexagon $< \pi r^2$ $<$ area of larger hexagon	M1	Forms inequality for πr^2 bounded between the areas of the two hexagons;
	$\Rightarrow \frac{3\sqrt{3}}{2} < \pi < 2\sqrt{3}$	A1	May set $r = 1$ or some specific value cao in surd form
14 (i)	$\frac{15}{10} = 1.5$ rad	B1	1.5 or 0.75 for angle at B
	$2 \times 10 \times \sin 0.75 = 13.63$ cm	M1	Calculates length of chord AC
	7 cm	B1	Finds radius of small circle (seen or implied)
	$14 \sin \theta = 13.63$, so $2\theta = 2.681$ rad	A1	2.68 to 2.69 or 1.34 to 1.35
	$2\pi - 2\theta = 3.60$ rad	M1	Finds angle, 2π – their 2.682
	$7 \times 3.60 = 25.2$ cm	M1	$7 \times$ their 3.60
	$15 + 25.2 = 40.2$ cm	A1	40 to 40.5
(ii)	$\frac{1}{2} \times 7^2(3.60 - \sin 3.60) = 99.04$ cm ²	M1	Finds area of smaller segment ft their 7 and their 3.60
	$\frac{1}{2} \times 10^2(1.5 - \sin 1.5) = 25.13$ cm ²	B1	Finds area of larger segment cao
	$= 99.07 - 25.13 = 73.9$ cm ²	A1	Finds area of crescent shape cao

3 Sequences and series

- 1 Substitute $n = 1, 2, 3, 4, 5$ into the expression for the n th term:

$$4 - 7 \times 1 = -3$$

$$4 - 7 \times 2 = -10$$

$$4 - 7 \times 3 = -17$$

$$4 - 7 \times 4 = -24$$

$$4 - 7 \times 5 = -31$$

- 2 Using the formula for the general term, $u_n = a + d(n - 1)$, with $a = -2$, $d = 1.5$:

$$u_n = -2 + 1.5(n - 1) \Rightarrow u_n = -3.5 + 1.5n$$

- 3 Substituting:

$$2 \times 0.5 = 1$$

$$2 \times 0.5^2 = 0.5$$

$$2 \times 0.5^3 = 0.25$$

$$2 \times 0.5^4 = 0.125$$

$$2 \times 0.5^5 = 0.0625$$

- 4 Write the information given using algebra:

$$u_3 = ar^2 = 2; u_4 = ar^3 = 4$$

$$\Rightarrow r = 2; a = \frac{1}{2}; u_n = \frac{1}{2} \times 2^{n-1} = \frac{1}{4} \times 2^n$$

- 5 $1 + 2 + 3 + \dots + 100$ is an arithmetic series with $a = 1$ and $d = 1$

$$S_n = \frac{n}{2}(2a + (n-1)d) \text{ and } n = 100 \Rightarrow S_{100} = \frac{100}{2}(2 + 99 \times 1) = 5050$$

- 6 Geometric series converges if $r^n \rightarrow 0$ as $n \rightarrow \infty$

This only happens if $|r| < 1$

- 7 Sum of a geometric series $= \frac{a(r^n - 1)}{r - 1}$

As $n \rightarrow \infty$, if $|r| < 1$, $r^n \rightarrow 0$

$$\text{So, } S_\infty = \frac{-a}{r - 1} = \frac{a}{1 - r}$$

- 8 Sum of a geometric series $= \frac{a(r^n - 1)}{r - 1}$, $a = 3$, $r = 1.5$

$$\frac{3(1.5^n - 1)}{0.5} > 100$$

$$1.5^n > \frac{100}{6} + 1 \left(= 17\frac{2}{3} \right)$$

$$1.5^7 \approx 17.09, 1.5^8 \approx 25.63$$

Eight terms are needed for the sum to exceed 100.

- 9 (i) General arithmetic series is $S_n = \frac{n}{2}(2a + (n-1)d)$

$$\text{General geometric series is } S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\text{Sum of first ten terms is equal, so } 5(2 + 9d) = \frac{(r^{10} - 1)}{r - 1}$$

$$2 + 9d = \frac{r^{10} - 1}{5(r - 1)} \Rightarrow 9d = \frac{r^{10} - 1}{5(r - 1)} - 2$$

$$\text{So, } d = \frac{1}{9} \left(\frac{(r^{10} - 1)}{5(r - 1)} - 2 \right) \text{ as required.}$$

$$(ii) \text{ When } r = 2, d = \frac{1}{9} \left(\frac{1024 - 1}{5} - 2 \right) = \frac{1013}{45} = 22.51$$

- 10 (i) Each person gets $\frac{1}{4}$ of the cake, then $\frac{1}{4} \times \frac{1}{4} = \left(\frac{1}{4}\right)^2$, then $\frac{1}{4} \times \left(\frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^3$, etc.

$$\text{Total amount of cake per person is } \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots$$

This is a geometric series, $a = \frac{1}{4}$, $r = \frac{1}{4} < 1$

$$\text{Sum to infinity of a geometric series} = \frac{a}{1 - r}$$

$$\begin{aligned} \text{In this case the sum is } \frac{\frac{1}{4}}{1 - \frac{1}{4}} &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{3} \end{aligned}$$

This shows that each person will eventually receive a third of the cake.

- (ii) If the cake is divided into $n + 1$ pieces, and each person is given a piece, the remaining piece can be divided into $n + 1$ pieces, and so on.

$$\begin{aligned}\text{Total amount of cake per person} &= \frac{1}{n+1} + \left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{n+1}\right)^3 + \dots \\ &= \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} \\ &= \frac{\frac{1}{n+1}}{\frac{n}{n+1}} \\ &= \frac{1}{n}\end{aligned}$$

- 11 (i) Sum of an arithmetic series is $\frac{n}{2}(2a + (n-1)d)$

If the sum is 0:

$$2a + (n-1)d = 0 \Rightarrow a = \frac{-(n-1)d}{2}$$

- (ii) n is positive as it is the number of terms.

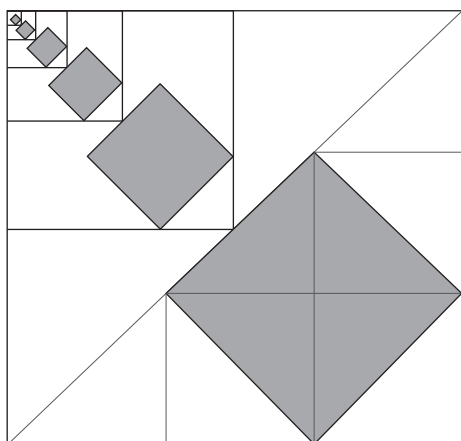
(a) If d is positive, then a must be negative, so (a) is not possible.

(b) If d is a positive multiple of 2, then a is a negative integer, so (b) is possible.

(c) If d is negative, then a is positive and is an integer if n is odd (making $n-1$ even), so (c) is possible.

(d) If d is positive, then a is negative if n is even or odd, so (d) is possible.

- 12 (i) Added lines show that the shaded square is $\frac{4}{9}$ of half the square i.e. $\frac{2}{9}$ of the entire square as required.



Or

$$\text{half the diagonal of the largest square} = \frac{1}{2}\sqrt{2}$$

$$\begin{aligned}\text{side length for largest shaded square} &= \frac{2}{3} \times \frac{1}{2}\sqrt{2} \\ &= \frac{\sqrt{2}}{3}\end{aligned}$$

$$\text{Length scale factor of } \frac{\sqrt{2}}{3} \text{ means area scale factor of } \left(\frac{\sqrt{2}}{3}\right)^2 = \frac{2}{9}$$

- (ii) Each unshaded square is one quarter of its predecessor, so the fraction shaded is:

$$\frac{2}{9} + \frac{1}{4} \times \frac{2}{9} + \frac{1}{4} \times \frac{1}{4} \times \frac{2}{9} + \dots$$

This is a geometric series with $a = \frac{2}{9}$, $r = \frac{1}{4}$

$$\begin{aligned}\text{Sum to infinity is } \frac{\frac{2}{9}}{1 - \frac{1}{4}} &= \frac{\frac{2}{9}}{\frac{3}{4}} \\ &= \frac{8}{27}\end{aligned}$$

13 The arithmetic sequence has terms $u_n = a + (n - 1)d$ in this case:

$$a, a + 5d, a + 25d$$

The geometric sequence has terms a, ar, ar^2

$$ar = a + 5d; ar^2 = a + 25d$$

$$r(a + 5d) = a + 25d \text{ and } r = \frac{a + 5d}{a}$$

$$(a + 5d)^2 = a(a + 25d)$$

$$a^2 + 10ad + 25d^2 = a^2 + 25ad$$

$$25d^2 - 15ad = 0$$

$$d(5d - 3a) = 0$$

$$d = 0 \text{ or } \frac{3a}{5}, \text{ but } d \neq 0$$

$$\text{Substituting into } r = \frac{a + 5d}{a}, r = 4$$

14 The geometric sequence has terms $a, ar, ar^2, \dots$

$$a + ar^2 = 6ar$$

$$r^2 - 6r + 1 = 0$$

$$(r - 3)^2 - 9 + 1 = 0$$

$$(r - 3)^2 = 8$$

$$r - 3 = \pm\sqrt{8}$$

$$r = 3 \pm 2\sqrt{2}$$

Mark scheme

	Answer	Marks	Guidance
1	$-3, -10, -17, -24, -31$	B1	All five terms
2	$u_n = -3.5 + 1.5n$	B1	oe (e.g. $u_n = -2 + 1.5(n - 1)$)
3	$1, 0.5, 0.25, 0.125, 0.0625$	B1	oe (e.g. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$)
4	$r = 2, a = \frac{1}{2}$	M1	Solves simultaneous equations
	$u_n = \frac{1}{2} \times 2^{n-1} = \frac{1}{4} \times 2^n$	A1	Accept $u_n = \frac{1}{2}(2)^{n-1}$
5	$a = 1, d = 1, n = 100$	M1	Identifies AP
	$\frac{100}{2}(2 + 99 \times 1) (= 5050)$	A1	Substitutes correctly AG
6	$ r < 1$	B1	Accept $-1 < r < 1$
7	$\text{sum} = \frac{a(r^n - 1)}{r - 1}, r < 1$	B1	States condition on r
	$r^n \rightarrow 0 \text{ as } n \rightarrow \infty$	M1	States limiting argument
	$S_\infty = \frac{a}{1 - r}$	A1	Completes argument

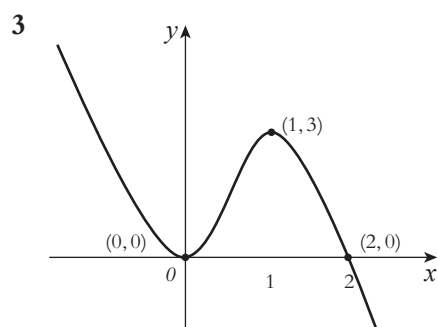
	Answer	Marks	Guidance
8	$\frac{3(1.5^n - 1)}{0.5} > 100$	M1 A1	Derives inequality Correct, including >
	$1.5^7 < 17\frac{2}{3}, 1.5^8 > 17\frac{2}{3}$	M1	Solves for n
	Eight terms are needed	A1	Conclusion must be drawn
9 (i)	Sum of AP and GP	B1	Uses sums of sequences
	$5(2 + 9d) = \frac{(r^{10} - 1)}{r - 1}$	M1	Establishes equation
	$d = \frac{1}{9} \left(\frac{(r^{10} - 1)}{5(r - 1)} - 2 \right)$	M1	Rearranges; accept substitution for d in the LHS to verify equality
		A1	Answer given
(ii)	$\frac{1}{9} \left(\frac{2^{10} - 1}{5} \right) - 20$	M1	Substitutes $r = 2$
	$d = \frac{1013}{45} = 22.5\bar{1}$	A1	Accept any equivalent form
10 (i)	$a = \frac{1}{4}, r = \frac{1}{4}$	M1 A1	Establishes the geometric progression (GP) Correct a and r
	$= \frac{\frac{1}{4}}{1 - \frac{1}{4}}$	M1	Uses sum to infinity
	$\frac{1}{3}$	A1	Concluding statement
(ii)	$a = \frac{1}{n+1}, r = \frac{1}{n+1}$	B1	Identifies GP
	$\frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}}$	M1	Uses sum to infinity
	$\frac{1}{n}$	A1	cao
11 (i)	$\frac{n}{2}(2a + (n-1)d) = 0$	B1	Sets sum of arithmetic progression (AP) to 0
	$a = \frac{-(n-1)d}{2}$	M1 A1	Rearranges... ...in this form with a as subject
(ii)	n is positive as it is the number of terms		
	d positive $\Rightarrow a$ negative, so (a) is not possible	B1	
	d is a positive multiple of 2 $\Rightarrow a$ negative integer, so (b) is possible	B1	
	d negative $\Rightarrow a$ is positive and an integer if n odd, so (c) is possible	B1	
	d positive $\Rightarrow a$ is negative if n is even or odd, so (d) is possible	B1	
			oe statements that demonstrate understanding of the general expression
12 (i)	Identifies a square	M1	Forms a strategy
	Calculates a relevant amount	M1	Makes a calculation
	Leads to $\frac{2}{9}$	A1	All correct

	Answer	Marks	Guidance
(ii)	$= \frac{2}{9} + \frac{1}{4} \times \frac{2}{9} + \dots$	M1	Identifies area sequence
	$a = \frac{2}{9}, r = \frac{1}{4}$	M1	Interprets as a GP
	$\frac{8}{27}$	A1	Correct sum to infinity
13	$a, a + 5d, a + 25d$	B1	AP
	a, ar, ar^2	B1	GP
	$(a + 5d)^2 = a(a + 25d)$	M1	Equates terms and derives an expression in two variables
	$d = 0, \frac{3a}{5}$	M1	Solves for d
		M1	Substitutes for r
	$r = 4$	A1	cao
14	$a, ar, ar^2, \dots$	B1	GP
	$a + ar^2 = 6ar$	B1	Forms equation
	$r^2 - 6r + 1 = 0$	B1	Simplifies to quadratic
		M1	Solves quadratic
	$r = 3 \pm 2\sqrt{2}$	A1	cao

4 Functions

- 1 A is many-to-one and is also a function, as it is 'to-one'.
 B is many-to-many and is **not** a function, as it is not 'to-one'.
 C is one-to-many and is **not** a function, as it is not 'to-one'.

2 $f(x) = x^2 - 5$, since x^2 is always ≥ 0 , $f(x) \geq -5$



$y = f(x + 1)$ means that the function, $f(x)$ has been translated by 1, in the negative x direction.
 Passes through $(0,0)$, $(1,3)$ and $(2,0)$.

- 4 Substitute f into g , as f is implemented first:

$$gf(x) = (3x - 7)^2 + 1 = 9x^2 - 42x + 50$$

- 5 Finding the composite function: $fg(x) = 3(x^2 + 1) - 7$

Forming the equation:

$$3(x^2 + 1) - 7 = 14 \Rightarrow 3(x^2 + 1) = 21 \Rightarrow x^2 + 1 = 7 \Rightarrow x^2 = 6$$

$$x = \pm\sqrt{6}$$

- 6 The inequality requires the same size value on each side.

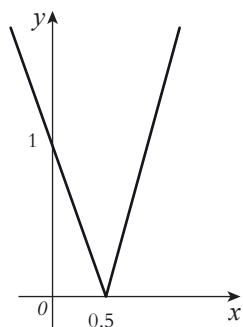
Subtracting the mean of 2 and 5 from each term,

$2 - 3.5 < x - 3.5 < 5 - 3.5$, gives:

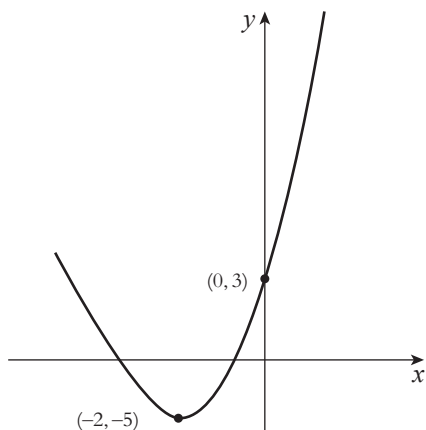
$$-1.5 < x - 3.5 < 1.5$$

$$|x - 3.5| < 1.5, a = 3.5 \text{ and } b = 1.5$$

- 7 Drawing the graph of $y = 2x - 1$, then reflecting the part below the x -axis, in the x -axis.



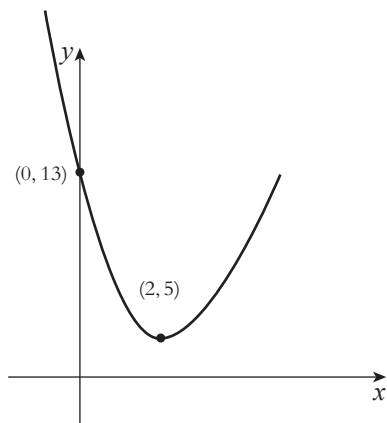
- 8 (i) $f(x)$ is the graph of $y = x^2$ translated horizontally to the left 2 units, stretched vertically scale factor 2, and translated vertically down 5 units.



$$f(0) = 2(0 + 2)^2 - 5 = 8 - 5 = 3$$

Range is $f(x) \geq -5$.

- (ii) $g(x)$ is the graph of $y = x^2$ translated horizontally to the right 2 units, stretched vertically scale factor 2, and translated vertically up 5 units.

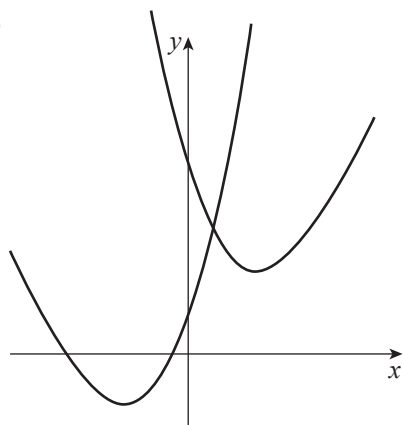


Minimum point is at (2, 5).

$$2(x - 2)^2 \geq 0 \text{ so } 2(x - 2)^2 + 5 \geq 5$$

Range is $g(x) \geq 5$.

(iii)



$$f(x) = g(x) \text{ when } 2(x+2)^2 - 5 = 2(x-2)^2 + 5$$

$$2x^2 + 8x + 3 = 2x^2 - 8x + 13$$

$$16x = 10$$

$$x = 0.625 \text{ (or using a solver)}$$

$$\text{So, } g(x) > f(x) \text{ when } 0 < x < 0.625$$

- 9 (i) (a) Curve is translated horizontally 1 unit to the left and vertically 1 unit up.

$$y = (x+1)^2(2(x+1)-3) + 1 = (x+1)^2(2x-1) + 1$$

Or

$x^2(2x-3)$ has roots $x = 0, 0, 1.5$, so translating 1 unit to the left gives roots $-1, -1, 0.5$

$$y = (x+1)^2(2x-1).$$

Then translating 1 unit upwards gives $y = (x+1)^2(2x-1) + 1$

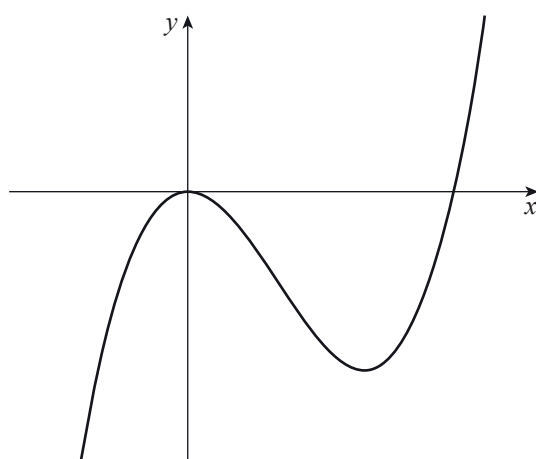
Or

$$x^2(2x-3) = -1 \text{ when } 2x^3 - 3x^2 + 1 = 0 \Rightarrow (x-1)^2(2x+1) = 0 \Rightarrow x = 1, 1, -0.5$$

So, translated curve has roots $0, 0, -1.5$ and hence $y = x^2(2x+3)$.

$$\begin{aligned} \text{(b)} \quad y &= -((x+1)^2(2x-1) + 1) \\ &= -(x+1)^2(2x-1) - 1, \quad y = -2x^3 - 3x^2 \end{aligned}$$

(ii) (a)



Function is many-to-one (graph goes up and down), so no inverse.

- (b) Need to restrict domain so that function is one-to-one, e.g. $x > 1.5$

10 (i) (a)

x	0	2	5
$f(x)$	3	1	-2

So,

x	2	4	7
$x - 2$	0	2	5
$f(x - 2)$	3	1	-2

Points (2,3), (4,1), (7,-2) are on $y = f(x - 2)$

(b)

x	0	2	5
$f(x)$	3	1	-2

So,

x	0	2	5
$f(x)$	3	1	-2
$3f(x)$	9	3	-6

Points (0,9), (2,3), (5,-6) are on $y = 3f(x)$

(c)

x	0	2	5
$f(x)$	3	1	-2

So,

x	0	2	5
$f(x)$	3	1	-2
$f(x) - 1$	2	0	-3

Points (0,2), (2,0), (5,-3) are on $y = f(x) - 1$

(d)

x	0	2	5
$f(x)$	3	1	-2

So,

x	0	1	2.5
$2x$	0	2	5
$f(2x)$	3	1	-2

Points (0,3), (1,1), (2.5,-2) are on $y = f(2x)$ (ii) (a) $f(0) = f(3)$, so f is many-to-one and $f^{-1}(x)$ does not exist (unless the domain of $f(x)$ is restricted).(b) $f^{-1}(x)$ may or may not exist. If it does exist then,

x	0	2	5
$f(x)$	3	1	-2

So,

x	3	1	-2
$f^{-1}(x)$	0	2	5

If $f^{-1}(x)$ exists then points (3,0), (1,2), (-2,5) are on $y = f^{-1}(x)$

- 11 (i) (a) $fgh(1) = fg(h(1)) = fg(2^1) = fg(2) = f(g(2)) = f(3 \times 2^2 + 1) = f(13) = 5 \times 13 + 2 = 67$
 (b) $h(x) > 1 \Rightarrow gh(x) > 4 \Rightarrow fgh(x) > 22$
 (ii) (a) $fg(x) = f(3x^2 + 1) = 5(3x^2 + 1) + 2 = 15x^2 + 7$
 $gf(x) = g(5x + 2) = 3(5x + 2)^2 + 1 = 75x^2 + 60x + 13$
 (b) $15x^2 + 7 = 75x^2 + 60x + 13 \Rightarrow 60x^2 + 60x + 6 = 0 \Rightarrow 10x^2 + 10x + 1 = 0$

$$x = \frac{-10 \pm \sqrt{60}}{20} = -0.113, -0.887$$

12 (i) $f(0) = \frac{0 - a}{0 - ka} = \frac{1}{k}$

Curve crosses y -axis at $\left(0, \frac{1}{k}\right)$

$f(x) = 0$ when $x - a = 0 \Rightarrow x = a$

Curve crosses x -axis at $(a, 0)$

(ii) Function $y = \frac{x - a}{x - ka}$ has inverse $x = \frac{y - a}{y - ka}$

$\Rightarrow x(y - ka) = y - a$ (multiply through by $y - ka$)

$\Rightarrow xy - xka = y - a$ (expand brackets)

$\Rightarrow xy - y = xka - a$ (collect terms involving y together)

$\Rightarrow y(x - 1) = xka - a$ (bring out y as a common factor)

$\Rightarrow y = \frac{a(kx - 1)}{x - 1}$

$f^{-1}(x) = \frac{a(kx - 1)}{x - 1}$

(iii) $\frac{4x - 2}{x - 1} = \frac{x - 4}{x - 2}$

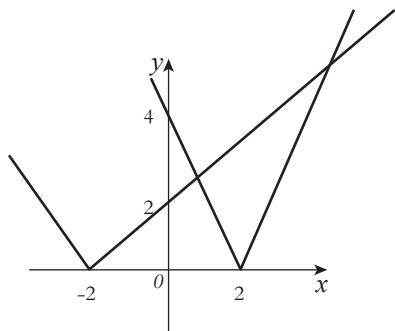
$\Rightarrow (x - 2)(4x - 2) = (x - 1)(x - 4)$

$\Rightarrow 4x^2 - 10x + 4 = x^2 - 5x + 4$

$\Rightarrow 3x^2 - 5x = 0$

$x = 0$ or $\frac{5}{3}$

13 (i)



$y = |x + 2|$ touches the x -axis at -2 , cuts the y -axis at $(0, 2)$ and has gradient ± 1

$y = |2x - 4|$ touches x -axis at 2 , cuts the y -axis at $(0, 4)$ and has gradient ± 2

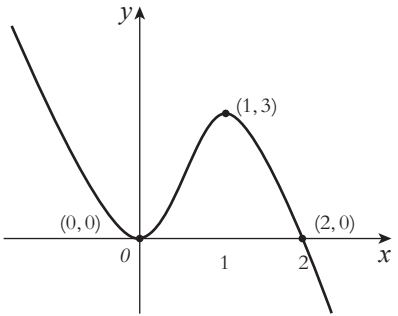
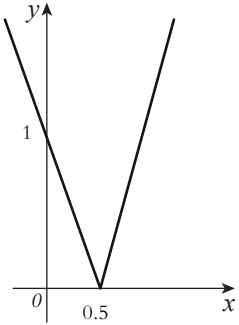
(ii) Graphs cross when $x + 2 = -(2x - 4)$ and $x + 2 = 2x - 4$

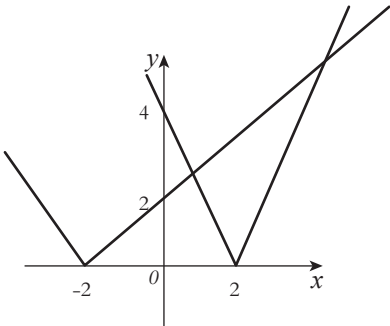
$\Rightarrow x = \frac{2}{3}$ and $x = 6$

From the graph, $|x + 2| > |2x - 4|$ between the points where the graphs cross.

So, when $\frac{2}{3} < x < 6$

Mark scheme

	Answer	Marks	Guidance
1	A is many-to-one and is also a function B is many-to-many and is not a function C is one-to-many and is not a function	B1 B1 B1	Both required Both required Both required
2	$f(x) \geq -5$	B1	cao
3	 Passes through (0,0), (1,3) and (2,0)	B1 B1 B1	Maximum value at (1,3) Passes through (0,0) or (2,0) Shape correct
4	$gf(x) = (3x - 7)^2 + 1$ $= 9x^2 - 42x + 50$	M1 A1	Substitutes f into g cao
5	$3(x^2 + 1) - 7 = 14$ $x = \pm\sqrt{6}$	M1 M1 A1	Forms equation Solves equation cao
6	$2 - 3.5 < x - 3.5 < 5 - 3.5$ $ x - 3.5 < 1.5, a = 3.5 \text{ and } b = 1.5$	M1 A1	Reduces the terms by 3.5 cao
7		B1 B1	Negative part reflected above x-axis All correct
8 (i)	$f(x) \geq 3$	M1 A1	Sketches graph or implied from sight of 3 $f(x) \geq 3$ or $y \geq 3$
(ii)	$g(x) \geq 5$	M1 A1	Sketches graph or implied from sight of 5 $g(x) \geq 5$ or $y \geq 5$
(iii)	$2(x - 2)^2 + 5 > 2(x + 2)^2 - 5$ $2x^2 - 8x + 13 > 2x^2 + 8x + 3$ $10 > 16x$ $0 < x < \frac{5}{8}$	M1 M1 A1	Solves $g(x) > f(x)$ or $g(x) = f(x)$ Broadly correct algebraically or using graph cao in fractions or decimal
9 (i)(a)	$y = (x + 1)^2(2(x + 1) - 3) + 1$ $= (x + 1)^2(2x - 1) + 1$	M1 A1	Substitutes $x + 1$ for x Or $y = 2x^3 + 3x^2$ or equivalent
(b)	$y = -((x + 1)^2(2x - 1) + 1)$ $= -(x + 1)^2(2x - 1) - 1$	M1 A1	-(their function from part (i)(a)) Or $y = -2x^3 - 3x^2$ oe, cao

	Answer	Marks	Guidance
(ii)(a)	Function is many-to-one, e.g. $x = 0$, $x = 1.5$ both give $y = 0$	B1	Many-to-one oe Accept not one-to-one
(b)	e.g. $x > 1.5$	B1	Any suitable interval (e.g. a subset of $x \leq 0$, or a subset of $0 \leq x \leq 1$, or a subset of $x > 1.5$)
10 (i) (a)	(2,3), (4,1), (7,-2)	B1	cao
(b)	(0,9), (2,3), (5,-6)	B1	cao
(c)	(0,2), (2,0), (5,-3)	B1	cao
(d)	(0,3), (1,1), (2.5,-2)	B1	cao
(ii)(a)	$f^{-1}(x)$ does not exist	B1	f is many-to-one
(b)	If $f^{-1}(x)$ exists then (3,0), (1,2), (-2,5) are on $y = f^{-1}(x)$	B1 B1	$f^{-1}(x)$ may or may not exist (3,0), (1,2), (-2,5)
11 (i) (a)	67	B1	cao
(b)	$fgh(x) > 22$	M1 A1	Calculates 22 $y > 1$ or $fgh(x) > 1$ (not ≥ 22)
(ii)(a)	$fg(x) = 15x^2 + 7$ $gf(x) = 3(5x + 2)^2 + 1$ $= 75x^2 + 60x + 13$	B1 M1 A1	$15x^2 + 7$ $3(5x + 2)^2 + 1$ $75x^2 + 60x + 13$
(b)	$10x^2 + 10x + 1 = 0$ $x = \frac{-10 \pm \sqrt{60}}{20} = -0.113, -0.887$	M1 A1	$10x^2 + 10x + 1 = 0$ oe Both solutions as surds or to 3 s.f. or better
12 (i)	$\left(0, \frac{1}{k}\right)$ (a, 0)	B1 B1	Accept $y = \frac{1}{k}$ Accept $x = a$
(ii)	$x = \frac{y-a}{y-ka} \Rightarrow y(x-1) = a(kx-1)$ $f^{-1}(x) = \frac{a(kx-1)}{x-1}$	M1 A1	Swaps x and y and rearranges $\frac{a(kx-1)}{x-1}$ oe
(iii)	$\frac{4x-2}{x-1} = \frac{x-4}{x-2}$ $x = 0$ or $\frac{5}{3}$	M1 A1 A1	Sets $f^{-1}(x) = \frac{1}{f(x)}$ with $a = 2$ and $k = 2$ $x = 0$ $x = \frac{5}{3}$
13 (i)		M1 M1 A1	V-shaped graph with vertex at (0,-2) V-shaped graph with vertex at (0,2) Cutting y -axis above first graph
(ii)	$x + 2 = -(2x - 4)$ when $x = \frac{2}{3}$ $x + 2 = 2x - 4$ when $x = 6$ $ x + 2 > 2x - 4 $ when $\frac{2}{3} < x < 6$	M1 A1 A1 A1	Solves $x + 2 = \pm(2x - 4)$ cao cao Correct region

5 Differentiation

1 At a point of inflection, the second derivative is zero, i.e. $\frac{d^2y}{dx^2} = 0$.

2 If $\frac{d^2y}{dx^2} < 0$, the stationary point is a maximum.

3 The point is a stationary point and a point of inflection if

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{d^2y}{dx^2} = 0$$

4 The function may be stationary or decreasing for it not to be increasing.

$$\text{So, } f'(x) \leq 0$$

$$f'(x) = 3x^2 - 12 = 0 \quad \text{when } x = \pm 2$$

The function $f'(x)$ is a quadratic with a positive coefficient of x^2 so has negative values between the roots.

$$\text{So, } -2 \leq x \leq 2$$

$$5 \quad y = \sqrt{3x + \frac{1}{x}} = \left(3x + \frac{1}{x}\right)^{\frac{1}{2}}$$

$$\text{Let } u = 3x + \frac{1}{x} = 3x + x^{-1} \Rightarrow \frac{du}{dx} = 3 - x^{-2}$$

$$y = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

Using the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{3 - \frac{1}{x^2}}{2\sqrt{3x + \frac{1}{x}}} \end{aligned}$$

6 Product rule: $y = x \times \sqrt{x-5}$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$v = \sqrt{x-5} = (x-5)^{\frac{1}{2}} \Rightarrow \frac{dv}{dx} = \frac{1}{2}(x-5)^{-\frac{1}{2}}$$

Substituting into the formula gives:

$$\begin{aligned} \frac{dy}{dx} &= x \times \frac{1}{2}(x-5)^{-\frac{1}{2}} + \sqrt{x-5} \times 1 \\ &= \frac{x}{2\sqrt{x-5}} + \sqrt{x-5} \end{aligned}$$

$$\text{So, } \frac{dy}{dx} = \frac{3x-10}{2\sqrt{x-5}}$$

7 Quotient rule (the formula is given in the paper):

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$y = \frac{x}{x^2 + 3x - 2}$$

$$u = x, \quad \frac{du}{dx} = 1$$

$$v = x^2 + 3x - 2, \quad \frac{dv}{dx} = 2x + 3$$

Substituting into the formula gives:

$$\frac{dy}{dx} = \frac{(x^2 + 3x - 2) \times 1 - x(2x + 3)}{(x^2 + 3x - 2)^2}$$

$$\begin{aligned}\text{So, } \frac{dy}{dx} &= \frac{x^2 + 3x - 2 - 2x^2 - 3x}{(x^2 + 3x - 2)^2} \\ &= \frac{-x^2 - 2}{(x^2 + 3x - 2)^2}\end{aligned}$$

8 $y = x^4 - 2x^3 - 12x^2 + 2x - 1$

The second derivative is needed to determine points of inflection:

$$\frac{dy}{dx} = 4x^3 - 6x^2 - 24x + 2$$

$$\frac{d^2y}{dx^2} = 12x^2 - 12x - 24$$

This is zero for points of inflection.

$$12x^2 - 12x - 24 = 0 \Rightarrow x^2 - x - 2 = 0$$

This factorises to give: $(x + 1)(x - 2) = 0$

So, the points of inflection are $(-1, -12)$ and $(2, -45)$.

9 (i) $f(x) = x^3 - x + 10$

Differentiating once: $f'(x) = 3x^2 - 1$

Differentiating again: $f''(x) = 6x$

$f''(x) < 0$ for the curve to be concave downwards.

$$6x < 0 \Rightarrow x < 0$$

So, concave downwards when $x < 0$.

(ii) At the point of inflection, $f''(x) = 0 \Rightarrow x = 0$, $f(0) = 10$

Coordinates are $(0, 10)$.

10 Volume of a cone, V , is $\frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = 7.5 \text{ cm}^3 \text{ s}^{-1}$

(i) To find the rate of change of h , substitute for r to form equations in h :

$$r:h = 2:3 \Rightarrow r = \frac{2}{3}h$$

$$V = \frac{1}{3}\pi \left(\frac{2}{3}h\right)^2 h = \frac{4\pi}{27}h^3$$

$$\frac{dV}{dh} = \frac{4\pi}{9}h^2$$

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{9}{4\pi h^2} \times 7.5\end{aligned}$$

When $r = 50$, $h = 75 \text{ cm}$

$$\begin{aligned}\text{So, } \frac{dh}{dt} &= \frac{9}{4\pi \times 75^2} \times 7.5 \\ &= 0.000955 \text{ cm s}^{-1}\end{aligned}$$

(ii) The shape of the cone may need to change, with radius to height ratio increasing, as the sides of the pond will probably get less steep.

11 $x = y + \frac{3}{2y^2 + 1}$

So, differentiate with respect to y to get:

$$\frac{dx}{dy} = 1 - \frac{12y}{(2y^2 + 1)^2}$$

Curve crosses y -axis when $x = 0$,

$$0 = y + \frac{3}{2y^2 + 1} \Rightarrow 2y^3 + y + 3 = 0$$

By inspection, $y = -1$ is a root of this equation. (Or use the Factor theorem.)

$$\begin{aligned}\text{When } y = -1, \frac{dx}{dy} &= 1 - \frac{12 \times -1}{(2(-1)^2 + 1)^2} \\ &= \frac{7}{3}\end{aligned}$$

So, gradient of curve $\frac{dy}{dx} = \frac{3}{7}$

- 12** Volume of water in cylindrical tank, V , is $\pi r^2 h$

$$\frac{dV}{dh} = \pi r^2 = 4\pi, \text{ since } r = 2$$

Rate of change of volume of water, $\frac{dV}{dt} = 6 \text{ m}^3 \text{ min}^{-1}$

Using the chain rule:

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

Using the inverse of $\frac{dV}{dh}$:

$$\begin{aligned}\frac{dh}{dt} &= \frac{1}{\left(\frac{dV}{dh}\right)} \times \frac{dV}{dt} \\ &= \frac{6}{4\pi}\end{aligned}$$

$$\frac{dh}{dt} = \frac{3}{2\pi} \text{ m min}^{-1}$$

- 13** $y = 2x^4 + ax^2 + 2x + b$ has a point of inflection at $\left(\frac{1}{2}, 1\right)$

$$\frac{dy}{dx} = 8x^3 + 2ax + 2$$

$$\frac{d^2y}{dx^2} = 24x^2 + 2a$$

At a point of inflection $\frac{d^2y}{dx^2} = 0$

$$\text{So, } 24x^2 + 2a = 0 \Rightarrow a = -\frac{1}{2} \times 24 \times \left(\frac{1}{2}\right)^2 = -3$$

$$\text{This gives } 24x^2 - 6 = 0 \Rightarrow x = \pm \frac{1}{2}$$

So, $x = -\frac{1}{2}$ is the x coordinate of the other point of inflection.

To find b , substitute the known coordinates into y .

$$1 = 2\left(\frac{1}{2}\right)^4 + (-3) \times \left(\frac{1}{2}\right)^2 + 2 \times \frac{1}{2} + b \Rightarrow b = \frac{5}{8}$$

$$\begin{aligned}\text{When } x = -\frac{1}{2}, y &= 2\left(-\frac{1}{2}\right)^4 + (-3) \times \left(-\frac{1}{2}\right)^2 + 2 \times \left(-\frac{1}{2}\right) + \frac{5}{8} \\ &= -1\end{aligned}$$

So, the other point of inflection is $\left(-\frac{1}{2}, -1\right)$.

- 14 (i)** Volume of a sphere, V , is $\frac{4}{3}\pi r^3$, so differentiating: $\frac{dV}{dr} = 4\pi r^2$

$$\text{Using the chain rule: } \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\begin{aligned}\frac{dr}{dt} &= \frac{1}{\frac{dV}{dr}} \times \frac{dV}{dt} \\ &= \frac{-2}{4\pi \times r^2} \\ &= -\frac{1}{200\pi}, \text{ when } r = 10 \text{ cm}\end{aligned}$$

Which means r is decreasing at $\frac{1}{200\pi} \text{ cm s}^{-1}$

(ii) Surface area $= 4\pi r^2$, so differentiating: $\frac{dS}{dr} = 8\pi r$

$$\begin{aligned}\text{Using the chain rule: } \frac{dS}{dt} &= \frac{dS}{dr} \times \frac{dr}{dt} \times \frac{dV}{dt} \\ &= 8\pi r \times \frac{1}{4\pi r^2} \times -2 \\ &= \frac{-4}{r}\end{aligned}$$

When $r = 10 \text{ cm}$, $\frac{dS}{dt} = -\frac{4}{10} \text{ cm}^2 \text{ s}^{-1}$, so the surface area is decreasing at $\frac{2}{5} \text{ cm}^2 \text{ s}^{-1}$

15 A general cubic is of the form $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

At the stationary points $\frac{dy}{dx} = 0$, $3ax^2 + 2bx + c = 0$

$$\text{So, } x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

The x coordinate of the mid-point is $\frac{1}{2} \left(\frac{-b + \sqrt{b^2 - 3ac}}{3a} + \frac{-b - \sqrt{b^2 - 3ac}}{3a} \right) = -\frac{b}{3a}$

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

At a point of inflection $\frac{d^2y}{dx^2} = 0$

$$\text{So, } 6ax + 2b = 0 \Rightarrow x = -\frac{b}{3a}$$

Hence result.

16 (i) An increasing function has a positive gradient:

$$\frac{dy}{dx} = 1 + \frac{4}{(x+1)^2} > 0 \text{ for all } x$$

So, the function is always increasing as x increases.

$$\text{(ii)} \quad \frac{dy}{dx} = 1 + \frac{4}{(x+1)^2} = 2$$

$$\text{So, } \frac{4}{(x+1)^2} = 1$$

$$(x+1)^2 = 4 \Rightarrow x+1 = \pm 2 \Rightarrow x = 1, -3$$

Coordinates are $(1, 1)$ and $(-3, 1)$

(iii) Tangents are $y = 2x - 1$ and $y = 2x + 7$

They cut the x -axis at $P\left(\frac{1}{2}, 0\right)$ and $Q\left(-\frac{7}{2}, 0\right)$

So, $PQ = 4$

Mark scheme

	Answer	Marks	Guidance
1	$\frac{d^2y}{dx^2} = 0$	B1	Accept $f''(x) = 0$
2	maximum	B1	
3	$\frac{dy}{dx} = 0$ (and $\frac{d^2y}{dx^2} = 0$)	B1	Second derivative = 0 may be assumed
4	$\frac{dy}{dx} = 3x^2 - 12 = 0$ $x = \pm 2$ $-2 \leq x \leq 2$	M1 A1 A1	Differentiates Identifies critical points Correct inequality
5	$\frac{1}{2}\left(3x + \frac{1}{x}\right)^{-\frac{1}{2}} \times \left(3 - \frac{1}{x^2}\right)$ $\frac{dy}{dx} = \frac{3 - \frac{1}{x^2}}{2\sqrt{3x + \frac{1}{x}}}$	M1 A1 A1	Differentiates using chain rule Derivative $\left(3 - \frac{1}{x^2}\right)$ correct All correct; accept equivalent forms
6	$x \times \frac{1}{2}(x-5)^{-\frac{1}{2}} + \sqrt{x-5} \times 1$ $= \frac{x}{2\sqrt{x-5}} + \sqrt{x-5}$	M1 M1 A1 A1	Differentiates using product rule Differentiates using chain rule First term correct All correct, with square root signs, not indices
7	$\frac{dy}{dx} = \frac{(x^2 + 3x - 2) \times 1 - x(2x + 3)}{(x^2 + 3x - 2)^2}$ $\frac{-x^2 - 2}{(x^2 + 3x - 2)^2}$	M1 A1 A1	Differentiates using quotient rule Numerator correct All correct
8	$\frac{dy}{dx} = 4x^3 - 6x^2 - 24x + 2$ $\frac{d^2y}{dx^2} = 12x^2 - 12x - 24$ $x^2 - x - 2 = 0$ $(-1, -12)$ and $(2, -45)$	M1 M1 M1 A2	Differentiates once Differentiates again Solves equation = 0 A1 one correct; A2 both correct
9 (i)	$f'(x) = 3x^2 - 1$ $f''(x) = 6x$ $x < 0$	B1 B1 B1	Differentiates once Differentiates again Solves inequality
(ii)	$f''(x) = 0$ $(0, 10)$	M1 A1	Solves equation Both coordinates correct
10 (i)	$r = \frac{2}{3}h$ $V = \frac{4\pi}{27}h^3$ $\frac{dV}{dh} = \frac{4\pi}{9}h^2$	B1 B1 M1	Forms equation for r in terms of h Correct formula for volume in terms of h Differentiates V

	Answer	Marks	Guidance
	$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{9}{4\pi h^2} \times 7.5$ When $r = 50$, $h = 75$ cm $\frac{dh}{dt} = 0.000\,955 \text{ cm s}^{-1}$	M1 A1	Uses chain rule Correct to 3 s.f. cao
(ii)	The shape of the cone may need to change, with radius to height ratio increasing, as the sides of the pond will probably get less steep	E1	Or other appropriate comment
11	$\frac{dx}{dy} = 1 - \frac{12y}{(2y^2 + 1)^2}$ $2y^3 + y + 3 = 0$ $y = -1$ $\frac{dx}{dy} = 1 - \frac{12 \times -1}{(2(-1)^2 + 1)^2} = \frac{7}{3}$ So, gradient of curve, $\frac{dy}{dx} = \frac{3}{7}$	M1 A1 M1 A1 M1 A1	Differentiates wrt y Correct derivative Solves cubic Correct y intercept Evaluates $\frac{dx}{dy}$ Applies chain rule for inverses
12	$V = \pi r^2 h = 4\pi h$ $\frac{dV}{dh} = 4\pi$ $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $\frac{dh}{dt} = \frac{1}{\left(\frac{dV}{dh}\right)} \times \frac{dV}{dt} = \frac{6}{4\pi}$ $\frac{dh}{dt} = \frac{3}{2\pi} \text{ m min}^{-1}$	M1 M1 M1 M1 A1	Uses formula for volume Differentiates V Uses chain rule Uses inverse relationship Do not penalise lack of units or decimal equivalents
13	$\frac{dy}{dx} = 8x^3 + 2ax + 2$ $\frac{d^2y}{dx^2} = 24x^2 + 2a (= 0)$ $a = -3$ $24x^2 - 6 = 0 \left(\Rightarrow x = \pm \frac{1}{2} \right)$ $x = -\frac{1}{2}$ $1 = 2\left(\frac{1}{2}\right)^4 + (-3) \times \left(\frac{1}{2}\right)^2 + 2 \times \frac{1}{2} + b$ $b = \frac{5}{8}$ $\left(-\frac{1}{2}, -1\right)$	M1 M1 A1 M1 A1 M1 A1	Differentiates once Differentiates again Uses condition for a point of inflection Finds other root cao Finds b Coordinates correct

	Answer	Marks	Guidance
14 (i)	$\frac{dV}{dr} = 4\pi r^2$	M1	Differentiates volume
	$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt},$	M1	Uses chain rule
	$\frac{dr}{dt} = \frac{1}{\left(\frac{dV}{dr}\right)} \times \frac{dV}{dt}$ $-\frac{1}{200\pi}$	A1	oe (must be negative)
(ii)	$\frac{dS}{dr} = 8\pi r$	M1	Differentiates surface area
	$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$	M1	Uses chain rule
	$= 8\pi r \times \frac{1}{4\pi r^2} \times -2 \left(= \frac{-4}{r} \right)$	A1	Correct terms
	$-\frac{2}{5} \text{ cm}^2 \text{ s}^{-1}$	A1	oe (must be negative)
15	$\frac{dy}{dx} = 3ax^2 + 2bx + c$	M1	Differentiates once
	$3ax^2 + 2bx + c = 0$	M1	Uses condition for stationary points
	$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$	M1	Finds x coordinate of midpoint
	$\frac{1}{2} \left(\frac{-b + \sqrt{b^2 - 3ac}}{3a} + \frac{-b - \sqrt{b^2 - 3ac}}{3a} \right)$		
	$= -\frac{b}{3a}$	A1	Correct
	$\frac{d^2y}{dx^2} = 6ax + 2b$	M1	Differentiates again
	$6ax + 2b = 0$	M1	Uses condition for point of inflection
16 (i)	$\frac{dy}{dx} = 1 + \frac{4}{(x+1)^2}$	M1	Differentiates
	> 0	A1	Correct
	y increases as x increases	E1	Deduces derivative is always positive
(ii)	$\frac{dy}{dx} = 1 + \frac{4}{(x+1)^2} = 2$	M1	Forms equation
	$x = 1, -3$	M1	Solves equation
	(1,1) and (-3,1)	A1	Correct x coordinates
		A1	All correct
(iii)	$y = 2x - 1$ and $y = 2x + 7$	M1	Determines equations of tangents
	$P\left(\frac{1}{2}, 0\right)$ and $Q\left(-\frac{7}{2}, 0\right)$	A1	Correct x-intercepts
	PQ = 4	A1	Implies correct method

6 Trigonometric functions

- 1 The smallest angle is opposite the smallest side.

Using the cosine rule, with $a = 3$, $b = 5$, $c = 7$, gives:

$$3^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos A \Rightarrow \cos A = \frac{5^2 + 7^2 - 3^2}{2 \times 5 \times 7} = \frac{65}{70}$$

$$A = \arccos \frac{65}{70} = 0.38 = 0.4 \text{ radians, to the nearest } 0.1 \text{ radians.}$$

- 2 $\tan 2x = 1$

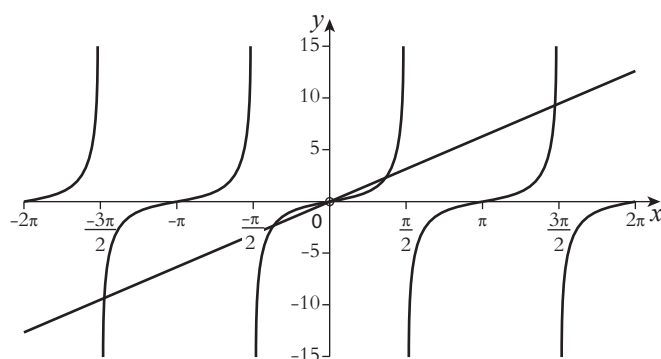
Using inverse tan gives:

$$2x = \arctan 1 = 45^\circ, \text{ or } 225^\circ, \text{ or } 405^\circ$$

$$\text{So, } x = 22.5^\circ, 112.5^\circ, 202.5^\circ$$

The only one in the interval $90^\circ \leq x < 180^\circ$ is 112.5°

- 3 Drawing both graphs:



Five solutions.

$$4 \quad \operatorname{cosec} \frac{\pi}{6} = \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$$5 \quad 2\cos^2 x - 13\cos x + 6 = 0$$

Factorising gives: $(2\cos x - 1)(\cos x - 6) = 0$

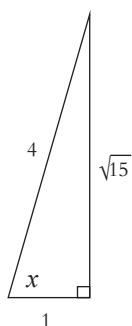
Either

$\cos x = 6$, which is not possible.

Or

$$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, -\frac{\pi}{3}$$

$$6 \quad \cos x = \frac{1}{4}, \text{ so the third side of the triangle} = \sqrt{4^2 - 1^2} = \sqrt{15}$$



So, $\tan x = \sqrt{15}$

7 $\cot x = 2 \cos x$

Substituting for $\cot x$ and rearranging:

$$\frac{\cos x}{\sin x} = 2 \cos x$$

$$\Rightarrow \cos x = 2 \cos x \sin x$$

$$\Rightarrow \cos x - 2 \cos x \sin x = 0$$

$$\Rightarrow \cos x(1 - 2 \sin x) = 0$$

Either

$$\cos x = 0, x = -\frac{\pi}{2}, \frac{\pi}{2}$$

Or

$$1 - 2 \sin x = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

8 Using $C = \cos x$, $2C^2 - 3C + 1 = 0 \Rightarrow (2C - 1)(C - 1) = 0$

$$2\cos^2 x - 3 \cos x + 1 = 0 \Rightarrow (2\cos x - 1)(\cos x - 1) = 0$$

Either $(2\cos x - 1) = 0$ or $(\cos x - 1) = 0$

$$\text{So, } \cos x = \frac{1}{2} \text{ or } \cos x = 1$$

$$x = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \text{ or } x = \arccos(1) = 0$$

9 To solve $\sin\left(3x - \frac{\pi}{6}\right) = 1$, using the inverse function, gives:

$$\arcsin(1) = \frac{\pi}{2}$$

Other solutions are obtained by adding or subtracting integer multiples of 2π .

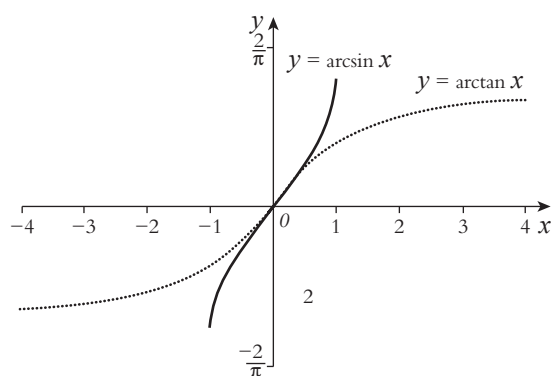
$$\text{So, } 3x - \frac{\pi}{6} = \dots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots$$

$$\text{Add } \frac{\pi}{6} \text{ to get } 3x = \dots, -\frac{10\pi}{3}, -\frac{4\pi}{3}, \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \dots$$

$$\text{Divide by 3 to get } x = \dots, -\frac{10\pi}{9}, -\frac{4\pi}{9}, \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{14\pi}{9}, \dots$$

Reject solutions that are not in the range $-\pi < x \leq \pi$ to get $x = -\frac{4\pi}{9}, \frac{2\pi}{9}, \frac{8\pi}{9}$

10 (i)



(ii) (a) $\sin \frac{\pi}{2} = 1$, so $\arcsin(1) = \frac{\pi}{2}$ (Or using 90° from calculator and converting to radians.)

$\tan \frac{\pi}{4} = 1$, so $\arctan(1) = \frac{\pi}{4}$ (Or using 45° from calculator and converting to radians.)

$$(b) \frac{\arcsin(1)}{\arctan(1)} = \frac{\pi}{2} \times \frac{4}{\pi} = 2$$

But, $\arccos(1) = 0$

Or

$$\frac{\arcsin(1)}{\arctan(1)} = \frac{\pi}{2} \times \frac{4}{\pi} = 2 = \arccos(-0.416), \text{ not } \arccos(1)$$

Or

$$\arccos(1) = 0 \text{ but } \arcsin(1) \neq 0 \text{ so } \frac{\arcsin(1)}{\arctan(1)} = \frac{\pi}{2} \times \frac{4}{\pi} \neq 0$$

11 Equation becomes: $\frac{\cos x}{\sin x} = \frac{1}{\cos x} \Rightarrow \cos^2 x = \sin x$

Using identity, so quadratic is in just one trigonometric ratio:

$$\cos^2 x = 1 - \sin^2 x \Rightarrow 1 - \sin^2 x = \sin x \Rightarrow \sin^2 x + \sin x - 1 = 0$$

Solving quadratic using the formula:

$$\sin x = \frac{-1 \pm \sqrt{5}}{2}$$

Since $\sin x = \cos^2 x$ it is not possible for $\sin x$ to be negative.

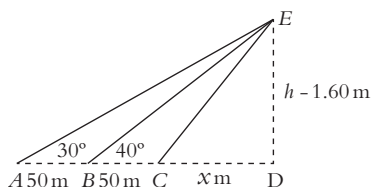
$$\frac{-1 - \sqrt{5}}{2} < -1, \text{ so } \sin x = \frac{-1 + \sqrt{5}}{2}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{2}{-1 + \sqrt{5}}$$

Rationalise the denominator:

$$\operatorname{cosec} x = \frac{2}{\sqrt{5} - 1} \times \frac{1 + \sqrt{5}}{1 + \sqrt{5}} = \frac{\sqrt{5} + 1}{2}$$

12 (i)



Angle $ABE = 140^\circ$ so angle $AEB = 10^\circ$

Using the sine rule on triangle ABE :

$$\frac{BE}{\sin 30^\circ} = \frac{50}{\sin 10^\circ}$$

$$BE = 143.97 \text{ m}$$

(ii) (a) $h - 1.60 = BE \sin 40^\circ = 92.54 \text{ m}$

So, $h = 94.1 \text{ m}$

(b) $x + 50 = BE \cos 40^\circ = 110.29 \text{ m}$

So, $x = 60.29 \text{ m}$

Edge length $= 2x = 120.6 \text{ m}$

13 (i) For example:

$$\frac{\sec^2 \theta - 2}{\cos \theta + \sin \theta} \equiv \frac{1 + \tan^2 \theta - 2}{\cos \theta + \sin \theta} \equiv \frac{\tan^2 \theta - 1}{\cos \theta + \sin \theta}$$

$$\equiv \frac{(\tan \theta - 1)(\tan \theta + 1)}{\cos \theta(1 + \tan \theta)} \equiv \frac{\tan \theta - 1}{\cos \theta}, \text{ provided } \tan \theta + 1 \neq 0 \text{ (i.e. } \cos \theta + \sin \theta \neq 0)$$

$$\equiv \sec \theta(\tan \theta - 1)$$

Or

$$\sec \theta(\tan \theta - 1)(\cos \theta + \sin \theta) \equiv (\tan \theta - 1)(1 + \tan \theta)$$

$$\equiv \tan^2 \theta - 1 \equiv \sec^2 \theta - 1 - 1 \equiv \sec^2 \theta - 2$$

Hence result.

(ii) For example:

$$\sec \theta (\tan \theta - 1) = \tan \theta - \sec^2 \theta$$

$$\Rightarrow \frac{1}{\cos \theta} \left(\frac{\sin \theta}{\cos \theta} - 1 \right) = \frac{\sin \theta}{\cos \theta} - \frac{1}{\cos^2 \theta}, \cos \theta \neq 0$$

Multiply through by $\cos^2 \theta$

$$\sin \theta - \cos \theta = \sin \theta \cos \theta - 1$$

$$\sin \theta - \sin \theta \cos \theta - \cos \theta + 1 = 0$$

$$(\sin \theta + 1)(1 - \cos \theta) = 0$$

$$\sin \theta = -1 \text{ or } \cos \theta = 1$$

But, $\cos \theta \neq 0$ so $\sin \theta = -1$ is not possible.

So, $\cos \theta = 1$

$\theta = 0, 2\pi$ are the only solutions in the given range.

14 (i) For example:

$$\operatorname{cosec}^2 x \equiv 1 + \cot^2 x$$

$$\operatorname{cosec}^2 x = 2 \cot x$$

$$\Rightarrow 1 + \cot^2 x = 2 \cot x$$

$$\Rightarrow (\cot x - 1)^2 = 0$$

$$\Rightarrow \cot x = 1$$

(ii) Using definition of cot gives:

$$\cot x = 1$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \dots, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$

$$\cot(4\theta + \pi) = 1$$

$$\Rightarrow (4\theta + \pi) = \dots, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$

$$\Rightarrow 4\theta = \dots, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$

$$\Rightarrow \theta = -\frac{7\pi}{16}, -\frac{3\pi}{16}, \frac{\pi}{16}, \frac{5\pi}{16}$$

(iii) $\operatorname{cosec}^2(5\theta) \tan(5\theta) = 2$

$$\Rightarrow \operatorname{cosec}^2(5\theta) = 2 \cot(5\theta)$$

$$\Rightarrow \cot(5\theta) = 1$$

$$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow 0 \leq 5\theta \leq \frac{5\pi}{2}$$

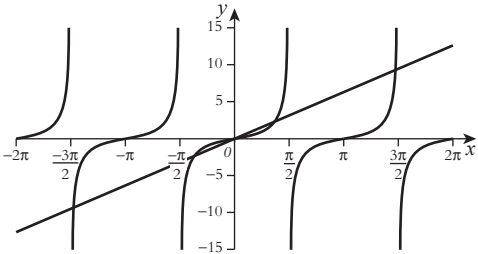
$$5\theta = \frac{\pi}{4}, \frac{\pi}{4} + \pi, \frac{\pi}{4} + 2\pi = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$$

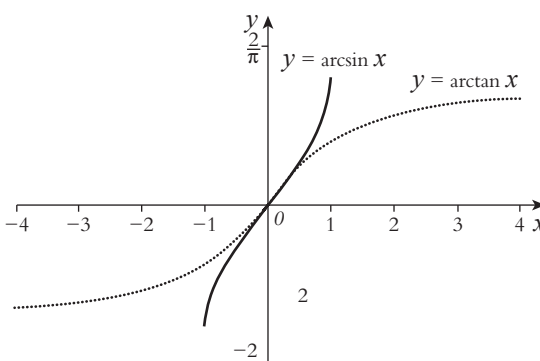
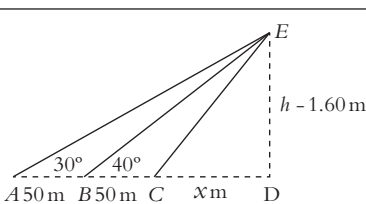
or

$$\theta = \frac{\pi}{20}, \frac{\pi}{20} + \frac{\pi}{5}, \frac{\pi}{20} + \frac{2\pi}{5} = \frac{\pi}{20}, \frac{\pi}{4}, \frac{9\pi}{20}$$

So, three solutions.

Mark scheme

	Answer	Marks	Guidance
1	$\cos A = \frac{5^2 + 7^2 - 3^2}{2 \times 5 \times 7}$ 0.4 rad	M1 A1	Uses cosine rule appropriately cao
2	$2x = \arctan 1 = 45^\circ$, or 225° 112.5°	B1 B1	Finds principal value cao
3	 Five solutions	B1 B1 B1	Graph of $y = 2x$ Graph of $y = \tan x$ cao
4	$\frac{1}{\sin \frac{\pi}{6}}$ 2	M1 A1	Uses $\frac{1}{\sin}$ cao
5	$(2\cos x - 1)(\cos x - 6) = 0$ $\cos x = 6$, which is not possible Or $\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, -\frac{\pi}{3}$	M1 B1 A1	Factorises Eliminates root Both roots
6	$\sqrt{4^2 - 1^2}$ $\sqrt{15}$	M1 A1	Uses Pythagoras cao
7	$\frac{\cos x}{\sin x} = 2 \cos x$ $\Rightarrow \cos x(1 - 2 \sin x) = 0$ $-\frac{\pi}{2}, \frac{\pi}{2}$ $\frac{\pi}{6}, \frac{5\pi}{6}$	M1 M1 A1 A1	Uses definition of $\cot x$ Factorises Both values Both values
8	$(2\cos x - 1)(\cos x - 1) = 0$ $\cos x = \frac{1}{2}$ or $\cos x = 1$ $x = \frac{\pi}{3}$ or 0	M1 A1 B1	Factorises or uses quadratic formula Both values correct or implied from answers Both values in radians; accept 1.05 (or better) for $\frac{\pi}{3}$
9	$\arcsin(1) = \frac{\pi}{2}$ $3x - \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow x = \frac{2\pi}{9}$ $3x - \frac{\pi}{6} = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$ $x = -\frac{4\pi}{9}, \frac{2\pi}{9}, \frac{8\pi}{9}$	B1 M1 M1 A1	Principal solution Adds $\frac{\pi}{6}$ then divides by 3 Considers (some) other solutions These solutions and no others

	Answer	Marks	Guidance
10 (i)		B1 B1	Both graphs have rotational symmetry and pass through the origin y -axis from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$; $\arcsin(x)$ above $\arctan(x)$ for $0 < x < 1$ graphs have roughly the right shapes
(ii)(a)	$\arcsin(1) = \frac{\pi}{2}$ $\arctan(1) = \frac{\pi}{4}$	B1 B1	cao in exact form cao in exact form
(b)	$\frac{\arcsin(1)}{\arctan(1)} = 2$ but $\arccos(1) = 0$	M1 A1	Checks ratio Or, $\arccos(1) = 0$ and $\arcsin(1) \neq 0$
11	$\frac{\cos x}{\sin x} = \frac{1}{\cos x} \Rightarrow 1 - \sin^2 x = \sin x$ $\sin^2 x + \sin x - 1 = 0$ $\sin x = \frac{-1 \pm \sqrt{5}}{2}$ $\sin x = \frac{-1 + \sqrt{5}}{2}$ $\operatorname{cosec} x = \frac{2}{-1 + \sqrt{5}}$ $= \frac{1 + \sqrt{5}}{2}$	M1 A1 M1 A1 M1 A1	Uses $\cos^2 x + \sin^2 x = 1$, oe, to form an equation that involves just one trig function (sin, cos, tan oe) of x oe Solves (their) quadratic equation Rejects any infeasible solutions (or implied) to get a single value for $\sin x$ or $\operatorname{cosec} x$ Derives a correct expression for $\operatorname{cosec} x$ cao, in this form
12 (i)	 Angle $AEB = 10^\circ$ $\frac{BE}{\sin 30^\circ} = \frac{50}{\sin 10^\circ}$ $BE = 143.97 \text{ m} = 144 \text{ m (3 s.f.)}$	B1 M1 A1	Diagram optional Seen or implied, may be shown on diagram Uses the sine rule on triangle ABE 143.9 to 144.0

	Answer	Marks	Guidance
12 (ii)(a)	$h - 1.60 = BE \sin 40^\circ = 92.54 \text{ m}$	M1	Uses (their) $BE \sin 40^\circ$ or 92.5 oe seen
	$h = 94.1 \text{ m}$	A1	
	(b) $x + 50 = BE \cos 40^\circ = 110.29 \text{ m}$	M1	Uses (their) $BE \cos 40^\circ$ or 60.3 oe seen
	Edge length = $120.6 \text{ m} = 121 \text{ m}$ (3 s.f.)	A1	cao (3 s.f. or better)
13 (i)	$\frac{\sec^2 \theta - 2}{\cos \theta + \sin \theta} \equiv \frac{1 + \tan^2 \theta - 2}{\cos \theta + \sin \theta} \equiv \frac{\tan^2 \theta - 1}{\cos \theta + \sin \theta}$	M1	Simplifies LHS using $\sec^2 \theta \equiv 1 + \tan^2 \theta$ or $\cos^2 \theta + \sin^2 \theta \equiv 1$
	$\equiv \frac{(\tan \theta - 1)(\tan \theta + 1)}{\cos \theta (1 + \tan \theta)}$	M1	Factorises
	$\equiv \frac{\tan \theta - 1}{\cos \theta} \equiv \sec \theta (\tan \theta - 1)$	A1	Cancels and simplifies to achieve RHS
(ii)	$\sec \theta (\tan \theta - 1) = \tan \theta - \sec^2 \theta$	M1	Uses the result from part (i)
	$\frac{1}{\cos \theta} \left(\frac{\sin \theta}{\cos \theta} - 1 \right) = \frac{\sin \theta}{\cos \theta} - \frac{1}{\cos^2 \theta}$		
	$\Rightarrow \sin \theta - \cos \theta = \sin \theta \cos \theta - 1$	M1	Simplifies algebraically or uses an appropriate graph
	$\Rightarrow (\sin \theta + 1)(1 - \cos \theta) = 0$	A1	$\cos \theta = 1$ or $\sec \theta = 1$ or $\theta = 0$ oe
	$\sin \theta = -1$ or $\cos \theta = 1$	B1	Excluding $\theta = \frac{3\pi}{2}$ (may be by having a vertical asymptote at $\theta = \frac{3\pi}{2}$ on graph)
	$\cos \theta \neq 0$ so $\sin \theta = -1$ is not possible		
	$\theta = 0, 2\pi$	B1	These solutions and no others
14 (i)	$\operatorname{cosec}^2 x \equiv 1 + \cot^2 x$	M1	Uses $\operatorname{cosec}^2 x \equiv 1 + \cot^2 x$
	$\Rightarrow 1 + \cot^2 x = 2 \cot x$ $\Rightarrow (\cot x - 1)^2 = 0$ $\Rightarrow \cot x = 1$	A1	Deduces $(\cot x - 1)^2 = 0$, oe, and hence given result
(ii)	$\operatorname{arccot} 1 = \frac{\pi}{4}$ or $\arctan 1 = \frac{\pi}{4}$	B1	$\frac{\pi}{4}$ seen or implied as a solution of $\cot x = 1$
	$(4\theta + \pi) = \frac{\pi}{4} \Rightarrow 4\theta = -\frac{3\pi}{4} \Rightarrow \theta = -\frac{3\pi}{16}$	M1	Subtracts π and then divides by 4 or a correct value for θ
	$(4\theta + \pi) = \dots, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \dots$		
	$\Rightarrow 4\theta = \dots, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \dots$	A1	At least three correct values for 4θ or for θ
	$\Rightarrow \theta = -\frac{7\pi}{16}, -\frac{3\pi}{16}, \frac{\pi}{16}, \frac{5\pi}{16}$	A1	cao, these four values and no others

7 Further algebra

- $$\begin{aligned}\frac{x^3 + 2x^2 + x}{2x^4 + 2x^3 - x - 1} &= \frac{x(x+1)^2}{(x+1)(2x^3 - 1)} \\ &= \frac{x(x+1)}{2x^3 - 1}\end{aligned}$$

$$\begin{aligned}
 6 \quad \text{Check left-hand side } (1-x)^{\frac{1}{2}} &= \left(1 - \frac{1}{50}\right)^{\frac{1}{2}} \\
 &= \frac{7}{\sqrt{50}} \\
 &= \frac{7}{5\sqrt{2}} \\
 &= \frac{7\sqrt{2}}{10}
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } \sqrt{2} &= \frac{10}{7} \left(1 - \frac{1}{100} - \frac{1}{20000}\right) \\
 &= \frac{10}{7} \times \frac{19799}{20000} \\
 &= \frac{19799}{14000}
 \end{aligned}$$

7 Expanding binomial using formula gives:

$$(1 - 2x + 3x^2 - 4x^3 + \dots)$$

Calculating product:

$$(1 + 3x)(1 - 2x + 3x^2 - 4x^3 + \dots) = 1 + x - 3x^2 + 5x^3$$

Valid for $|x| < 1$

8 Combining the fractions gives:

$$\frac{A}{1-3x} + \frac{B}{3+x} + \frac{C}{(3+x)^2} \equiv \frac{A(3+x)^2 + B(1-3x)(3+x) + C(1-3x)}{(1-3x)(3+x)^2}$$

Numerators are equal because denominators are the same.

$$A(3+x)^2 + B(1-3x)(3+x) + C(1-3x) \equiv 2x + 5$$

Substituting $x = \frac{1}{3}$:

$$\frac{100}{9}A = 5\frac{2}{3} \Rightarrow A = 0.51$$

Substituting $x = -3$:

$$10C = -1 \Rightarrow C = -0.1$$

Equating coefficients of x^2 :

$$A - 3B = 0 \Rightarrow B = 0.17$$

Hence, $A = 0.51$, $B = 0.17$, $C = -0.1$

$$\frac{2x+5}{(1-3x)(3+x)^2} \equiv \frac{0.51}{1-3x} + \frac{0.17}{3+x} - \frac{0.1}{(3+x)^2}$$

9 (i) Using the formula for the binomial expansion gives:

$$\begin{aligned}
 (1+2x)^{\frac{1}{3}} &\approx 1 + \frac{1}{3} \times 2x + \frac{\frac{1}{3} \times -\frac{2}{3}}{2!} (2x)^2 \\
 &= 1 + \frac{1}{3} \times 2x - \frac{1 \times 2}{3^2 \times 2!} (4x^2) \\
 &= 1 + \frac{2}{3}x - \frac{8}{18}x^2 \\
 &= 1 + \frac{2}{3}x - \frac{4}{9}x^2
 \end{aligned}$$

(ii) $|2x| < 1$ so $|x| < \frac{1}{2}$

10 $\frac{4x^3 - x + 6}{2x^2 + x - 1}$ can be written in the form $Ax + B + \frac{C}{mx + n} + \frac{D}{px + q}$

$4x^3 - x + 6 \equiv (2x^2 + x - 1)(2x - 1) + 2x + 5$ by inspection.

So,

$$\begin{aligned}\frac{4x^3 - x + 6}{2x^2 + x - 1} &= \frac{(2x^2 + x - 1)(2x - 1)}{2x^2 + x - 1} + \frac{2x + 5}{2x^2 + x - 1} \\ &\equiv 2x - 1 + \frac{2x + 5}{2x^2 + x - 1} \quad A = 2, B = -1\end{aligned}$$

$$2x^2 + x - 1 \equiv (2x - 1)(x + 1)$$

$$\frac{2x + 5}{2x^2 + x - 1} \equiv \frac{2x + 5}{(2x - 1)(x + 1)} \equiv \frac{C}{2x - 1} + \frac{D}{x + 1} \quad m = 2, n = -1, p = 1, q = 1$$

$$2x + 5 \equiv C(x + 1) + D(2x - 1)$$

Equating terms gives $C = 4, D = -1$

Hence,

$$f(x) = 2x - 1 + \frac{4}{2x - 1} - \frac{1}{x + 1}$$

11 Using the formula for the binomial expansion gives:

$$(8 - x)^{-\frac{1}{3}} = \frac{1}{(8 - x)^{\frac{1}{3}}} = \frac{1}{8^{\frac{1}{3}} \left(1 - \frac{x}{8}\right)^{\frac{1}{3}}} = \frac{1}{2} \left(1 - \frac{x}{8}\right)^{-\frac{1}{3}}$$

$$\left(1 - \frac{x}{8}\right)^{-\frac{1}{3}} = 1 + \left(-\frac{1}{3}\right) \times \left(-\frac{x}{8}\right) + \frac{\left(-\frac{1}{3}\right) \times \left(-\frac{4}{3}\right)}{2!} \left(-\frac{x}{8}\right)^2 + \dots = 1 + \frac{1}{24}x + \frac{1}{288}x^2 + \dots$$

$$\text{So, } (8 - x)^{-\frac{1}{3}} \approx \frac{1}{2} + \frac{1}{48}x + \frac{1}{576}x^2 \text{ for } |x| < 8$$

12 (i) Combining the fractions gives:

$$\begin{aligned}\frac{3x + 1}{x^2 - 1} + \frac{4}{x - 1} &= \frac{3x + 1}{(x - 1)(x + 1)} + \frac{4(x + 1)}{(x - 1)(x + 1)} \\ &= \frac{7x + 5}{(x - 1)(x + 1)}\end{aligned}$$

(ii) Expanding the binomial gives:

$$\begin{aligned}\frac{3x + 1}{x^2 - 1} + \frac{4}{x - 1} &= \frac{7x + 5}{x^2 - 1} \\ &= -(7x + 5)(1 - x^2)^{-1} \\ &= -(7x + 5)(1 + x^2 + \dots) \\ &= -5 - 7x - 5x^2 - 7x^3 + \dots\end{aligned}$$

Cubic approximation is $-5 - 7x - 5x^2 - 7x^3$

(iii) $x = 0.1$ in $\frac{3x + 1}{x^2 - 1} + \frac{4}{x - 1}$ gives $-5.75757575 \dots$ or $-5\frac{25}{33}$

$x = 0.1$ in $-5 - 7x - 5x^2 - 7x^3$ gives -5.757

Absolute error = $0.00057575 \dots$

So, percentage absolute error = $\frac{0.00057575}{5.75757575} \times 100\% = 0.01\%$

- 13 (i)** Expanding using the binomial formula gives:

$$\begin{aligned}\frac{(6-x)(1+2x)}{1+x} &= (6-x)(1+2x)(1+x)^{-1} \\ &= (6+11x-2x^2)(1-x+x^2-\dots) \\ &= 6+5x-7x^2+\dots\end{aligned}$$

Quadratic approximation is $6+5x-7x^2$

- (ii)** Expansion of $(1+x)^{-1}$ is valid when $|x| < 1$

- (iii)** Combining the fractions gives:

$$\begin{aligned}(6+5x-7x^2) - \frac{(6-x)(1+2x)}{1+x} \\ &= \frac{(6+5x-7x^2)(1+x) - (6-x)(1+2x)}{1+x} \\ &= \frac{(6+11x-2x^2-7x^3) - (6+11x-2x^2)}{1+x} \\ &= \frac{-7x^3}{1+x}\end{aligned}$$

For positive x , absolute error $= \frac{7x^3}{1+x}$

$$\frac{7x^3}{1+x} < \frac{x}{1+x} \Rightarrow 7x^2 < 1, \text{ since } x > 0$$

$$\Rightarrow 0 < x < \frac{1}{\sqrt{7}} \text{ or } \frac{\sqrt{7}}{7} \text{ or } 0.378 \text{ (3 s.f.)}$$

- 14 (i)** Forming the partial fractions:

$$\frac{4x+5}{(1+5x)(1-x)} = \frac{A}{1+5x} + \frac{B}{1-x}$$

$$4x+5 \equiv A(1-x) + B(1+5x)$$

Setting $x = 1$:

$$9 = 6B \Rightarrow B = 1.5$$

Equating constant terms:

$$5 = A + B \Rightarrow A = 3.5$$

Hence,

$$\frac{4x+5}{(1+5x)(1-x)} = \frac{7}{2(1+5x)} + \frac{3}{2(1-x)}$$

- (ii)** Expanding using the binomial formula gives:

$$\begin{aligned}(1+5x)^{-1} &= 1 + (-1) \times (5x) + \frac{(-1) \times (-2)}{2!} (5x)^2 + \frac{(-1) \times (-2) \times (-3)}{3!} (5x)^3 + \dots \\ &\approx 1 - 5x + 25x^2 - 125x^3\end{aligned}$$

$$\begin{aligned}(1-x)^{-1} &= 1 + (-1) \times (-x) + \frac{(-1) \times (-2)}{2!} (-x)^2 + \frac{(-1) \times (-2) \times (-3)}{3!} (-x)^3 + \dots \\ &\approx 1 + x + x^2 + x^3\end{aligned}$$

So,

$$\begin{aligned}\frac{4x+5}{(1+5x)(1-x)} &\approx \frac{7}{2}(1-5x+25x^2-125x^3) + \frac{3}{2}(1+x+x^2+x^3) \\ &= 5 - 16x + 89x^2 - 436x^3\end{aligned}$$

- (iii)** $|5x| < 1$ and $|-x| < 1 \Rightarrow |x| < 0.2$ and $|x| < 1 \Rightarrow |x| < 0.2$

Mark scheme

	Answer	Marks	Guidance
1	$ x < \frac{1}{3}$ $ x < \frac{1}{5}$ $ x < 0.2$	M1 A1	Identifies validity of one Selects correct one
2	$\frac{1}{\sqrt{9+6x}} = \frac{1}{\sqrt{9}\sqrt{1+\frac{6x}{9}}}$ $= \frac{1}{3}\left(1+\frac{6x}{9}\right)^{-\frac{1}{2}}$ $= \frac{1}{3}\left(1+\frac{2x}{3}\right)^{-\frac{1}{2}}$	M1 M1 A1	Extracts common factor Expresses using an index Simplifies, AG
3	$x^2 + 4x + 5 + \frac{5}{x-1}$	M1 A1	Divides or by inspection cao
4	$= 1 - 2x + 3x^2 - 4x^3$	B1 B1 B1	x^3 term correct x^2 term correct All correct
5	$= \frac{x(x+1)^2}{(x+1)(2x^3-1)}$ $= \frac{x(x+1)}{2x^3-1}$	B1 B1 B1	Numerator correct Denominator correct Cancels correctly
6	$\frac{10}{7}\left(1 - \frac{1}{100} - \frac{1}{20000}\right)$ $= \frac{10}{7} \times \frac{19799}{20000}$ $= \frac{19799}{14000}$	B1 M1 A1	Deals with LHS Substitutes into RHS cao
7	$(1+3x)(1-2x+3x^2-4x^3+\dots)$ $\approx 1+x-3x^2+5x^3$ Valid for $ x < 1$	M1 A2 B1	Expands binomial A1 for two terms correct Validity correct
8	$A(3+x)^2 + B(1-3x)(3+x) + C(1-3x)$ $\equiv 2x+5$ $A=0.51; B=0.17$ $C=-0.1$	B1 B1 M1 A1 B1	$A(3+x)^2$ and $C(1-3x)$ $B(1-3x)(3+x)$ Substitutes values or equates coefficients A and B both correct C correct
9 (i)	$1 + \frac{1}{3} \times 2x + \frac{\left(\frac{1}{3}\right) \times \left(-\frac{2}{3}\right)}{2!} (2x)^2$ $= 1 + \frac{2}{3}x - \frac{4}{9}x^2$	M1 M1 B1 A1	$\frac{\left(\frac{1}{3}\right) \times \left(-\frac{2}{3}\right)}{2!}$ seen or implied $(2x)^2$ seen or implied $1 + \frac{2}{3}x$ Correct expansion
(ii)	$ x < \frac{1}{2}$	B1	Or $-0.5 < x < 0.5$

	Answer	Marks	Guidance
10	$4x^3 - x + 6$	M1	Divides out
	$\equiv (2x^2 + x - 1)(2x - 1) + 2x + 5$		
	$f(x) = 2x - 1 + \frac{2x + 5}{2x^2 + x - 1}$	A1	$A = 2, B = -1$
	$2x^2 + x - 1 \equiv (2x - 1)(x + 1)$		
	$\frac{2x + 5}{(2x - 1)(x + 1)} \equiv \frac{C}{2x - 1} + \frac{D}{x + 1}$	B1	$m = 2, n = -1, p = 1, q = 1$
	$2x + 5 \equiv C(x + 1) + D(2x - 1)$	M1	Substitutes or equates terms
	$f(x) = 2x - 1 + \frac{4}{2x - 1} - \frac{1}{x + 1}$	A1	$C = 4, D = -1$
11	$(8 - x)^{-\frac{1}{3}} = \frac{1}{2} \left(1 - \frac{x}{8}\right)^{-\frac{1}{3}}$	B1	Extracts factor $\frac{1}{2}$
	$\left(1 - \frac{x}{8}\right)^{-\frac{1}{3}} =$	M1	$-\frac{1}{3}$ and $\frac{\left(-\frac{1}{3}\right) \times \left(-\frac{4}{3}\right)}{2!}$
	$1 + \left(-\frac{1}{3}\right) \times \left(-\frac{x}{8}\right) + \frac{\left(-\frac{1}{3}\right) \times \left(-\frac{4}{3}\right)}{2!} \left(-\frac{x}{8}\right)^2$	M1	$\left(-\frac{x}{8}\right)^2$ oe
	$= 1 + \frac{1}{24}x + \frac{1}{288}x^2$	M1	$\frac{1}{24}$ and $\frac{1}{288}$ seen or implied
	$(8 - x)^{-\frac{1}{3}} \approx \frac{1}{2} + \frac{1}{48}x + \frac{1}{576}x^2$	A1	Correct expansion
12 (i)	$\frac{3x + 1}{(x - 1)(x + 1)} + \frac{4(x + 1)}{(x - 1)(x + 1)}$	M1	Writes terms with a common denominator, which may be $(x^2 - 1)(x - 1)$
	$= \frac{7x + 5}{(x - 1)(x + 1)}$	A1	Adds and simplifies numerator as $7x + 5$ Accept $\frac{7x + 5}{x^2 - 1}$ but not $\frac{7x^2 - 2x - 5}{(x^2 - 1)(x - 1)}$
(ii)	$\frac{7x + 5}{x^2 - 1} = -(7x + 5)(1 - x^2)^{-1}$	M1	$(x^2 - 1)^{-1} = -(1 - x^2)^{-1}$ seen or implied
	Hence $-5 - 7x - 5x^2 - 7x^3$	A1	ft
(iii)	$\frac{(-5.757) - (-5.75757...)}{(-5.75757...)}$	M1	Substitutes $x = 0.1$ in both expressions and calculating a relative error
	0.01%	A1	0.01% from correct working seen
13 (i)	$(6 + 11x - 2x^2)(1 - x + x^2 - \dots)$	M1	$(1 + x)^{-1} \approx 1 - x + x^2$
	$6 + 5x - 7x^2$	A1	$-7x^2 + 5x + 6$
(ii)	$ x < 1$	B1	Or $-1 < x < 1$

	Answer	Marks	Guidance
(iii)	$6 + 5x - 7x^2 - \frac{(6-x)(1+2x)}{1+x}$ $= \frac{(6 + 5x - 7x^2)(1+x) - (6-x)(1+2x)}{1+x}$ $= \frac{-7x^3}{1+x}$ $\frac{7x^3}{1+x} < \frac{x}{1+x} \Rightarrow 7x^2 < 1, \text{ since } x > 0$ $0 < x < \frac{\sqrt{7}}{7}$	M1 A1 M1 A1	Calculates error $\frac{-7x^3}{1+x}$ or $\frac{7x^3}{1+x}$ Compares with $\frac{x}{1+x}$ and simplifies $\frac{1}{\sqrt{7}}$ or $\frac{\sqrt{7}}{7}$ or 0.378 (3 s.f.)
14 (i)	$\frac{4x+5}{(1+5x)(1-x)} = \frac{A}{1+5x} + \frac{B}{1-x}$ $4x+5 \equiv A(1-x) + B(1+5x)$ $A = 3.5, B = 1.5$ $\frac{7}{2(1+5x)} + \frac{3}{2(1-x)} \text{ or } \frac{3.5}{1+5x} + \frac{1.5}{1-x}$	B1 M1 A1	Forms partial fractions Substitutes or equates terms Both constants correct
(ii)	$(1+5x)^{-1} \approx 1 - 5x + 25x^2 - 125x^3$ $(1-x)^{-1} \approx 1 + x + x^2 + x^3$ $\frac{4x+5}{(1+5x)(1-x)}$ $\approx \frac{7}{2}(1 - 5x + 25x^2 - 125x^3) +$ $\frac{3}{2}(1 + x + x^2 + x^3)$ $= 5 - 16x + 89x^2 - 436x^3$	B1 B1 M1 B1 A1	$1, \pm 5x, \pm 25x^2, \pm 125x^3$ $1 + x + x^2 + x^3$ Uses their A and B from part (i) and their cubic expansions Constant term = 5 Other coefficients correct
(iii)	$ x < 0.2$	B1	Or $-0.2 < x < 0.2$, but not with $ x < 1$ as well

8 Trigonometric identities

- Using the compound angle identity:
 $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $A = B = \theta$
Substituting gives:
 $\sin 2\theta = \sin \theta \cos \theta + \cos \theta \sin \theta$
 $= 2 \sin \theta \cos \theta$
- Using the compound angle identity:
 $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $A = B = \theta$
Substituting gives:
 $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
 $= \cos^2 \theta - (1 - \cos^2 \theta)$
 $= 2\cos^2 \theta - 1$
- $y = R \cos(x - \alpha)$ has a maximum when $\cos(x - \alpha) = 1$
So, when $x - \alpha = 0$, the maximum value is R , so the coordinates are (α, R) .

- 4 Translation of $y = \cos x$ by $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

Stretch of $y = \cos(x - \alpha)$ by a factor R parallel to the y -axis.

- 5 By inspection:

$\tan 3\theta = \tan \theta$ has one solution when $3\theta = \theta$ ($n = 0$) and the general solution is $3\theta = n\pi + \theta$, $n = 0, 1, 2, 3, \dots$

Subtracting:

$$2\theta = n\pi \Rightarrow \theta = 0, \frac{\pi}{2}, \pi$$

- 6 By inspection (or solving $A + B = 75$, $A - B = 15$):

$$\begin{aligned} \sin 75^\circ - \sin 15^\circ &= \sin(45^\circ + 30^\circ) - \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ - \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= 2 \cos 45^\circ \sin 30^\circ \\ &= 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

- 7 To show $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ using double angle identity for $\cos 2\theta$ involving $\cos$, as the right-hand side involves only $\cos\theta$:

$$\begin{aligned} \cos 4\theta &= 2\cos^2 2\theta - 1 \\ &= 2(2\cos^2\theta - 1)^2 - 1 \\ &= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 \\ &= 8\cos^4\theta - 8\cos^2\theta + 1 \text{ as required.} \end{aligned}$$

- 8 (i) $R \cos(x + \alpha) = R(\cos x \cos \alpha - \sin x \sin \alpha)$, $R, \alpha > 0$

Equating coefficients:

$$R \cos \alpha = 1, \quad R \sin \alpha = 1$$

Solving the simultaneous equations by squaring and adding:

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 1^2 + 1^2$$

$$R^2(\cos^2 \alpha + \sin^2 \alpha) = 2$$

$$R^2 = 2 \Rightarrow R = \sqrt{2}$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{1}{1}$$

$$\tan \alpha = 1 \Rightarrow \alpha = 45^\circ$$

$$\text{So, } \cos x - \sin x = \sqrt{2} \cos(x + 45^\circ)$$

- (ii) The minimum value of cosine is -1 , so the minimum value of $\sqrt{2} \cos(x + 45^\circ)$ is $-\sqrt{2}$

- (iii) To solve $\cos x - \sin x = 0.1$ for $0 \leq x \leq 360^\circ$

$$\sqrt{2} \cos(x + 45^\circ) = 0.1$$

$$\Rightarrow \cos(x + 45^\circ) = \frac{0.1}{\sqrt{2}} = 0.07071\dots$$

$$\Rightarrow x + 45^\circ = \cos^{-1} 0.07071\dots = 85.945^\circ \text{ or } 274.055^\circ$$

$$\Rightarrow x = 40.9^\circ \text{ or } 229.1^\circ$$

9 (i) $2 \sin x - 3 \cos x = R \sin(x - \alpha) = R \sin x \cos \alpha - R \cos x \sin \alpha$

Comparing the two expressions gives:

$$R \cos \alpha = 2, R \sin \alpha = 3$$

Solving, by squaring and adding to get R , and dividing to get α , gives:

$$R = \sqrt{2^2 + 3^2} = \sqrt{13}, \tan \alpha = \frac{3}{2} \Rightarrow \alpha = \arctan \frac{3}{2} = 0.983$$

$$\text{So, } 2 \sin x - 3 \cos x = \sqrt{13} \sin(x - 0.983)$$

(ii) Using the previous result:

$$\begin{aligned} f(x) &= \frac{1}{k + 2 \sin x - 3 \cos x} \\ &= \frac{1}{k + \sqrt{13} \sin(x - 0.983)} \end{aligned}$$

This is a minimum when $\sin(x - 0.983)$ is a maximum, 1

This only occurs once in the interval as $\sin(x - 0.983)$ is periodic with period 2π .

$$\text{So, the minimum value of } f(x) = \frac{1}{k + \sqrt{13}}$$

(iii) The minimum value is $\frac{1}{2}$

$$\frac{1}{k + \sqrt{13}} = \frac{1}{2} \Rightarrow k + \sqrt{13} = 2 \Rightarrow k = 2 - \sqrt{13}$$

10 $\tan \theta = \frac{a}{b}$

Using Pythagoras theorem:

$$\sin \theta = \frac{a}{\sqrt{a^2 + b^2}}, \cos \theta = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\text{So, } \cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{b^2}{a^2 + b^2} - \frac{a^2}{a^2 + b^2} \text{ and } \sin 2\theta = 2 \sin \theta \cos \theta = \frac{2ab}{a^2 + b^2}$$

Substituting:

$$\begin{aligned} a \cos 2\theta - b \sin 2\theta &= a \left(\frac{b^2}{a^2 + b^2} - \frac{a^2}{a^2 + b^2} \right) - b \left(\frac{2ab}{a^2 + b^2} \right) \\ &= \frac{1}{a^2 + b^2} (ab^2 - a^3 - 2ab^2) \\ &= \frac{-a(a^2 + b^2)}{a^2 + b^2} \\ &= -a \end{aligned}$$

11 Use $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$ and $\tan 30^\circ = \frac{1}{\sqrt{3}}$

Let $t = \tan 15^\circ$ and substitute into the double angle identity:

$$\frac{1}{\sqrt{3}} = \frac{2t}{1 - t^2}$$

$$\Rightarrow 1 - t^2 = 2\sqrt{3}t$$

$$\Rightarrow t^2 + 2\sqrt{3}t - 1 = 0$$

Completing the square:

$$(t + \sqrt{3})^2 - 1 - 3 = 0$$

$$(t + \sqrt{3})^2 = 4$$

$$t + \sqrt{3} = \pm 2$$

$$t = \pm 2 - \sqrt{3}$$

$$\tan 15^\circ > 0, \tan 15^\circ = 2 - \sqrt{3}$$

- 12** Solving $\cos 2\theta - \sin \theta - 1 = 0$, use the substitution for $\cos 2\theta$ that involves $\sin \theta$ to get a quadratic in $\sin \theta$.

$$1 - 2\sin^2 \theta - \sin \theta - 1 = 0$$

$$\Rightarrow -2\sin^2 \theta - \sin \theta = 0$$

$$\Rightarrow \sin \theta (2\sin \theta + 1) = 0$$

Either

$$\sin \theta = 0 \Rightarrow \theta = 0, \pi, 2\pi$$

Or

$$2\sin \theta + 1 = 0 \Rightarrow \sin \theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

- 13 (i)** $\sqrt{3}\cos x + \sin x$ in the form $R\cos(x - \alpha) = R(\cos x \cos \alpha + \sin x \sin \alpha)$

Equating coefficients:

$$\sqrt{3} = R\cos \alpha, \quad 1 = R\sin \alpha \Rightarrow R = 2$$

$$\tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 30^\circ$$

$$\sqrt{3}\cos x + \sin x = 2\cos(x - 30^\circ)$$

(ii) $2\cos(x - 30^\circ) = \sqrt{2}$

$$\Rightarrow \cos(x - 30^\circ) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x - 30^\circ = 360^\circ k \pm 45^\circ \quad (k = 0 \text{ or } 1) = 45^\circ, 315^\circ$$

$$\Rightarrow x = 75^\circ, 345^\circ$$

- 14** Using the compound angle identity:

$$\cos\left(\theta + \frac{\pi}{6}\right) = \cos \theta \cos \frac{\pi}{6} - \sin \theta \sin \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2}\cos \theta - \frac{1}{2}\sin \theta$$

$$2\sin\left(\theta + \frac{\pi}{3}\right) = 2\left(\sin \theta \cos \frac{\pi}{3} + \cos \theta \sin \frac{\pi}{3}\right)$$

$$= \sin \theta + \sqrt{3}\cos \theta$$

Solving the equation, having substituted:

$$\frac{\sqrt{3}}{2}\cos \theta - \frac{1}{2}\sin \theta = \sin \theta + \sqrt{3}\cos \theta$$

$$\Rightarrow \frac{3}{2}\sin \theta + \frac{\sqrt{3}}{2}\cos \theta = 0$$

$$\Rightarrow 3\sin \theta + \sqrt{3}\cos \theta = 0$$

$$\Rightarrow \cos \theta (3\tan \theta + \sqrt{3}) = 0$$

Either

$$\cos \theta = 0, \quad \theta = \pm \frac{\pi}{2}$$

Or

$$3\tan \theta + \sqrt{3} = 0, \quad \tan \theta = -\frac{\sqrt{3}}{3}, \quad \theta = -\frac{\pi}{6}, \frac{5\pi}{6}$$

$$-\pi \leq \theta \leq \pi, \quad \theta = -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

Mark scheme

	Answer	Marks	Guidance
1	$\sin 2\theta = \sin \theta \cos \theta + \cos \theta \sin \theta$ $2 \sin \theta \cos \theta$	M1 A1	Substitutes $A = B = \theta$ into compound angle identity Answer given
2	$\cos 2\theta = \cos \theta \cos \theta - \sin \theta \sin \theta$ $= \cos^2 \theta - \sin^2 \theta$ $= \cos^2 \theta - (1 - \cos^2 \theta)$ $= 2\cos^2 \theta - 1$	M1 M1 A1	Substitutes $A = B = \theta$ into compound angle identity Uses identity Answer given
3	$\cos(x - \alpha) = 1, x - \alpha = 0$ Maximum is (α, R)	M1 A1	Identifies maximum value of 1 Both correct
4	Translation by $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ Stretch by a factor R Parallel to the y -axis	B1 M1 A1	Identifies translation Identifies stretch and scale factor All correct
5	$3\theta = \theta$ $3\theta = n\pi + \theta, n = 0, 1, 2, 3, \dots$ $\theta = 0, \frac{\pi}{2}, \pi$	M1 M1 A1 A1	Identifies principal solution Generalises One correct All correct
6	$= \sin(45^\circ + 30^\circ) - \sin(45^\circ - 30^\circ)$ $= 2 \cos 45^\circ \sin 30^\circ$ $= 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{2}$ $= \frac{\sqrt{2}}{2}$	M1 M1 M1 A1	Converts to compound angle form Simplifies expanded form Substitutes values cao
7	$\cos 4\theta = 2\cos^2 2\theta - 1$ $= 2(2\cos^2 \theta - 1)^2 - 1$ $= 8\cos^4 \theta - 8\cos^2 \theta + 1$	M1 M1 A1	Uses cos version of identity Uses cos version again Answer given
8 (i)	$R(\cos x \cos \alpha - \sin x \sin \alpha), R, \alpha > 0$ $R \cos \alpha = 1, R \sin \alpha = 1$ $R = \sqrt{2}$ $\alpha = 45^\circ$	M1 M1 A1 A1	Uses identity Solves equations Correct R Correct angle
(ii)	$-\sqrt{2}$	B1	cao
(iii)	$\cos(x + 45^\circ) = \frac{0.1}{\sqrt{2}} (= 0.07071\dots)$ $x + 45^\circ = 85.945^\circ \text{ or } 274.055^\circ$ $x = 40.9^\circ \text{ or } 229.1^\circ$	M1 M1 A1 A1	Uses previous result Finds solutions in range One correct Both correct

	Answer	Marks	Guidance
9 (i)	$R \sin x \cos \alpha - R \cos x \sin \alpha$	M1	Uses identity
	$R \cos \alpha = 2, R \sin \alpha = 3$	M1	Solves equations
	$R = \sqrt{2^2 + 3^2} = \sqrt{13}$	A1	Correct R
	$\alpha = \arctan \frac{3}{2} = 0.983$	A1	Accept either version
(ii)	$\frac{1}{k + \sqrt{13} \sin(x - 0.983)}$	M1	Uses previous result
	minimum when $\sin(x - 0.983) = 1$	M1	Identifies when minimum occurs
	$\frac{1}{k + \sqrt{13}}$	A1	Correct
(iii)	$\frac{1}{k + \sqrt{13}} = \frac{1}{2}$	M1	Forms equation
	$k = 2 - \sqrt{13}$	A1	Fully correct
10	$\sin \theta = \frac{a}{\sqrt{a^2 + b^2}}$	B1	Expression for $\sin \theta$
	$\cos \theta = \frac{b}{\sqrt{a^2 + b^2}}$	B1	Expression for $\cos \theta$
	$\cos^2 \theta - \sin^2 \theta = \frac{b^2}{a^2 + b^2} - \frac{a^2}{a^2 + b^2}$	M1	Uses trig identities for $\cos 2\theta, \sin 2\theta$
	$2 \sin \theta \cos \theta = \frac{2ab}{a^2 + b^2}$	A1	Correct
	$a \cos(2\theta) - b \sin(2\theta) = \frac{1}{a^2 + b^2} (ab^2 - a^3 - 2ab^2)$	M1	Simplifies
	$= -a$	A1	cao
11	$\tan 30^\circ = \frac{1}{\sqrt{3}}$	B1	$\tan 30^\circ$
	$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$	M1	Uses trig identity
	$\tan^2 15^\circ + 2\sqrt{3} \tan 15^\circ - 1 = 0$	M1	Forms quadratic
		A1	Correct quadratic
	$\tan 15^\circ = 2 - \sqrt{3}$	M1	Solves quadratic
		A1	Selects positive solution
12	$1 - 2 \sin^2 \theta - \sin \theta - 1 = 0$	M1	Uses appropriate identity
	$\Rightarrow -2 \sin^2 \theta - \sin \theta = 0$	A1	Correct equation
	$\Rightarrow \sin \theta (2 \sin \theta + 1) = 0$	M1	Factorises expression
	$\sin \theta = 0$	M1	Solves equation
	$\Rightarrow \theta = 0, \pi, 2\pi$	A1	All three values, expressed exactly
	$2 \sin \theta + 1 = 0 \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$	A1	Both values
13 (i)	$\sqrt{3} = R \cos \alpha$	M1	Uses identity
	$1 = R \sin \alpha$	M1	Solves equations
	$2 \cos(x - 30^\circ)$	A1	Value for R
		A1	Value for angle

	Answer	Marks	Guidance
(ii)	$\cos(x - 30^\circ) = \frac{1}{\sqrt{2}}$	M1	Uses result from part (i)
	$x - 30^\circ = 45^\circ, 315^\circ$	M1	Derives second value
	$x = 75^\circ, 345^\circ$	A1	Both correct
14	$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta$	M1	Uses identity
	$2\sin\left(\theta + \frac{\pi}{3}\right) = \sin\theta + \sqrt{3}\cos\theta$	M1	Uses identity
	$\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \sin\theta + \sqrt{3}\cos\theta$	M1	Substitutes to form equation
	$3\sin\theta + \sqrt{3}\cos\theta = 0$	A1	Correct equation
	$\cos\theta(3\tan\theta + \sqrt{3}) = 0$	M1	Factorises
	Either		
	$\cos\theta = 0$	M1	Solves for cos
	$\theta = \pm\frac{\pi}{2}$	A1	Correct values
	Or		
	$3\tan\theta + \sqrt{3} = 0$	M1	Solves for tan
	$\theta = -\frac{\pi}{6}, \frac{5\pi}{6}$	A1	Fully correct

9 Further Differentiation

- 1 If $y = \ln x$ then $x = e^y$

$$\frac{dx}{dy} = e^y = x$$

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{x}$$

So, the derivative of $y = \ln x$ is $\frac{1}{x}$

- 2 $y = \sin kx$

$$\frac{dy}{dx} = k \cos kx = 0 \text{ when } kx = \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{2k} = \frac{\pi}{6} \text{ so } k = 3$$

- 3 $y = \ln 2x$

The curve cuts the x -axis when:

$$y = 0 \Rightarrow \ln 2x = 0 \Rightarrow 2x = e^0 = 1 \Rightarrow x = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{2}{2x} = \frac{1}{x} = 2 \text{ when } x = \frac{1}{2}$$

So, the gradient is 2 where the curve crosses the x -axis.

- 4 $y = x \cos x$

Differentiating the product with respect to x :

$$\frac{dy}{dx} = \cos x - x \sin x$$

For a stationary point:

$$\cos x - x \sin x = 0$$

Rearranging gives:

$$x \sin x = \cos x$$

Dividing both sides by $\sin x$ gives:

$$x = \cot x \text{ as required}$$

- 5 Differentiating $ax^2 - by^2 = c$ implicitly with respect to x :

$$2ax - 2by \frac{dy}{dx} = 0$$

Rearranging gives:

$$\frac{dy}{dx} = \frac{ax}{by}$$

- 6 Differentiating gives:

$$\frac{dy}{dx} = -2x ke^{1-x^2}$$

This is zero when $x = 0$

$$f(0) = ke = 1, \text{ so } k = \frac{1}{e}$$

- 7 If $y = \tan^{-1} 3x$ then $3x = \tan y$

Differentiating with respect to y :

$$\frac{dx}{dy} = \frac{1}{3} \sec^2 y$$

Using a Pythagorean identity:

$$\frac{dx}{dy} = \frac{1}{3} (1 + \tan^2 y)$$

Substituting:

$$\frac{dx}{dy} = \frac{1}{3} (1 + (3x)^2)$$

Using the chain rule:

$$\frac{dy}{dx} = \left(\frac{dx}{dy} \right)^{-1} = \frac{3}{1 + 9x^2}$$

- 8 If $y = \sin^{-1} x$, then $x = \sin y$

Differentiating with respect to y :

$$\frac{dx}{dy} = \cos y$$

Using a Pythagorean identity:

$$\frac{dx}{dy} = \sqrt{1 - (\sin y)^2}$$

Substituting:

$$\frac{dx}{dy} = \sqrt{1 - x^2}$$

Using the chain rule:

$$\frac{dy}{dx} = \left(\frac{dx}{dy} \right)^{-1} = \frac{1}{\sqrt{1 - x^2}}$$

- 9 (i) Differentiating $\frac{dy}{dx} = \cos x - k(\sec x)^2 = 0$ at stationary points gives:

$$\cos x - k(\cos x)^{-2} = 0, (\cos x)^3 = k$$

since $|\cos x| \leq 1$, this will only have solutions for $-1 \leq k \leq 1$

- (ii) $(\cos x)^3 = \frac{1}{8}$, $\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}, \frac{5\pi}{3}$ in the given interval.

10 Differentiating $y = x \ln x$ with respect to x gives:

$$\frac{dy}{dx} = x \times \frac{1}{x} + \ln x = 1 + \ln x$$

Stationary point:

$$\frac{dy}{dx} = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$$

Is it a minimum?

$$\frac{d^2y}{dx^2} = \frac{1}{x} = e > 0 \text{ when } x = e^{-1}, \text{ so it is a minimum.}$$

$$\text{Minimum value is } y = e^{-1} \ln e^{-1} = -e^{-1} = -\frac{1}{e}$$

11 (i) Differentiating $3x^2 + 4y^2 = 12$ implicitly with respect to x gives:

$$6x + 8y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3x}{4y}$$

$$\text{When } x = 1, 4y^2 = 9 \Rightarrow y = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}$$

$$\text{Equation of tangent at } \left(1, \frac{3}{2}\right) \text{ has gradient } -\frac{3}{6} = -\frac{1}{2}$$

$$\text{So, equation is } y = -\frac{1}{2}x + 2$$

$$\text{Equation of tangent at } \left(1, -\frac{3}{2}\right) \text{ has gradient } \frac{3}{6} = \frac{1}{2}$$

$$\text{So, equation is } y = \frac{1}{2}x - 2$$

(ii) Tangents meet when $-\frac{1}{2}x + 2 = \frac{1}{2}x - 2 \Rightarrow x = 4, y = 0$ (i.e. on the x -axis).

12 (i) $x^2 + y^2 = r^2$ is the equation of the circle.

So, differentiating implicitly with respect to x gives:

$$2x + 2y \frac{dy}{dx} = 0$$

$$\text{So, the gradient is } \frac{dy}{dx} = -\frac{x}{y}$$

$$\text{At } (a, b) \text{ the gradient of the tangent is } -\frac{a}{b}$$

Using the product of gradient of perpendicular lines is -1 , the gradient of the normal is $\frac{b}{a}$

Using $y - y_1 = m(x - x_1)$:

$$y - b = \frac{b}{a}(x - a)$$

Rearranging gives:

$$y = \frac{b}{a}x$$

(ii) The equation of the normal is $y = \frac{b}{a}x$ which passes through $(0, 0)$ – the centre of the circle.

13 $y = x^2 \ln(x + 1)$

Differentiating with respect to x gives:

$$\frac{dy}{dx} = 2x \ln(x + 1) + \frac{x^2}{x + 1}$$

$$x = t^2 - 2t$$

Differentiating with respect to t gives:

$$\frac{dx}{dt} = 2t - 2$$

Using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \left(2x \ln(x+1) + \frac{x^2}{x+1} \right) (2t-2)$$

When $t = 3$, $x = 3$:

$$\frac{dy}{dt} = 4 \left(6 \ln 4 + \frac{9}{4} \right) = 24 \ln 4 + 9$$

14 $y = e^x \sin 3x$

Differentiating the product gives:

$$\frac{dy}{dx} = e^x \times 3 \cos 3x + e^x \sin 3x = e^x (3 \cos 3x + \sin 3x)$$

Differentiating $y = \cos 2t$ gives:

$$\frac{dy}{dt} = -2 \sin 2t$$

Using the chain rule:

$$\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx} = \frac{-2 \sin 2t}{e^x (\sin 3x + 3 \cos 3x)}$$

For x to be stationary, $\frac{dx}{dt} = 0$ as required.

When $t = 2\pi$, $\frac{dy}{dt} = 0$, $\frac{dy}{dx} \neq 0$, so $\frac{dx}{dt} = 0$ as required.

Mark scheme

	Answer	Marks	Guidance
1	$x = e^y$	M1	Uses inverse function
	$\frac{dx}{dy} = e^y = x$	M1	Differentiates inverse function
	$\frac{dy}{dx} = \left(\frac{dx}{dy} \right)^{-1} = \frac{1}{x}$ as required	A1	Uses chain rule and states conclusion
2	$\frac{dy}{dx} = k \cos kx = 0$	M1	Differentiates
	$x = \frac{\pi}{2k}$	M1	Solves equation
	$k = 3$	A1	Finds k
3	$x = \frac{1}{2}$	B1	Identifies intercept
	$\frac{dy}{dx} = \frac{2}{2x}$	M1	Differentiates
	2	A1	cao
4	$\frac{dy}{dx} = \cos x - x \sin x$	M1	Differentiates product
	$= 0$	A1	Correct
	$x \sin x = \cos x$	M1	Uses condition for stationary point
	$x = \cot x$	A1	States conclusion

	Answer	Marks	Guidance
5	$2ax - 2by \frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{ax}{by}$	M1 A1 A1	Differentiates implicitly Correct expression cao
6	$\frac{dy}{dx} = -2x ke^{1-x^2}$ $= 0 \Rightarrow x = 0$ $f(0) = ke = 1$ $k = \frac{1}{e}$	M1 M1 A1	Differentiates Solves equation cao
7	$3x = \tan y$ $\frac{dx}{dy} = \frac{1}{3} \sec^2 y$ $\frac{1}{3}(1 + \tan^2 y) = \frac{1}{3}(1 + (3x)^2)$ $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{3}{1 + 9x^2}$	M1 M1 M1 A1	Uses inverse function Differentiates Uses trig identity Correct expression
8	$x = \sin y, \frac{dx}{dy} = \cos y$ $\frac{dx}{dy} = \sqrt{1 - (\sin y)^2}$ $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$ $= \frac{1}{\sqrt{1 - x^2}}$	M1 M1 M1 A1	Uses inverse function to differentiate Uses trig identity Uses chain rule All correct
9 (i)	$\frac{dy}{dx} = \cos x - k(\sec x)^2 (= 0)$ $(\cos x)^3 = k$ $-1 \leq k \leq 1$ as required	M1 M1 A1	Differentiates Uses condition for stationary points oe
(ii)	$\cos x = \frac{1}{2}$ $x = \frac{\pi}{3}, \frac{5\pi}{3}$	M1 A1	Solves equation Both values
10	$\frac{dy}{dx} = x \times \frac{1}{x} + \ln x = 1 + \ln x$ $\frac{dy}{dx} = 0, \ln x = -1$ $x = e^{-1} = \frac{1}{e}$	M1 A1 M1 A1	Differentiates product Correct Uses condition for stationary point Correct value for x

	Answer	Marks	Guidance
	$\frac{d^2y}{dx^2} = \frac{1}{x} = e$ when $x = e^{-1} > 0$	M1	Checks for minimum
	Minimum value is $-\frac{1}{e}$	A1	Finds minimum value
11 (i)	$6x + 8y \frac{dy}{dx} = 0$	M1	Differentiates implicitly
	$\frac{dy}{dx} = -\frac{3x}{4y}$	A1	Correct expression for gradient
	$y = \pm \frac{3}{2}$	M1	Finds y -values
	$y = -\frac{1}{2}x + 2$	M1	Finds tangent equations
	$y = \frac{1}{2}x - 2$	A1	oe; allow ft
(ii)	$-\frac{1}{2}x + 2 = \frac{1}{2}x - 2$	M1	Solves simultaneously
	$x = 4, y = 0$ i.e. on x -axis	A1	Draws conclusion
12 (i)	$2x + 2y \frac{dy}{dx} = 0$	M1	Differentiates implicitly
	$\frac{dy}{dx} = -\frac{x}{y}$	A1	Finds expression for gradient
	At (a, b) the gradient of the tangent is $-\frac{a}{b}$	B1	Substitutes coordinates
	gradient of normal is $\frac{b}{a}$	M1	Uses perpendicular gradient property
	$y - b = \frac{b}{a}(x - a)$	M1	Finds equation of straight line
	$y = \frac{b}{a}x$	A1	Correct answer
(ii)	The equation of the normal is $y = \frac{b}{a}x$ which passes through $(0,0)$ – the centre of the circle	E1	Or other appropriate explanation
13	$\frac{dy}{dx} = 2x \ln(x+1) + \frac{x^2}{x+1}$	M1 A1	Differentiates product with respect to x
	$\frac{dx}{dt} = 2t - 2$	M1	Differentiates with respect to t
	$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} =$		
	$\left(2x \ln(x+1) + \frac{x^2}{x+1}\right)(2t-2)$	M1	Uses the chain rule
	$t = 3, x = 3$	A1	All correct
	$\frac{dy}{dt} = 24 \ln 4 + 9$	A1	Evaluates rate of change

	Answer	Marks	Guidance
14	$\frac{dy}{dx} = e^x \times 3 \cos 3x + e^x \sin 3x$	M1 A1	Differentiates the product
	$\frac{dy}{dt} = -2 \sin 2t$	M1	Differentiates y
	$\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx}$ $= \frac{-2 \sin 2t}{e^x (\sin 3x + 3 \cos 3x)}$	M1	Uses the chain rule
	For x to be stationary requires $\frac{dx}{dt} = 0$	M1	Uses condition for stationary points
	When $t = 2\pi$, $\frac{dy}{dt} = 0$, $\frac{dy}{dx} \neq 0$, so $\frac{dx}{dt} = 0$ as required	A1	Correct conclusion AG

10 Integration

- 1 Using the formula $\int (e^{kx}) dx = \frac{1}{k} e^{kx} + c$:

$$\begin{aligned} \int_0^1 5e^{3x} dx &= \left[\frac{5}{3} e^{3x} \right]_0^1 \\ &= \frac{5}{3} e^3 - \frac{5}{3} e^0 \\ &= \frac{5}{3} (e^3 - 1) \end{aligned}$$

- 2 Using the formulae $\int \cos kx dx = \frac{1}{k} \sin kx + c$ and $\int \sin kx dx = -\frac{1}{k} \cos kx + c$:

$$\int (\cos 3x + 2 \sin 3x) dx = \frac{1}{3} \sin 3x - \frac{2}{3} \cos 3x + c$$

- 3 Using the result $\int \sec^2 x dx = \tan x + c$:

$$\begin{aligned} \int_0^{\frac{\pi}{6}} \sec^2 2t dt &= \left[\frac{1}{2} \tan 2t \right]_0^{\frac{\pi}{6}} \\ &= \frac{1}{2} \tan \frac{\pi}{3} - 0 \\ &= \frac{1}{2} \sqrt{3} \end{aligned}$$

- 4 Integrating powers of x :

$$\begin{aligned} \int \left(2t - 3t^{\frac{2}{3}} \right) dt &= t^2 - 3 \times \frac{t^{\frac{5}{3}}}{\left(\frac{5}{3}\right)} + c \\ &= t^2 - \frac{9}{5} t^{\frac{5}{3}} + c \end{aligned}$$

- 5 Using the substitution $u = 5 - 2t$:

$$\frac{du}{dt} = -2$$

$$\begin{aligned}\int \frac{1}{5-2t} dt &= \int \left(\frac{1}{u}\right) \times \left(-\frac{1}{2}\right) du \\ &= -\frac{1}{2} \ln u + c \\ &= -\frac{1}{2} \ln(5-2t) + c\end{aligned}$$

You can just recognise this integral and write down the answer; take care with the factor of $-\frac{1}{2}$

- 6 Integrating by parts with $u = \ln x$, $\frac{dv}{dx} = 1$:

$$\frac{du}{dx} = \frac{1}{x}, \quad v = x$$

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int \left(\frac{1}{x}\right)x \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + c\end{aligned}$$

Notice the question requires the use of integration. Had the question asked 'Show that ...' then differentiation of the answer to get $\ln x$ would also be a valid method.

- 7 $f(x) = \int f'(x) \, dx = \int x(3+x^2)^4 \, dx$

You can just write the integral down if you recognise the form of the integral; otherwise, use the substitution $u = 3 + x^2$.

$$\frac{du}{dx} = 2x$$

$$\int u^4 \left(\frac{1}{2}\right) du = \frac{1}{10} u^5 + c = \frac{1}{10} (3+x^2)^5 + c$$

$$f(0) = 24 = \frac{1}{10} 3^5 + c = \frac{243}{10} + c$$

$$c = -\frac{3}{10}$$

$$\text{Giving } f(x) = \frac{1}{10} (3+x^2)^5 - \frac{3}{10}$$

- 8 Partial fractions:

$$\frac{3t+6}{t(t+3)} = \frac{A}{t} + \frac{B}{(t+3)}$$

$$3t+6 = A(t+3) + Bt$$

When $t = 0$, $6 = A(0+3)$, giving $A = 2$

When $t = -3$, $-9+6 = B(-3)$, giving $B = 1$

$$\begin{aligned}\int \frac{3t+6}{t(t+3)} dt &= \int \left(\frac{2}{t} + \frac{1}{t+3}\right) dt \\ &= 2 \ln t + \ln(t+3) + \ln a * \\ &= \ln(at^2(t+3))\end{aligned}$$

*Note: $+c$ can be written in the form $\ln a$.

9 $\int_0^1 (5x + 3)^5 dx$

Using the substitution $u = 5x + 3$:

$$\frac{du}{dx} = 5$$

When $x = 0$, $u = 3$

When $x = 1$, $u = 8$

$$\text{Integral becomes } \int_3^8 (u^5)^{\frac{1}{5}} du = \frac{1}{5} \left[\frac{u^6}{6} \right]_3^8 = \left[\frac{u^6}{30} \right]_3^8$$

$$\begin{aligned} \text{Substituting limits} &= \left[\frac{u^6}{30} \right]_3^8 \\ &= \left(\frac{8^6}{30} \right) - \left(\frac{3^6}{30} \right) \\ &= \frac{52\,283}{6} \text{ as required.} \end{aligned}$$

10 Curves intersect when $x^3 + 2x^2 - 3x + 1 = 2x^2 + x + 1$

$$x^3 - 4x = 0$$

$$x(x + 2)(x - 2) = 0$$

$$x = 0, \pm 2$$

$$\begin{aligned} \text{Required area} &= \int_0^2 \left((2x^2 + x + 1) - (x^3 + 2x^2 - 3x + 1) \right) dx \\ &= \int_0^2 (4x - x^3) dx \\ &= \left[2x^2 - \frac{x^4}{4} \right]_0^2 \\ &= \left(2 \times 2^2 - \frac{2^4}{4} \right) - 0 \\ &= 8 - 4 \\ &= 4 \end{aligned}$$

11 Using the substitution $u = \ln x$:

$$\frac{du}{dx} = \frac{1}{x} \text{ so } du = \frac{1}{x} dx$$

$$\begin{aligned} \int \frac{1}{x} \ln x \, dx &= \int u \, du \\ &= \frac{1}{2} u^2 + c \\ &= \frac{1}{2} (\ln x)^2 + c \end{aligned}$$

12 Using the substitution $u = 4 + 2 \sin 3x$:

$$\frac{du}{dx} = 6 \cos 3x, \text{ so } du = 6 \cos 3x \, dx$$

$$\text{When } x = 0, u = 4$$

$$\text{When } x = \frac{\pi}{6}, u = 6$$

$$\begin{aligned}
 \int_0^{\frac{\pi}{6}} \frac{\cos 3x}{4 + 2 \sin 3x} dx &= \frac{1}{6} \int_4^6 \frac{1}{u} du \\
 &= \left[\frac{1}{6} \ln u \right]_4^6 \\
 &= \frac{1}{6} (\ln 6 - \ln 4) \\
 &= \frac{1}{6} \ln \left(\frac{6}{4} \right) \\
 &= \frac{1}{6} \ln \left(\frac{3}{2} \right)
 \end{aligned}$$

13 $y = 4\sqrt{x} - x^2\sqrt{x}$ crosses x -axis when $y = 0$

$$\sqrt{x}(4 - x^2) = 0$$

$$x = 0, \pm 2$$

$$\begin{aligned}
 \int (4\sqrt{x} - x^2\sqrt{x}) dx &= \int \left(4x^{\frac{1}{2}} - x^{\frac{5}{2}} \right) dx \\
 &= 4 \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{x^{\frac{7}{2}}}{\left(\frac{7}{2}\right)} + c \\
 &= \frac{8}{3} x^{\frac{3}{2}} - \frac{2}{7} x^{\frac{7}{2}} + c
 \end{aligned}$$

Area above the axis is given by $\int_0^2 (4\sqrt{x} - x^2\sqrt{x}) dx$.

$$\begin{aligned}
 \int_0^2 (4\sqrt{x} - x^2\sqrt{x}) dx &= \left[\frac{8}{3} x^{\frac{3}{2}} - \frac{2}{7} x^{\frac{7}{2}} \right]_0^2 \\
 &= \left(\frac{8}{3} \times 2^{\frac{3}{2}} - \frac{2}{7} \times 2^{\frac{7}{2}} \right) - 0 \\
 &= \frac{8}{3} \times 2\sqrt{2} - \frac{2}{7} \times 2^3\sqrt{2} \\
 &= \frac{64}{21} \sqrt{2}
 \end{aligned}$$

Negative area below the axis is given by $\int_2^4 (4\sqrt{x} - x^2\sqrt{x}) dx$.

$$\begin{aligned}
 \int_2^4 (4\sqrt{x} - x^2\sqrt{x}) dx &= \left[\frac{8}{3} x^{\frac{3}{2}} - \frac{2}{7} x^{\frac{7}{2}} \right]_2^4 \\
 &= \left(\frac{8}{3} \times 4^{\frac{3}{2}} - \frac{2}{7} \times 4^{\frac{7}{2}} \right) - \left(\frac{64}{21} \sqrt{2} \right) \\
 &= \left(\frac{8}{3} \times 8 - \frac{2}{7} \times 128 \right) - \frac{64}{21} \sqrt{2} \\
 &= -\frac{320}{21} - \frac{64}{21} \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Total area} &= \frac{64}{21} \sqrt{2} + \left| -\frac{320}{21} - \frac{64}{21} \sqrt{2} \right| \\
 &= \frac{320}{21} + \frac{128}{21} \sqrt{2}
 \end{aligned}$$

14 Expanding the brackets before integrating:

$$\begin{aligned}\int (e^x + 2)^2 dx &= \int \left((e^x)^2 + 4e^x + 4 \right) dx \\ &= \int (e^{2x} + 4e^x + 4) dx \\ &= \frac{1}{2}e^{2x} + 4e^x + 4x + c\end{aligned}$$

15 Using the trigonometrical identity $\cos 2A = 1 - 2\sin^2 A$ with $A = 2x$:

$$\begin{aligned}\int 6\sin^2 2x dx &= \int 3(1 - \cos 4x) dx \\ &= 3x - \frac{3}{4}\sin 4x + c\end{aligned}$$

16 (i) $\int_0^2 5xe^{x^2} dx$

Using the substitution $u = x^2$:

$$\frac{du}{dx} = 2x$$

When $x = 0$, $u = 0$

When $x = 2$, $u = 4$

So, the integral becomes $\int_0^4 (5e^u) \frac{1}{2} du$

$$\begin{aligned}\int_0^4 (5e^u) \frac{1}{2} du &= \left[\frac{5}{2} e^u \right]_0^4 \\ &= \frac{5}{2} (e^4 - e^0) \\ &= \frac{5}{2} (e^4 - 1)\end{aligned}$$

(ii) $\int_0^2 5xe^{2x} dx$ integration by parts with $u = 5x$, $\frac{dv}{dx} = e^{2x}$

$$\frac{du}{dx} = 5, \quad v = \frac{1}{2}e^{2x}$$

$$\begin{aligned}\int_0^2 5xe^{2x} dx &= \left[\frac{5}{2} xe^{2x} \right]_0^2 - \int_0^2 \frac{5}{2} e^{2x} dx \\ &= \left[\frac{5}{2} xe^{2x} \right]_0^2 - \left[\frac{5}{4} e^{2x} \right]_0^2 \\ &= \left(5e^4 - \frac{5}{4}e^4 \right) - \left(0 - \frac{5}{4} \right) \\ &= \frac{15}{4}e^4 + \frac{5}{4}\end{aligned}$$

17 Method 1

Using double angle formula $\sin 2A = 2\sin A \cos A$:

$$\int_0^{\frac{\pi}{2}} 12\sin x \cos x dx = \int_0^{\frac{\pi}{2}} 6\sin 2x dx$$

$$\begin{aligned}\left[-\frac{6}{2}\cos 2x \right]_0^{\frac{\pi}{2}} &= (-3\cos \pi) - (-3\cos 0) \\ &= 3 + 3 \\ &= 6\end{aligned}$$

Method 2

Using the substitution $u = \sin x$ (notice $u = \cos x$ would work in the same way):

$$\frac{du}{dx} = \cos x$$

When $x = 0$, $u = 0$

When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} 12 \sin x \cos x \, dx &= \int_0^1 12u \, du \\ &= \left[6u^2 \right]_0^1 \\ &= 6\end{aligned}$$

$$18 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} \, dx \text{ use the substitution } u = \sin x$$

$$\frac{du}{dx} = \cos x \text{ so } du = \cos x \, dx$$

When $x = \frac{\pi}{4}$, $u = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} \, dx &= \int_{\frac{1}{\sqrt{2}}}^1 \left(\frac{1}{u}\right) \, du \\ [\ln u]_{\frac{1}{\sqrt{2}}}^1 &= \ln 1 - \ln \frac{1}{\sqrt{2}} \\ &= -\ln(2)^{-\frac{1}{2}} \\ &= \frac{1}{2} \ln 2\end{aligned}$$

$$19 \text{ (i)} \int 6x\sqrt{1-2x} \, dx \text{ using the substitution } u = 1-2x:$$

$$\frac{du}{dx} = -2$$

Rearranging $2x = 1 - u$:

$$\begin{aligned}\int 6x\sqrt{1-2x} \, dx &= \int 3(1-u)\left(-\frac{1}{2}\right)u^{\frac{1}{2}} \, du \\ &= \frac{3}{2} \int \left(-(1-u)u^{\frac{1}{2}}\right) \, du \\ &= \frac{3}{2} \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \, du\end{aligned}$$

(ii) Curve crosses the x -axis when $y = 6x\sqrt{1-2x} = 0$
So, $x = 0, 0.5$

$$\text{Area given by } \int_0^{0.5} 6x\sqrt{1-2x} \, dx$$

When $x = 0$, $u = 1$

When $x = 0.5$, $u = 0$

$$\begin{aligned}
 A &= \frac{3}{2} \int_1^0 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) du \\
 &= \frac{3}{2} \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^0 \\
 &= \frac{3}{2} \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} \right]_1^0 \\
 &= \frac{3}{2} \left((0) - \left(\frac{2 \times 1^{\frac{5}{2}}}{5} - \frac{2 \times 1^{\frac{3}{2}}}{3} \right) \right) \\
 &= \frac{3}{2} \times \frac{4}{15} \\
 &= \frac{2}{5}
 \end{aligned}$$

20 Using partial fractions (take care with the form of the answer):

$$\frac{3x^2 - x}{(3x + 1)^2(x + 1)} = \frac{A}{(3x + 1)} + \frac{B}{(3x + 1)^2} + \frac{C}{(x + 1)}$$

$$3x^2 - x = A(3x + 1)(x + 1) + B(x + 1) + C(3x + 1)^2$$

$$\text{Substituting } x = -1 \text{ gives } 3(-1)^2 - (-1) = 4 = C(-2)^2$$

$$\text{So, } C = 1$$

$$\text{Substituting } x = -\frac{1}{3} \text{ gives } 3 \times \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) = \frac{2}{3} = B\left(-\frac{1}{3} + 1\right)$$

$$\text{So, } B = 1$$

$$\text{Equating the coefficient of } x^2 \text{ gives } 3 = 3A + 9C$$

$$\begin{aligned}
 \text{So, } A &= \frac{3 - 9}{3} \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{3x^2 - x}{(3x + 1)^2(x + 1)} dx &= \int \left(-\frac{2}{(3x + 1)} + \frac{1}{(3x + 1)^2} + \frac{1}{(x + 1)} \right) dx \\
 &= -\frac{2}{3} \ln(3x + 1) + \frac{1}{3} \frac{(3x + 1)^{-1}}{(-1)} + \ln(x + 1) + c \\
 &= \ln \left(\frac{(x + 1)}{(3x + 1)^{\frac{2}{3}}} \right) - \frac{1}{3(3x + 1)} + c
 \end{aligned}$$

21 (i) Using the substitution $u = \ln x$:

$$\frac{du}{dx} = \frac{1}{x}$$

$$\begin{aligned}
 \int \frac{1}{x \ln x} dx &= \int \left(\frac{1}{\ln x} \right) \frac{1}{x} dx \\
 &= \int \frac{1}{u} du \\
 &= \ln u + c \\
 &= \ln(\ln x) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int_e^{e^2} \left(\frac{1}{x \ln x} \right) dx &= [\ln(\ln x)]_e^{e^2} \\
 &= (\ln(\ln e^2)) - (\ln(\ln e)) \\
 &= \ln 2 - \ln 1 \\
 &= \ln 2
 \end{aligned}$$

22 Curve crosses the x -axis when $y = 0$

$$y = x^2 \sin x = 0$$

$$x = 0, \text{ or } \sin x = 0$$

$$x = 0, \text{ or } \pi$$

$$\text{area} = \int_0^\pi x^2 \sin x \, dx$$

Integrating by parts:

$$u = x^2 \quad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = 2x \text{ and } v = -\cos x$$

$$\int_0^\pi x^2 \sin x \, dx = [-x^2 \cos x]_0^\pi + \int_0^\pi 2x \cos x \, dx$$

Integrating by parts again:

$$u = 2x \quad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 2 \text{ and } v = \sin x$$

$$= [-x^2 \cos x]_0^\pi + [2x \sin x]_0^\pi - \int_0^\pi (2 \sin x) \, dx$$

$$= [-x^2 \cos x]_0^\pi + [2x \sin x]_0^\pi + [2 \cos x]_0^\pi$$

$$= -\pi^2(-1) + 0 + (-2) - (0 + 0 + 2)$$

$$= \pi^2 - 4$$

Mark scheme

	Answer	Marks	Guidance
1	$\left[\frac{5}{3} e^{3x} \right]_0^1 = \frac{5}{3} e^3 - \frac{5}{3} e^0 = \frac{5}{3} (e^3 - 1)$	B1 M1 A1	Indefinite integral correct Uses limits Any exact form
2	$\frac{1}{3} \sin 3x - \frac{2}{3} \cos 3x + c$	M1 A1 A1	Correct form; allow sign errors and incorrect coefficients First term correct Second term correct
3	$\left[\frac{1}{2} \tan 2t \right]_0^{\frac{\pi}{6}} = \frac{1}{2} \tan \frac{\pi}{3} - 0 = \frac{1}{2} \sqrt{3}$	M1 M1 A1	Allow for $k \tan 2t$ Uses limits Exact answer needed
4	$t^2 - 3 \times \frac{t^{\frac{5}{3}}}{\left(\frac{5}{3} \right)} + c = t^2 - \frac{9}{5} t^{\frac{5}{3}} + c$	B1 M1 A1	For first term For $kt^{\frac{5}{3}}$ seen Second term fully correct

	Answer	Marks	Guidance
5	$\int \frac{1}{5-2t} dt = -\frac{1}{2} \ln(5-2t) = c$	M1 M1 A1	Allow for $\ln u$ or $\ln(5-2t)$ seen Allow for factor of $-\frac{1}{2}$ in answer or $\frac{du}{dt}$ Fully correct
6	Integration by parts with: $u = \ln x, \frac{dv}{dx} = 1$ $\frac{du}{dx} = \frac{1}{x}, v = x$ $x \ln x - \int \left(\frac{1}{x}\right)x dx = x \ln x - \int 1 dx$ $= x \ln x - x + c$	M1 M1 A1	Uses factor of 1 for $\frac{dv}{dx}$ Uses the formula for integration by parts
7	$f(x) = \frac{1}{2} \times \frac{1}{5} (3+x^2)^5 + c$ $f(0) = 24 = \frac{1}{10} 3^5 + c = \frac{243}{10} + c$ $c = -\frac{3}{10}$ $f(x) = \frac{1}{10} (3+x^2)^5 - \frac{3}{10}$	M1 M1 A1 M1 A1	Allow for u^5 or $(3+x^2)^5$ seen Allow for factor of $\frac{1}{2}$ so All correct including $+c$ Uses $f(0)$
8	$\frac{3t+6}{t(t+3)} = \frac{A}{t} + \frac{B}{t+3}$ $A = 2, B = 1$ $\int \left(\frac{2}{t} + \frac{1}{t+3}\right) dt = 2 \ln t + \ln(t+3) + \ln a$ $\ln(at^2(t+3))$	M1 A1 M1 A1 M1 A1	Attempts to use partial fractions Both coefficients correct At least one $\ln$ term seen All correct including $+c$ oe At least one law of logs used All correct; allow use of e^c
9	Use the substitution $u = 5x + 3$ $\int_3^8 (u^5)^{\frac{1}{5}} du = \frac{1}{5} \left[\frac{u^6}{6} \right]_3^8 = \left[\frac{u^6}{30} \right]_3^8$ $= \left(\frac{8^6}{30} \right) - \left(\frac{3^6}{30} \right)$ $= \frac{52\,283}{6}$	M1 A1 B1 M1 E1	Need not be explicitly seen; may be implied Factor of $\frac{1}{5}$ or $\frac{du}{dx} = 5$ must be seen Integrates the power correctly Use of limits (for x or u) must be explicitly seen Conclusion stated
10	$x^3 + 2x^2 - 3x + 1 = 2x^2 + x + 1$ $x = 0, \pm 2$ $\int_0^2 ((2x^2 + x + 1) - (x^3 + 2x^2 - 3x + 1)) dx$	M1 A1 M1	Attempts to find the limits Condone -2 missing Allow difference of two integrals; allow sign reversed

	Answer	Marks	Guidance
	$= \int_0^2 (4x - x^3) dx = \left[2x^2 - \frac{x^4}{4} \right]_0^2$ $= \left(2 \times 2^2 - \frac{2^4}{4} \right) - 0 = 8 - 4 = 4$	M1 M1 A1	Attempts to integrate Use of limits seen Do not allow for -4
11	Use the substitution $u = \ln x$ $\frac{du}{dx} = \frac{1}{x} \text{ so } du = \frac{1}{x} dx$ $\int \frac{1}{x} \ln x \, dx = \int u \, du$ $= \frac{1}{2} u^2 + c = \frac{1}{2} (\ln x)^2 + c$	M1 M1 A1	May be implied by answer of the form $k(\ln x)^2$ Integrates Must include + c
12	Use the substitution $u = 4 + 2 \sin 3x$ $\frac{du}{dx} = 6 \cos 3x$ When $x = 0$, $u = 4$ When $x = \frac{\pi}{6}$, $u = 6$ $\int_0^{\frac{\pi}{6}} \left(\frac{\cos 3x}{4 + 2 \sin 3x} \right) dx = \frac{1}{6} \int_4^6 \left(\frac{1}{u} \right) du = \left[\frac{1}{6} \ln u \right]_4^6$ $= \frac{1}{6} (\ln 6 - \ln 4) = \frac{1}{6} \ln \left(\frac{6}{4} \right) = \frac{1}{6} \ln \left(\frac{3}{2} \right)$	M1 A1 M1 M1 M1 A1	Uses the substitution (may be implied) Allow for factor of $\frac{1}{6}$ seen Changes limits or substitutes back in terms of x $\ln u$ or $\ln(4 + 2 \sin 3x)$ seen Limits for x or u used Answer given
13	$\sqrt{x}(4 - x^2) = 0$ $x = 0, \pm 2$ $\int \left(4x^{\frac{1}{2}} - x^{\frac{5}{2}} \right) dx$ $= 4 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + c = \frac{8}{3} x^{\frac{3}{2}} - \frac{2}{7} x^{\frac{7}{2}} + c$ Dividing the area into two separate integrals $\left[\frac{8}{3} x^{\frac{3}{2}} - \frac{2}{7} x^{\frac{7}{2}} \right]_0^2 = \left(\frac{8}{3} \times 2^{\frac{3}{2}} - \frac{2}{7} \times 2^{\frac{7}{2}} \right) - 0$ $= \frac{64}{21} \sqrt{2}$ $= \left[\frac{8}{3} x^{\frac{3}{2}} - \frac{2}{7} x^{\frac{7}{2}} \right]_2^4 = -\frac{320}{21} - \frac{64}{21} \sqrt{2}$ total area = $\frac{64}{21} \sqrt{2} + \left -\frac{320}{21} - \frac{64}{21} \sqrt{2} \right = \frac{320}{21} + \frac{128}{21} \sqrt{2}$	B1 B1 M1 M1 A1 A1 A1	Finds point of intersection with x -axis; need not be explicit; allow for use of 2 as limit Rewrites with correct fractional powers Attempts to integrate; allow without simplifying Need not be combined correctly for this mark Exact answer needed; any form, including index form Exact answer must include surd form cao
14	$\int (e^x + 2)^2 dx = \int (e^{2x} + 4e^x + 4) dx$ $= \frac{1}{2} e^{2x} + 4e^x + 4x + c$	M1 M1 A1 A1	Expands bracket Integrates at least one exponential term First term correct Other terms correct, including +c

	Answer	Marks	Guidance
15	$\int 6 \sin^2 2x \, dx = \int 3(1 - \cos 4x) \, dx$ $= 3x - \frac{3}{4} \sin 4x + c$	M1 A1 M1 A1	Uses double angle formula All correct Attempts to integrate both terms All correct; condone missing +c
16 (i)	$\int_0^4 (5e^u) \frac{1}{2} \, du$ $= \left[\frac{5}{2} e^u \right]_0^4$ $= \frac{5}{2} (e^4 - e^0)$ $= \frac{5}{2} (e^4 - 1)$	M1 A1 A1 M1 A1	Substitutes to get $\int ke^u \, du$ Factor $\frac{1}{2}$ seen or $\frac{du}{dx} = 2x$ All correct Uses new limits, or rewrites in terms of x and original limits used Exact answer needed
(ii)	$u = 5x, \frac{dv}{dx} = e^{2x}$ $\frac{du}{dx} = 5, v = \frac{1}{2} e^{2x}$ $\int_0^2 5xe^{2x} \, dx = \left[\frac{5}{2} xe^{2x} \right]_0^2 - \int_0^2 \left(\frac{5}{2} e^{2x} \right) dx$ $= \left[\frac{5}{2} xe^{2x} \right]_0^2 - \left[\frac{5}{4} e^{2x} \right]_0^2$ $= \left(5e^4 - \frac{5}{4} e^4 \right) - \left(0 - \frac{5}{4} \right)$ $= \frac{15}{4} e^4 + \frac{5}{4}$	M1 A1 A1 M1 A1	Uses integration by parts First term correct in a two term expression Second term correct Uses limits oe exact answer required
17	Method 1 using double angle formula $\int_0^{\frac{\pi}{2}} (12 \sin x \cos x) \, dx = \int_0^{\frac{\pi}{2}} (6 \sin 2x) \, dx$ $\left[-\frac{6}{2} \cos 2x \right]_0^{\frac{\pi}{2}}$ $= (-3 \cos \pi) - (-3 \cos 0)$ $= 6$ Method 2 using substitution $u = \sin x$ $\frac{du}{dx} = \cos x$ When $x = 0$, $u = 0$ When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$ $\int_0^{\frac{\pi}{2}} (12 \sin x \cos x) \, dx = \int_0^1 12u \, du$ $= \left[6u^2 \right]_0^1$ $= 6$	M1 A1 M1 M1 A1 M1 M1 A1 M1 A1	Uses double angle Correct integrand Integration attempted Uses limits Uses substitution Changes limits or substitutes back Correct integrand Uses limits (either x or u)
18	$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot x) \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{\cos x}{\sin x} \right) dx$ $\frac{du}{dx} = \cos x \text{ so } du = \cos x \, dx$	M1 M1	Uses trig identity Uses substitution

	Answer	Marks	Guidance
	When $x = \frac{\pi}{4}$, $u = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$ $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx = \int_{\frac{1}{\sqrt{2}}}^1 \left(\frac{1}{u}\right) du$ $[\ln u]_{\frac{1}{\sqrt{2}}}^1$ $= \ln 1 - \ln \frac{1}{\sqrt{2}}$ $= -\ln(2)^{-\frac{1}{2}} = \frac{1}{2} \ln 2$	B1 A1 M1 A1	Limits seen $\ln u$ seen Uses limits Must be in form $k \ln 2$
19 (i)	$\int 3(1-u)\left(-\frac{1}{2}\right)u^{\frac{1}{2}} du$ $= \frac{3}{2} \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$	B1 B1 B1	Substitutes for $6x$ any form For factor $-\frac{1}{2}$ and du seen Answer given
(ii)	$\int_0^{0.5} (6x\sqrt{1-2x}) dx$ $A = \frac{3}{2} \int_1^0 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du = \frac{3}{2} \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} \right]_1^0$ $\frac{3}{2} \left((0) - \left(\frac{2 \times 1^{\frac{5}{2}}}{5} - \frac{2 \times 1^{\frac{3}{2}}}{3} \right) \right) = \frac{2}{5}$	B1 M1 A1 M1 A1	Limits 0 and 0.5 seen Integrates powers Uses limits oe
20	$\frac{3x^2 - x}{(3x+1)^2(x+1)} = \frac{A}{(3x+1)} + \frac{B}{(3x+1)^2} + \frac{C}{(x+1)}$ $3x^2 - x = A(3x+1)(x+1) + B(x+1) + C(3x+1)^2$ $A = -2, B = 1, C = 1$ $\int \left(-\frac{2}{(3x+1)} + \frac{1}{(3x+1)^2} + \frac{1}{(x+1)} \right) dx$ $= -\frac{2}{3} \ln(3x+1) + \frac{1}{3} \frac{(3x+1)^{-1}}{(-1)}$ $+ \ln(x+1) + c$ $= \ln \left(\frac{(x+1)}{(3x+1)^{\frac{2}{3}}} \right) - \frac{1}{3(3x+1)} + c$	B1 M1 A1 A1 B1 B1 M1 A1	Correct form of partial fractions Multiplies by denominator First correct value Other two correct values At least one term $k \ln(x+1)$ or $k \ln(3x+1)$ Both $\ln$ terms correct; ft their constants Integrates the $(3x+1)^{-2}$ term to obtain $k(3x+1)^{-1}$ Fully correct; need not be simplified; isw if rearranged incorrectly

	Answer	Marks	Guidance
21 (i)	$\frac{du}{dx} = \frac{1}{x}$	M1	Uses the substitution appropriately
	$\int \left(\frac{1}{\ln x} \right) \frac{1}{x} dx = \int \left(\frac{1}{u} \right) du$	A1	$\int \left(\frac{1}{u} \right) du$
	$= \ln u + c$	A1	$\ln u$ oe seen
	$= \ln(\ln x) + c$	A1	exact answer including $+c$
(ii)	$[\ln(\ln x)]_e^{e^2}$	M1	Substitutes limits
	$= (\ln(\ln e^2)) - (\ln(\ln e))$	M1	Uses laws of logs to simplify at least one term
	$= \ln 2 - \ln 1$ $= \ln 2$	A1	cao
22	$y = x^2 \sin x = 0$ $x = 0, \text{ or } \pi$	B1	Both correct values seen
	$\int_0^\pi x^2 \sin x \, dx =$ Integration by parts		
	$u = x^2 \quad \frac{dv}{dx} = \sin x$	M1	Valid attempt at integration by parts; allow sign errors
	$\frac{du}{dx} = 2x \text{ and } v = -\cos x$		
	$[-x^2 \cos x]_0^\pi + \int_0^\pi 2x \cos x \, dx$	A1	All correct
	Integration by parts again		
	$u = 2x \quad \frac{dv}{dx} = \cos x$	M1	Second use of integration by parts
	$\frac{du}{dx} = 2 \text{ and } v = \sin x$		
	$= [-x^2 \cos x]_0^\pi + [2x \sin x]_0^\pi - \int_0^\pi (2 \sin x) \, dx$	A1	ft sign errors in previous part
	$= [-x^2 \cos x]_0^\pi + [2x \sin x]_0^\pi + [2 \cos x]_0^\pi$	A1	
	$= -\pi^2(-1) + 0 + (-2) - (0 + 0 + 2)$	M1	Uses limits
	$= \pi^2 - 4$	A1	oe; must be an exact form

11 Parametric equations

- 1 $x = 3t^2$, $y = 6t$, so use the expression for y to substitute into the expression for x .
 $t = \frac{y}{6} \Rightarrow x = 3 \times \left(\frac{y}{6}\right)^2 \Rightarrow x = \frac{y^2}{12}$
 So, $y^2 = 12x$ [parabola]
- 2 First differentiate with respect to t :
 $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6$
 Using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ gives $\frac{dy}{dx} = \frac{6}{6t} = \frac{1}{t}$
- 3 By inspection, it can be seen that multiplying the two parametric expressions means that t cancels:
 $xy = 2t \times \left(\frac{2}{t}\right) = 4 \Rightarrow y = \frac{4}{x}$
 Or, using a substitution equation, $t = \frac{x}{2}$, gives $y = \frac{2}{\frac{x}{2}} = \frac{4}{x}$
- 4 $x = 2\cos\theta$, $y = 2\sin\theta$
 ($\cos\theta$ and $\sin\theta$ are often referred to as circular functions)
- 5 $x = 1 + \cos\theta$, $y = 2 + \sin\theta$
 This can be interpreted as a translation of the circle, radius 1, centre the origin.
 $[x - 1 = \cos\theta, y - 2 = \sin\theta]$
- 6 On the y -axis the x coordinate is zero.
 So, solving $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$
- 7 First, differentiate with respect to θ :
 $\frac{dx}{d\theta} = -\sin\theta$ and $\frac{dy}{d\theta} = \cos\theta$
 Using $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ gives:
 $\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} = -\frac{1}{\tan\theta}$
 So, $\frac{dy}{dx} = 1$ when $\theta = -\frac{\pi}{4}$ and $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$
 (Or by considering the curve.)
- 8 The domain of x is limited to $x \geq 0$, the parametric equations do not generate points with negative x coordinates.
- 9 (i) As $t = \frac{y}{2}$, $x = \left(\frac{y}{2}\right)^3 + \frac{1}{2} \times \left(\frac{y}{2}\right)^2$ and so $y^3 + y^2 = 8x$
 (ii) Differentiating implicitly:
 $3y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 8$
 When $\frac{dy}{dx} = 1$, $3y^2 + 2y - 8 = 0$, so that $(3y - 4)(y + 2) = 0$
 Hence $y = \frac{4}{3}$ or -2 and $x = \frac{1}{8}(y^3 + y^2)$

So, either

$$x = \frac{1}{8} \left(\left(\frac{4}{3} \right)^3 + \left(\frac{4}{3} \right)^2 \right) = \frac{1}{27} (8 + 6) = \frac{14}{27}$$

Or

$$x = \frac{1}{8} \left((-2)^3 + (-2)^2 \right) = -\frac{1}{2}$$

So, the two points are $\left(\frac{14}{27}, \frac{4}{3} \right)$ and $\left(-\frac{1}{2}, -2 \right)$

(iii) $\frac{dx}{dt} = 3t^2 + t$ and $\frac{dy}{dt} = 2$

$$\text{So, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{3t^2 + t}$$

(iv) When $\frac{dy}{dx} = 1$, $\frac{2}{3t^2 + t} = 1$, so $3t^2 + t - 2 = 0$

$$\text{And } (3t - 2)(t + 1) = 0, \text{ so } t = \frac{2}{3} \text{ or } -1$$

(Can be checked with the values of y found in part (ii).)

10 (i) $x = \cos 3t \Rightarrow \frac{dx}{dt} = -3 \sin 3t$

$$y = \sin 2t \Rightarrow \frac{dy}{dt} = 2 \cos 2t$$

$$\text{speed squared, } s^2 = \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = 9(\sin 3t)^2 + 4(\cos 2t)^2$$

(ii) $\frac{d(s^2)}{dt} = 54 \cos 3t \sin 3t - 16 \sin 2t \cos 2t$

$$\text{Stationary point when } \frac{d(s^2)}{dt} = 0$$

Substitute $t = 2\pi$:

$$\begin{aligned} \frac{d(s^2)}{dt} &= 54 \cos 6\pi \sin 6\pi - 16 \sin 4\pi \cos 4\pi \\ &= 54 \times 1 \times 0 - 16 \times 0 \times 1 = 0 \end{aligned}$$

So, the function has a stationary point when $t = 2\pi$.

Using the double angle formula for $\sin 2x$ to simplify the first derivative before differentiating again:

$$\frac{d(s^2)}{dt} = 54 \cos 3t \sin 3t - 16 \sin 2t \cos 2t = 27 \sin 6t - 8 \sin 4t$$

Check for a minimum by differentiating again:

$$\frac{d^2(s^2)}{dt^2} = 162 \cos 6t - 32 \cos 4t$$

$$\text{When } t = 2\pi, \frac{d^2(s^2)}{dt^2} = 162 - 32 > 0, \text{ hence a minimum.}$$

11 (i) As $\tan^2 \theta + 1 = \sec^2 \theta = \frac{1}{\cos^2 \theta}$, $x^2 + 1 = \frac{1}{y^2}$

(ii) $\frac{dx}{d\theta} = \sec^2 \theta$ and $\frac{dy}{d\theta} = -\sin \theta$, so $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin \theta}{\sec^2 \theta} = -\sin \theta \cos^2 \theta$

$$\frac{dy}{dx} = 0 \Rightarrow \sin \theta = 0 \text{ or } \cos \theta = 0$$

$$\Rightarrow \theta = 0 \text{ or } \pi, \text{ or } \frac{\pi}{2}$$

$$\theta = 0 \Rightarrow x = 0 \text{ and } y = 1$$

$$\theta = \pi \Rightarrow x = 0 \text{ and } y = -1$$

$$\theta = \frac{\pi}{2} : \text{reject, as } x \text{ is undefined.}$$

Required coordinates are (0, 1) and (0, -1).

12 Differentiating with respect to t :

$$\frac{dx}{dt} = -2\sin 2t + \cos t, \frac{dy}{dt} = -e^{-t}$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t}}{-2\sin 2t + \cos t} = \frac{e^{-t}}{2\sin 2t - \cos t}$$

$$\text{When } t = \pi, \frac{dy}{dx} = \frac{e^{-\pi}}{2\sin 2\pi - \cos \pi} = \frac{e^{-\pi}}{1} = \frac{1}{e^{\pi}}$$

13 (i) Using the Pythagorean identity to eliminate the parameter:

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = \cos^2 \theta + \sin^2 \theta = 1; \text{ so } \frac{x^2}{9} + \frac{y^2}{4} = 1 \text{ [an ellipse]}$$

(ii) Differentiating with respect to θ :

$$\frac{dx}{d\theta} = -3\sin \theta \text{ and } \frac{dy}{d\theta} = 2\cos \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\cos \theta}{-3\sin \theta} = -\frac{2}{3}\cot \theta$$

$$\text{gradient of the perpendicular} = -\frac{1}{-\frac{2}{3}\cot \theta} = \frac{3}{2}\tan \theta$$

$$\text{The equation of the normal at the point } (3\cos \theta, 2\sin \theta) \text{ is } \frac{y - 2\sin \theta}{x - 3\cos \theta} = \frac{3}{2}\tan \theta$$

$$\text{i.e. } y - 2\sin \theta = \frac{3}{2}\tan \theta \cdot x - \frac{9}{2}\sin \theta \text{ or } y = \frac{3}{2}\tan \theta \cdot x - \frac{5}{2}\sin \theta$$

$$\text{This passes through } (1, 0) \text{ when } 0 = \frac{3}{2}\tan \theta - \frac{5}{2}\sin \theta$$

$$\text{So, } \frac{3\sin \theta}{\cos \theta} = 5\sin \theta \Rightarrow \sin \theta(3 - 5\cos \theta) = 0$$

$$\text{So, } \sin \theta = 0 \text{ or } \cos \theta = \frac{3}{5}$$

When $\sin \theta = 0$, $\cos \theta = \pm 1$, so the point on the curve has coordinates (3, 0) or (-3, 0).

When $\cos \theta = \frac{3}{5}$, $\sin \theta = \pm \frac{4}{5}$, so the point on the curve has coordinates $(\frac{9}{5}, \frac{8}{5})$ or $(\frac{9}{5}, -\frac{8}{5})$.

14 Eliminating parameter:

$$x + y = 2t; x - y = \frac{2}{t}$$

$$\Rightarrow (x + y)(x - y) = 4$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{4} = 1 \text{ (rectangular hyperbola)}$$

15 Eliminating the parameter:

$$\frac{y}{x} = t \Rightarrow x = \frac{\left(\frac{y}{x}\right)}{3 - \frac{y}{x}} = \frac{y}{3x - y}$$

$$\Rightarrow 3x^2 - xy = y \Rightarrow y(1 + x) = 3x^2$$

$$\Rightarrow y = \frac{3x^2}{1 + x}$$

Mark scheme

	Answer	Marks	Guidance
1	$t = \frac{y}{6}$	M1	Finds substitution equation
	$x = \frac{y^2}{12}$	A1	oe
2	$\frac{dx}{dt} = 6t$	B1	Differentiates x
	$\frac{dy}{dt} = 6$	B1	Differentiates y
	$\frac{dy}{dx} = \frac{1}{t}$	B1	Correct answer
3	$xy = 2t \times \left(\frac{2}{t}\right)$	M1	Eliminates parameter
	$y = \frac{4}{x}$	A1	oe
4	$x = 2 \cos \theta$	B1	cao
	$y = 2 \sin \theta$	B1	cao
5	$x = 1 + \cos \theta$	B1	cao
	$y = 2 + \sin \theta$	B1	cao
6	$\theta = \frac{\pi}{2}$	B1	cao
	$\theta = \frac{3\pi}{2}$	B1	cao
7	$\frac{dx}{d\theta} = -\sin \theta$	M1	Differentiates
	$\frac{dy}{d\theta} = \cos \theta$		
	$\frac{dy}{dx} = -\frac{\cos \theta}{\sin \theta} = -\frac{1}{\tan \theta}$	M1	Uses chain rule
	$\theta = -\frac{\pi}{4}$	M1	Solves equation
	$-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$	A1	Both correct
8	The domain of x is limited to $x \geq 0$	E1	Appropriate reason
9 (i)	$x = \left(\frac{y}{2}\right)^3 + \frac{1}{2} \times \left(\frac{y}{2}\right)^2$	B1	Eliminates t
	$y^3 + y^2 = 8x$	B1	Coefficient of x must be 8 or -8

	Answer	Marks	Guidance
(ii)	$3y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 8$	M1A1	Differentiates implicitly
	$3y^2 + 2y - 8 = 0$	B1	Correct quadratic
	$y = \frac{4}{3}$ or -2	B1	Both coordinates
	$\left(\frac{14}{27}, \frac{4}{3}\right)$ and $\left(-\frac{1}{2}, -2\right)$	B1	Both coordinates
(iii)	$\frac{dx}{dt} = 3t^2 + t$ and $\frac{dy}{dt} = 2$	B1	
	$\frac{dy}{dx} = \frac{2}{3t^2 + t}$	B1	
(iv)	$3t^2 + t - 2 = 0$	B1	Correct quadratic
	$t = \frac{2}{3}$ or -1	B2	B1 for each correct value
10 (i)	$\frac{dx}{dt} = -3 \sin 3t$	M1	Differentiates parametrically
	$\frac{dy}{dt} = 2 \cos 2t$	A1	Both correct
	$s^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$	M1	Uses Pythagoras
	$= 9(\sin 3t)^2 + 4(\cos 2t)^2$	A1	oe
(ii)	$\frac{d(s^2)}{dt} = 54 \cos 3t \sin 3t - 16 \sin 2t \cos 2t$	M1	Differentiates s^2
		A1	Correct
	$\frac{d(s^2)}{dt} = 0$, when $t = 2\pi$	A1	Substitutes $t = 2\pi$ to check stationary point
	$\frac{d^2(s^2)}{dt^2} = 162 \cos 6t - 32 \cos 4t > 0$, when $t = 2\pi$, hence a minimum	M1 A1 E1	Finds second derivative Correct Checks for minimum
11 (i)	$\tan^2 \theta + 1 = \sec^2 \theta$	B1	Or other valid method
	$x^2 + 1 = \frac{1}{y^2}$	B1	
(ii)	$\frac{dx}{d\theta} = \sec^2 \theta$ and $\frac{dy}{d\theta} = -\sin \theta$	B1B1	
	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\sin \theta \cos^2 \theta$	M1A1	
	$\frac{dy}{dx} = 0 \Rightarrow \sin \theta = 0$ or $\cos \theta = 0$	M1A1	
	$\Rightarrow \theta = 0$ or π , or $\frac{\pi}{2}$	B2, 1, 0	B1 for one value; B2 for all values; allow ft
	reject $\theta = \frac{\pi}{2}$	B1	Allow ft
	$(0, 1)$ and $(0, -1)$	B1	Allow ft

	Answer	Marks	Guidance
12	$\frac{dx}{dt} = -2\sin 2t + \cos t$	M1	Differentiates parameters
	$\frac{dy}{dt} = -e^{-t}$	A1	Both correct
	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$	M1	Finds $\frac{dy}{dx}$
	$= \frac{e^{-t}}{2\sin 2t - \cos t}$	A1	Allow ft
	$\frac{e^{-\pi}}{2\sin 2\pi - \cos \pi}$	M1	Substitutes $t = \pi$
	$\frac{dy}{dx} = \frac{1}{e^{\pi}}$	A1	Allow ft
13	(i) $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$	M1 A1	Eliminates parameter All correct
	(ii) $\frac{dy}{d\theta} = 2\cos\theta, \frac{dx}{d\theta} = -3\sin\theta$ $\frac{dy}{dx} = -\frac{2}{3}\cot\theta$ Gradient of normal $= \frac{3}{2}\tan\theta$ normal: $\frac{y - 2\sin\theta}{x - 3\cos\theta} = \frac{3}{2}\tan\theta$ passes through (1,0) when $0 = \frac{3}{2}\tan\theta - \frac{5}{2}\sin\theta$ $\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{3}{5}$ (3, 0), (-3, 0) $\left(\frac{9}{5}, \frac{8}{5}\right), \left(\frac{9}{5}, -\frac{8}{5}\right)$	M1 A1 A1 M1 M1 A1 M1 A1 B1B1 B1 B1	Differentiates At least one of $\frac{dx}{d\theta}$ or $\frac{dy}{d\theta}$ correct All correct; need not be simplified Uses negative reciprocal Equation of straight line Need not be simplified Applies constraint Need not be simplified ft for both For both; can be obtained by reasoning For both
14	$x + y = 2t; x - y = \frac{2}{t}$	M1	Attempts to eliminate t
	$\Rightarrow (x + y)(x - y) = 4$	M1	Completes method
	$\Rightarrow \frac{x^2}{4} - \frac{y^2}{4} = 1$	A1	Any form

	Answer	Marks	Guidance
15	$\frac{y}{x} = t$	M1	Makes t the subject
	$\Rightarrow x = \frac{\left(\frac{y}{x}\right)}{3 - \frac{y}{x}}$	M1	Eliminates t
	$x = \frac{y}{3x - y}$	M1	Simplifies fraction
	$3x^2 - xy = y$	M1	Uses strategy to isolate y
	$y(1 + x) = 3x^2$ $\Rightarrow y = \frac{3x^2}{1 + x}$	A1	Makes y the subject

12 Vectors

$$\begin{aligned}
 1 \quad |\mathbf{a}| &= \sqrt{10^2 + 24^2} \\
 &= 2\sqrt{5^2 + 12^2} \\
 &= 2 \times 13 \\
 &= 26
 \end{aligned}$$

Direction: $\tan^{-1}\left(\frac{24}{10}\right) = 67.4^\circ$ (3 s.f.) to the positive x -axis.

2 Using trigonometry:

$$\begin{aligned}
 \mathbf{F} &= -10\cos 30^\circ \mathbf{i} + 10\sin 30^\circ \mathbf{j} \\
 &= -10 \times \frac{\sqrt{3}}{2} \mathbf{i} + 10 \times \frac{1}{2} \mathbf{j} \\
 &= -5\sqrt{3}\mathbf{i} + 5\mathbf{j} \text{ N}
 \end{aligned}$$

3 Using Pythagoras' theorem:

$$\begin{aligned}
 |\mathbf{b}| &= \sqrt{(-1)^2 + 2^2 + (-2)^2} \\
 &= \sqrt{9} \\
 &= 3
 \end{aligned}$$

4 Using Pythagoras' theorem in three dimensions:

$$\begin{aligned}
 \sqrt{(-5-3)^2 + (6-[-1])^2 + (3-[-4])^2} &= \sqrt{64 + 49 + 49} \\
 &= \sqrt{162} \\
 &= 9\sqrt{2}
 \end{aligned}$$

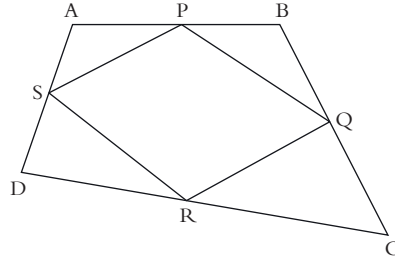
5 The vector joining $(1, -2, 3)$ and $(-3, 1, -2)$ is $-4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

$$\begin{aligned}
 |-4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}| &= \sqrt{(-4)^2 + 3^2 + (-5)^2} \\
 &= \sqrt{50} \\
 &= 5\sqrt{2}
 \end{aligned}$$

So, the unit vector is:

$$-\frac{4}{5\sqrt{2}}\mathbf{i} + \frac{3}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$$

6 Referring to the diagram (where $\mathbf{a} = \overrightarrow{OA}$, etc.):



The vector $\mathbf{p} = \overrightarrow{OP} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ as P is the midpoint of AB.

Similarly, $\mathbf{q} = \overrightarrow{OQ} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\mathbf{r} = \overrightarrow{OR} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$, $\mathbf{s} = \overrightarrow{OS} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$

The vector for $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$

$$\begin{aligned}\mathbf{q} - \mathbf{p} &= \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{b}) \\ &= \frac{1}{2}(\mathbf{c} - \mathbf{a})\end{aligned}$$

$$\begin{aligned}\text{And, similarly, } \mathbf{r} - \mathbf{s} &= \frac{1}{2}(\mathbf{c} + \mathbf{d}) - \frac{1}{2}(\mathbf{a} + \mathbf{d}) \\ &= \frac{1}{2}(\mathbf{c} - \mathbf{a}) \\ &= \mathbf{q} - \mathbf{p}\end{aligned}$$

So, the sides PQ and SR are of equal length and parallel.

This means that PQRS is a parallelogram.

$$\begin{aligned}7 \text{ Resultant force, } \mathbf{R} &= \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 20 \cos 45^\circ \\ 20 \sin 45^\circ \end{pmatrix} \\ &= \begin{pmatrix} 10 + \frac{20}{\sqrt{2}} \\ \frac{20}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 24.14214 \\ 14.14214 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}|\mathbf{R}| &= \sqrt{24.14214^2 + 14.14214^2} \\ &= 27.979 \\ &= 28.0 \text{ N (3 s.f.)}\end{aligned}$$

$$\begin{aligned}\text{Direction of } \mathbf{R} &= \tan^{-1} \left(\frac{14.14214}{24.14214} \right) \\ &= \tan^{-1}(0.58579) \\ &= 30.361^\circ \\ &= 30.4^\circ \text{ (3 s.f.) to the } x\text{-axis.}\end{aligned}$$

$$\begin{aligned} 8 \quad \text{Resultant} &= \begin{pmatrix} \mathbf{a} \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ \mathbf{b} \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{a} + 3 \\ \mathbf{b} + 7 \end{pmatrix} \end{aligned}$$

As the direction is 45° to the x -axis, $\mathbf{a} + 3 = \mathbf{b} + 7$

$$\text{Also, } (\mathbf{a} + 3)^2 + (\mathbf{b} + 7)^2 = 50$$

$$\text{So, } 2(\mathbf{a} + 3)^2 = 50 \Rightarrow \mathbf{a} + 3 = \pm 5$$

Hence, either $\mathbf{a} = 2$ and $\mathbf{b} = -2$, or $\mathbf{a} = -8$ and $\mathbf{b} = -12$

But, $\mathbf{a} = -8$ and $\mathbf{b} = -12$ gives a resultant of $\begin{pmatrix} -5 \\ -5 \end{pmatrix}$, which is not at 45° to the x -axis.

So, $\mathbf{a} = 2$ and $\mathbf{b} = -2$

9 Using the symmetry of the regular hexagon:

$\mathbf{d}$ has twice the magnitude of $\mathbf{b}$, and is in the same direction, so that $\mathbf{d} = 2\mathbf{b}$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{d}, \text{ so that } \mathbf{c} = \mathbf{d} - \mathbf{a} - \mathbf{b} = 2\mathbf{b} - \mathbf{a} - \mathbf{b} = \mathbf{b} - \mathbf{a}$$

10 Expressing the problem using vectors:

$$\overrightarrow{\text{OA}} = -2\mathbf{i} + 4\mathbf{j}$$

$$\overrightarrow{\text{OB}} = \mathbf{i} + 2\mathbf{j}$$

$$\overrightarrow{\text{OC}} = \frac{1}{5}(4\overrightarrow{\text{OA}} + \overrightarrow{\text{OB}})$$

$$= \frac{1}{5}(4(-2\mathbf{i} + 4\mathbf{j}) + \mathbf{i} + 2\mathbf{j})$$

$$= \frac{1}{5}(-7\mathbf{i} + 18\mathbf{j})$$

$$= -\frac{7}{5}\mathbf{i} + \frac{18}{5}\mathbf{j}$$

C has coordinates $\left(-\frac{7}{5}, \frac{18}{5}\right)$

Mark scheme

	Answer	Marks	Guidance
1	$ \alpha = \sqrt{10^2 + 24^2}$ 26 $\tan^{-1}\left(\frac{24}{10}\right)$ $= 67.4^\circ$ (3 s.f.) to the positive x -axis	M1 A1 M1 A1	Uses Pythagoras cao Uses inverse tan Correct to 3 s.f.
2	$-10\cos 30^\circ \mathbf{i} + 10\sin 30^\circ \mathbf{j}$ $-5\sqrt{3}\mathbf{i}$ $+5\mathbf{j}$ N	M1 A1 A1	Uses trigonometry Correct x component Correct y component
3	$\sqrt{(-1)^2 + 2^2 + (-2)^2}$ $= 3$	M1 A1	Uses Pythagoras Correct answer

	Answer	Marks	Guidance
4	$\sqrt{(-5-3)^2 + (6-[-1])^2 + (3-[-4])^2}$ $9\sqrt{2}$	M1 A1	Uses Pythagoras Correct answer
5	$-4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ $5\sqrt{2}$ $-\frac{4}{5\sqrt{2}}\mathbf{i} + \frac{3}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$	B1 B1 B1	Determines vector Finds magnitude oe
6	$\mathbf{p} = \overline{\mathbf{OP}} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ $\mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{b})$ $= \frac{1}{2}(\mathbf{c} - \mathbf{a})$ $\mathbf{r} - \mathbf{s} = \frac{1}{2}(\mathbf{c} + \mathbf{d}) - \frac{1}{2}(\mathbf{a} + \mathbf{d})$ $= \frac{1}{2}(\mathbf{c} - \mathbf{a}) = \mathbf{q} - \mathbf{p}$ PQ and SR are of equal length and parallel PQRS is a parallelogram	B1 M1A1 A1 M1 A1 E1 E1	Finds midpoint Finds PQ Simplifies Finds RS Identifies equal vectors Interprets result Complete explanation, with conclusion
7	$\mathbf{R} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 20 \cos 45^\circ \\ 20 \sin 45^\circ \end{pmatrix}$ $= \begin{pmatrix} 24.14214 \\ 14.14214 \end{pmatrix}$ $ \mathbf{R} = \sqrt{24.14214^2 + 14.14214^2}$ $= 27.979 = 28.0 \text{ N (3 s.f.)}$ $\tan^{-1}\left(\frac{14.14214}{24.14214}\right)$ $= 30.361 = 30.4^\circ \text{ (3 s.f.) [to the } x\text{-axis]}$	M1 A1 A1 M1 A1 M1 A1 A1	Resolves and adds vectors Correct components Correct to 3 s.f. Finds magnitude of resultant Correct to 3 s.f. Uses trigonometry Correct expression Correct to 3 s.f.
8	$\begin{pmatrix} \mathbf{a} \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ \mathbf{b} \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \mathbf{a} + 3 \\ \mathbf{b} + 7 \end{pmatrix}$ $\mathbf{a} + 3 = \mathbf{b} + 7$ $(\mathbf{a} + 3)^2 + (\mathbf{b} + 7)^2 = 50$ $\mathbf{a} + 3 = \pm 5$ Reject $\mathbf{a} = -8, \mathbf{b} = -12$ $\mathbf{a} = 2$ $\mathbf{b} = -2$	M1 A1 B1 M1 A1 B1 A1 A1	Forms vector equation Correct equation Correct equation Uses magnitude Solves equation With explanation cao cao

	Answer	Marks	Guidance
9	d has twice the magnitude of b , and the same direction	E1	Uses symmetry of the hexagon
	$\mathbf{d} = 2\mathbf{b}$	B1	Correct answer
	$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{d}$	M1	Forms vector equation
		A1	Correct equation
	$\mathbf{c} = \mathbf{b} - \mathbf{a}$	A1	Correct expression
10	$\overrightarrow{OA} = -2\mathbf{i} + 4\mathbf{j}, \overrightarrow{OB} = \mathbf{i} + 2\mathbf{j}$	B1	Expresses using vectors
	$\overrightarrow{OC} = \frac{1}{5}(4\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{5}(4(-2\mathbf{i} + 4\mathbf{j}) + \mathbf{i} + 2\mathbf{j})$	M1	Uses the ratio
	$= \frac{1}{5}(-7\mathbf{i} + 18\mathbf{j})$	A1	Correct vector
	$\left(-\frac{7}{5}, \frac{18}{5}\right)$	A1	cao

13 Differential equations

- 1 (i) Where there are no derivatives higher than the first.
(ii) The constant of integration is left in the solution, so it refers to a family of curves.
(iii) Additional information is used to find the constant of integration, so it refers to one curve.

2 Differentiating:

$$\begin{aligned}
 y = \frac{1}{4-x} &\Rightarrow \frac{dy}{dx} = -(4-x)^{-2}(-1) \\
 &= \frac{1}{(4-x)^2} \\
 &= y^2
 \end{aligned}$$

Also, checking that the point satisfies the function, $x = 3 \Rightarrow y = \frac{1}{4-3} = 1$

3 $\frac{dy}{dx} = 2x - 1$

Integrating both sides with respect to x :

$$y = x^2 - x + c$$

4 Rewrite as $\frac{1}{y} \frac{dy}{dx} = x$

Integrating both sides with respect to x :

$$\int \frac{1}{y} \frac{dy}{dx} dx = \int x dx$$

$$\Rightarrow \int \frac{1}{y} dy = \frac{1}{2}x^2 + c$$

$$\Rightarrow \ln|y| = \frac{1}{2}x^2 + c$$

$$\Rightarrow y = Ae^{\frac{1}{2}x^2}, \text{ where } A = e^c$$

5 From question 3, $y = x^2 - x + c$

$$\Rightarrow 1 = 1 - 1 + c \Rightarrow c = 1$$

So, the particular solution is $y = x^2 - x + 1$

- 6 Using the given information, and the answer to question 4, to evaluate c :

$$y = Ae^{\frac{1}{2}x^2} \Rightarrow 1 = Ae^2 \Rightarrow A = e^{-2}$$

So, the equation of the curve is $y = e^{\frac{1}{2}x^2 - 2}$

- 7 Rewrite as $\frac{1}{2y-1} \frac{dy}{dx} = 1$

Integrating both sides with respect to x :

$$\int \frac{1}{2y-1} \frac{dy}{dx} dx = \int 1 dx$$

$$\Rightarrow \int \frac{1}{2y-1} dy = x + c$$

$$\Rightarrow \frac{1}{2} \ln|2y-1| = x + c$$

$$\Rightarrow \ln|2y-1| = 2x + 2c$$

$$\Rightarrow 2y-1 = Ae^{2x}, \text{ where } A = e^{2c}$$

$$y = 3 \text{ when } x = 0 \Rightarrow 5 = A$$

$$\text{So, } y = \frac{1}{2}(5e^{2x} + 1)$$

- 8 (i) $\frac{dv}{dt} = \frac{k}{v}$

(ii) Substituting the known values:

$$2 = \frac{k}{10} \Rightarrow k = 20 \Rightarrow \frac{dv}{dt} = \frac{20}{v}$$

(iii) Rewrite as $v \frac{dv}{dt} = 20$

Integrating both sides with respect to t :

$$\int v \frac{dv}{dt} dt = \int 20 dt$$

$$\Rightarrow \int v dv = 20t + C$$

$$\Rightarrow \frac{1}{2}v^2 = 20t + C$$

$$\Rightarrow v = \sqrt{40t + 2C} \text{ or } v = \sqrt{40t + c} \ (v > 0)$$

(iv) $t = 0, v = 0 \Rightarrow c = 0$

So, particular solution is $v = \sqrt{40t}$

(v) $v = 20 \Rightarrow 400 = 40t \Rightarrow t = 10 \text{ s}$

- 9 Let T be the temperature in $^{\circ}\text{C}$ of the food at time t seconds.

Then, $\frac{dT}{dt} = k(175 - T)$

Also, initially $0.5 = k(175 - 15)$, so that $k = \frac{1}{320}$

So, the differential equation is $\frac{dT}{dt} = \frac{1}{320}(175 - T)$

- 10 Separating the variables:

$$\frac{dP}{dt} = kP \Rightarrow \int \frac{1}{P} \frac{dP}{dt} dt = \int k dt$$

$$\Rightarrow \int \frac{1}{P} dP = kt + C$$

$$\Rightarrow \ln P = kt + C \text{ (as } P > 0, \ln P \text{ is defined)}$$

$$\Rightarrow P = Ae^{kt}, \text{ where } A = e^C$$

$$\text{As } P = 7.5 \text{ when } t = 0 \text{ (measuring } t \text{ from 2017), } P = 7.5e^{kt}$$

$$\text{Also, } P = 12 \text{ when } t = 83, \text{ so that } 12 = 7.5e^{83k}$$

$$\text{Hence, } k = \frac{1}{83} \ln\left(\frac{12}{7.5}\right) = 0.0056627$$

$$\text{So, when } t = 183, P = 7.5e^{0.0056627 \times 183} = 21.14 = 21 \text{ (to the nearest billion).}$$

11 Separating the variables:

$$x \frac{dy}{dx} = y(2 - y) \Rightarrow \int \frac{1}{y(2 - y)} \frac{dy}{dx} dx = \int \frac{1}{x} dx$$

Using partial fractions:

$$\frac{1}{y(2 - y)} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{2 - y} \right)$$

Integrating:

$$\frac{1}{2} \int \left(\frac{1}{y} + \frac{1}{2 - y} \right) dy = \ln x + \ln C \text{ (as } x > 0, \ln x \text{ is defined)}$$

$$\Rightarrow \frac{1}{2} [\ln y - \ln(2 - y)] = \ln(Cx) \text{ (as } 0 < y < 2, \ln y \text{ and } \ln(2 - y) \text{ are both defined)}$$

$$\Rightarrow \ln\left(\frac{y}{2 - y}\right) = 2\ln(Cx)$$

$$\Rightarrow \ln\left(\frac{y}{2 - y}\right) = \ln(C^2 x^2)$$

$$\Rightarrow \frac{y}{2 - y} = C^2 x^2$$

$$\text{As } x = 1 \text{ when } y = 1, 1 = C^2$$

$$\text{So, } \frac{y}{2 - y} = x^2$$

$$\Rightarrow y = x^2(2 - y)$$

$$\Rightarrow y(1 + x^2) = 2x^2$$

$$\text{So, } y = \frac{2x^2}{1 + x^2}$$

Mark scheme

	Answer	Marks	Guidance
1 (i)	There are no derivatives higher than the first	E1	oe
(ii)	The constant of integration is left in the solution so it refers to a family of curves	E1	oe
(iii)	Additional information is used to find the constant of integration so it refers to one curve	E1	oe
2	$\frac{dy}{dx} = -(4 - x)^{-2}(-1)$ $= \frac{1}{(4 - x)^2}$ $x = 3 \Rightarrow y = 1$	M1 A1 A1	Differentiates function Correct derivative Checks point
3	$y = x^2 - x + c$	M1 A1	Integrates both sides Correct answer

	Answer	Marks	Guidance
4	$\frac{1}{y} \frac{dy}{dx} = x$	B1	Separates variables
	$\int \frac{1}{y} \frac{dy}{dx} dx = \int x dx$	M1	Integrates with respect to x
	$\ln y = \frac{1}{2}x^2 + c$	A1	Correct integral
	$y = Ae^{\frac{1}{2}x^2}$, where $A = e^c$	A1	Written with y as subject
5	$c = 1$	B1	Evaluates c
	$y = x^2 - x + 1$	B1	$x^2 - x + c$ taken from question 3 cao
6	$y = Ae^{\frac{1}{2}x^2} \Rightarrow 1 = Ae^2$	M1	Uses general solution from question 4
	$\Rightarrow A = e^{-2}$	A1	Evaluates A
	$y = e^{\frac{1}{2}x^2 - 2}$	B1	Writes solution with y as subject
7	$\frac{1}{2y-1} \frac{dy}{dx} = 1$	B1	Separates variables
	$\int \frac{1}{2y-1} \frac{dy}{dx} dx = \int 1 dx$	M1	Integrates both sides; need not see $\frac{dy}{dx}$
	$\frac{1}{2} \ln 2y-1 = x + c$	A1	
	$y = 3$ when $x = 0 \Rightarrow \frac{1}{2} \ln 5 = c$	M1	Condone missing moduli signs Uses initial conditions to find A
	$y = \frac{1}{2}(5e^{2x} + 1)$	M1	Makes y the subject
		A1	All correct
8	(i) $\frac{dv}{dt} = \frac{k}{v}$	B1 B1	Condone missing k Fully correct
	(ii) $2 = \frac{k}{10} \Rightarrow k = 20$	M1 A1	Substitutes known values cao
	(iii) $\int v \frac{dv}{dt} dt = \int 20 dt$	B1	Separates variables
	$\frac{1}{2}v^2 = 20t + C$	M1 A1	Integrates both sides Correct expression
	$v = \sqrt{40t + C}$	M1 A1	Makes v the subject Not $\sqrt{40t + C}$
	(iv) $v = \sqrt{40t}$	B1	Evaluates c
	(v) $400 = 40t$ $t = 10$ s	M1 A1	Establishes equation cao
9	$\frac{dT}{dt} = k(175 - T)$	M1 A1	Sets up differential equation Fully correct
	$0.5 = k(175 - 15) \Rightarrow k = \frac{1}{320}$	M1	Substitutes known values
	$\frac{dT}{dt} = \frac{1}{320}(175 - T)$	A1	cao

Answer	Marks	Guidance
10 $\int \frac{1}{P} \frac{dP}{dt} dt = k \int dt$ $\ln P = kt + C$ $P = Ae^{kt}$ $P = 7.5e^{kt}$ $12 = 7.5e^{83k}$ $k = 0.0056627$ $P = 21$	B1 M1 A1 A1 B1 M1 A1 M1 A1	Separates variables, condone no integral sign Integrates both sides P need not be subject at this stage P is subject Fully correct Uses information to find k May be implied by final answer Substitutes 183 To nearest whole number
11 $\frac{1}{y(2-y)} dy = \frac{1}{x} dx$ $\frac{1}{y(2-y)} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{2-y} \right)$ $\frac{1}{2} [\ln y - \ln(2-y)] = \ln(Cx)$ $\ln \left(\frac{y}{2-y} \right) = \ln(C^2 x^2)$ $\frac{y}{2-y} = C^2 x^2$ $x = 1$ when $y = 1$, $1 = C^2$ $y = \frac{2x^2}{1+x^2}$	B1 M1 A1 M1 A1 M1 A1 B1 M1 A1	Separates variables Uses partial fractions Correct fractions Integrates Correct Prepares to eliminate logs y need not be subject Uses known point Attempts to make y the subject Fully correct

14 Numerical methods

- Substitute $x = 1$ into $x^5 - x^2 - 3$, $1^5 - 1^2 - 3 = -3$
 Substitute $x = 2$ into $x^5 - x^2 - 3$, $2^5 - 2^2 - 3 = 25$
 The change in sign indicates that a root exists between $x = 1$ and $x = 2$.
- Solving the simultaneous equations is equivalent to solving the equation $e^{x^2} + x^3 - 4 = 0$
 When $x = 0.9$, $e^{x^2} + x^3 - 4 = e^{0.9^2} + 0.9^3 - 4 = -1.02\dots$
 When $x = 1.1$, $e^{x^2} + x^3 - 4 = e^{1.1^2} + 1.1^3 - 4 = 0.684\dots$
 The change in sign indicates that a root exists between $x = 0.9$ and $x = 1.1$ and so one of the points of intersection is in this interval.
 The estimate of the value is 1 ± 0.1 at this stage.
- The curve has a discontinuity when $x = \sqrt{2}$, so a decimal search may not identify that there is a root between 1 and 2.
- It is an overestimate because, either each trapezium includes some of the area above the curve, or the curve is concave (upwards).

- 5 Four strips require five ordinates, so they are at the integer values from 0 to 4.

The values of the function are:

x	0	1	2	3	4
y	1	2.6458	3.8730	5	6.0828

The width of each rectangular strip is 1, so the area is numerically equal to the ordinate.

For a lower bound, the left-hand ordinate for each strip is used:

$$\text{area} = 1 + 2.6458 + 3.8730 + 5 = 12.52$$

For an upper bound, the right-hand ordinate for each strip is used:

$$\text{area} = 2.6458 + 3.8730 + 5 + 6.0820 = 17.60$$

- 6 Neither of the integer points either side of it will result in an iteration that finds the negative root. Starting with -2 will not work as the function is undefined there.

Starting with -1 will give the root near 1.

- 7 (i) $x^5 + x^2 - 4 = 0$

Make x^5 the subject:

$$x^5 = 4 - x^2$$

Take the fifth root of each side:

$$x = \sqrt[5]{4 - x^2}$$

- (ii) (a) $x = 1$, it is the first value of x .

(b) Since it is the next value of x , it must be the calculated value from the line before, so ' $=B2$ '.

(c) 1.2056 is consistent for the final two values, so check 1.20555 and 1.20565:

$$1.20555^5 + 1.20555^2 - 4 = -0.000252$$

$$1.20565^5 + 1.20565^2 - 4 = 0.00105$$

The change of sign confirms that the root is 1.2056 (4 d.p.)

- 8 (i) The x values increase by 0.5 each time so the width of the strips is 0.5.

(ii) Column D contains the areas of each trapezium.

D5 is the third area so uses the ordinates at $x = 1$ and $x = 1.5$.

The formula is $=0.5 \times (C4 + C5)$, or $=0.5 \times 0.5 \times (C4 + C5)$ is acceptable.

(iii) The sum of the entries in column D gives the estimated area, 7.4375.

- 9 Sam is correct as $1.31105^3 - 4 \times 1.31105^2 + 2 \times 1.31105 + 2 = 0.0001926...$

and $1.31115^3 - 4 \times 1.31115^2 + 2 \times 1.31115 + 2 = -0.0001405...$ show there is a change of sign.

He is not justified, as he did not check for the change in sign before making his statement.

- 10 $\ln(x^2 + 1) - x^3 - 0.01 = 0$

(i) Substitute $x = 0.1$ into the equation:

$$\ln(0.1^2 + 1) - 0.1^3 - 0.01 = -0.001049... < 0$$

$$\ln(0.2^2 + 1) - 0.2^3 - 0.01 = 0.021220... > 0$$

The change in sign shows there is a root between 0.1 and 0.2.

- (ii) $x = 0.11$, $y = 0.0007$, so root lies between 0.10 and 0.11
 $x = 0.106$, $y = -0.00002$
 $x = 0.107$, $y = 0.00016$ so root lies between 0.106 and 0.107
 $x = 0.1065$, $y > 0$ so root is 0.106 (3 d.p.)

(iii) Using the table facility on a graphing calculator to find a change in sign:

X	Y ₁	
-3	29.293	
-2	9.5994	
-1	1.6831	
0	-.01	
1	-.3169	
2	-6.401	
3	-24.71	

X = -3

X	Y ₁	
-6	.51348	
-5	.33814	
-4	.20242	
-3	.10318	
-2	.03722	
-1	9.5E-4	
0	-.01	

X = -.6

There is a root in $[-1, 0]$, the root is in $[-0.1, 0]$.

Return to the integer search:

X	Y ₁	
0	-.01	
1	-.3169	
2	-6.401	
3	-24.71	
4	-61.18	
5	-121.8	
6	-212.4	

X = 0

X	Y ₁	
.4	.07442	
.5	.08814	
.6	.08148	
.7	.04578	
.8	-.0273	
.9	-.1457	
1	-.3169	

X = .4

There is no change in sign but there is a root in $[0.1, 0.2]$, so there must be another root in $[0, 1]$.

The root is in $[0.7, 0.8]$.

- (iv) As seen in part (iii), there are two roots in the interval $[0, 1]$; so, none is detected in an integer search.

11 (i) $0e^0 - 1 = -1$ and $1e^1 - 1 = 1.718$

Or, using a graphing calculator:

X	Y ₁	
-3	-1.149	
-2	-1.271	
-1	-1.368	
0	-1	
1	1.7183	
2	13.778	
3	59.257	

X = -3

Shows the change in sign so there is a root in $[0, 1]$.

(ii) $x = e^{-x}$

Multiply both sides of the equation by e^x :

$$xe^x = e^{-x}e^x = 1$$

So, $xe^x - 1 = 0$ as required.

(iii) $x_{n+1} = e^{-x_n}$

$x_0 = 0, x_1 = 1$ and using the iterative function on the calculator:

0
1
0.367879
0.692201
0.500474
0.606244
0.545396
0.579612
0.560115
0.571143
0.564879
0.568429
0.566415

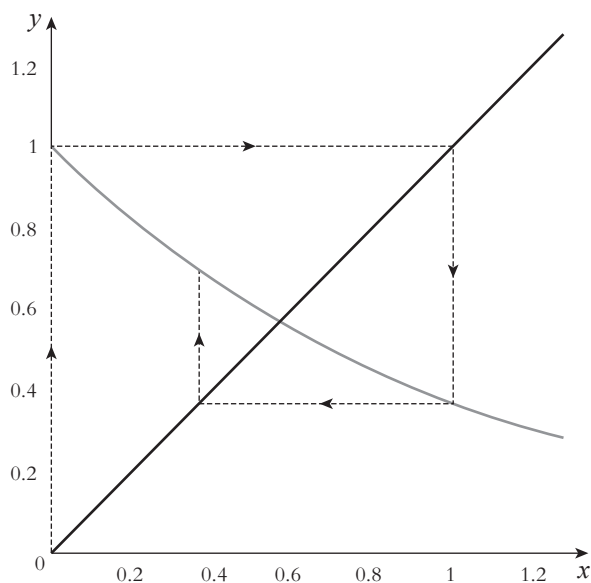
The root is 0.57 to two decimal places.

Check $0.565e^{0.565} - 1 = -0.0059$

$0.575e^{0.575} - 1 = 0.0219$

The change in sign confirms that the root is 0.57 (2 d.p.)

(iv)



(v) The iterations form a cobweb. The gradient of $g(x)$ is small, so the convergence is slow.

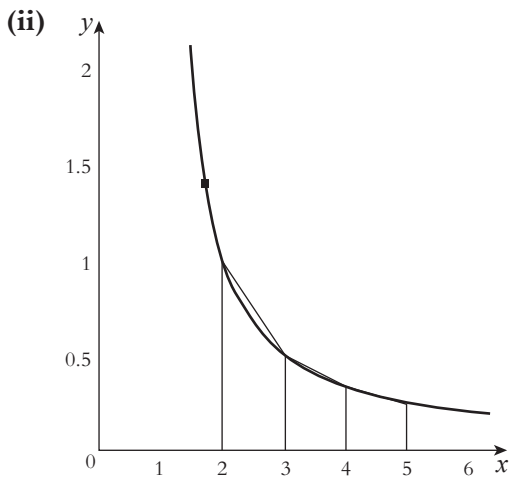
(vi) Only by one step since the second value is 1.

12 (i) Using a graphing calculator to obtain the values of the ordinates:

X	Y ₁	
2	1	
3	.5	
4	.33333	
5	.25	
6	.2	
7	.16667	
8	.14286	

$X=2$

area $\approx \frac{1}{2} \times 1 \times (1 + 0.25 + 2(0.5 + 0.33333\dots)) = 1.458$



The estimate is an overestimate because the curve is concave.

13 (i) Using a graphing calculator:

X	Y ₁	
-3	-233	
-2	-25	
-1	3	
0	1	
1	-1	
2	27	
3	235	

X = -3

There is a change of sign between -2 and -1 indicating the presence of a root.

(ii) $f(x) = x^5 - 3x + 1$

$$f'(x) = 5x^4 - 3$$

$$x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

$$x_{n+1} = \frac{x_n(5x_n^4 - 3) - (x_n^5 - 3x_n + 1)}{5x_n^4 - 3}$$

$$x_{n+1} = \frac{5x_n^5 - 3x_n - x_n^5 + 3x_n - 1}{5x_n^4 - 3}$$

$$x_{n+1} = \frac{4x_n^5 - 1}{5x_n^4 - 3} \text{ as required.}$$

(iii)

X	Y ₁	
-3	-233	
-2	-25	
-1	3	
0	1	
1	-1	
2	27	
3	235	

X = -3

Shows roots in $[0, 1]$ and $[1, 2]$ as well as $[-2, -1]$

For $[-2, -1]$ starting value -2:

```

2
(4*Ans^5-1)/(5*Ans^4-3)
-1.675324675
-1.47823803
-1.400445373
-1.389019864
-1.388792074

```

Root is -1.39 (2 d.p.)

Check $f(-1.395) = -0.098$, $f(-1.385) = 0.0588$ confirms -1.39 (2 d.p.)

For $[0, 1]$ starting value 0:

```

0
(4*Ans^5-1)/(5*Ans^4-3)
.3333333333
.3347338936
.3347341419

```

Root is 0.33 (2 d.p.)

Check $f(0.325) = 0.0286$, $f(0.335) = -0.00078$ confirms 0.33 (2 d.p.)

For $[1, 2]$ starting value 2:

```

2
(4*Ans^5-1)/(5*Ans^4-3)
1.649350649
1.406489089
1.268586604
1.220369383
1.214721237
1.214648055

```

Root is 1.21 (2 d.p.)

Check $f(1.205) = -0.0744$, $f(1.215) = 0.00278$ confirms 1.21 (2 d.p.)

- 14 (i) Using the ordinates when the distance is 0, 8 and 16:

$$\text{area} \approx \frac{1}{2} \times 8 \times (3 + 3 + 2 \times 10.7) = 109.6 \text{ m}^2$$

- (ii) Using all the ordinates in the table:

$$\text{area} \approx \frac{1}{2} \times 2 \times (3 + 3 + 2(7.8 + 9.6 + 10.4 + 10.7 + 10.4 + 9.6 + 7.8)) = 138.6 \text{ m}^2$$

- (iii) The estimate in part (ii) is better as it uses more strips.

- 15 (i) Using a graphing calculator:

X	Y ₁	
0	1	
1	-.3513	
2	-1.282	
3	-1.518	
4	-.6109	
5	2.1825	
6	8.0855	

X=0

There is a change of sign between 0 and 1, so there is a root in that interval.

There is another change of sign in the interval $[4, 5]$.

(ii) ① $e^{\frac{1}{2}x} - 2x = 0 \Rightarrow e^{\frac{1}{2}x} = 2x \Rightarrow x = \frac{1}{2}e^{\frac{1}{2}x}$

② $e^{\frac{1}{2}x} - 2x = 0 \Rightarrow e^{\frac{1}{2}x} = 2x \Rightarrow \frac{1}{2}x = \ln 2x \Rightarrow x = 2 \ln 2x$

(iii) $x_{n+1} = \frac{1}{2}e^{\frac{1}{2}x_n}$

Try $x_0 = 1$, using a graphing calculator:

```

1
0.5*e^(0.5*Ans)
.8243606354
.7550533554
.7293361786
.7200179827
.7166711511

```

The root is 0.7 (1 d.p.)

Check $0.65 \rightarrow 0.084$, $0.75 \rightarrow -0.045$, so the root of 0.7 is confirmed.

$$x_{n+1} = 2 \ln 2x_n$$

Try $x_0 = 4$, using a graphing calculator:

```

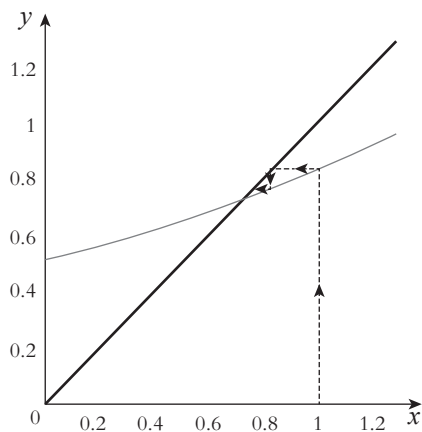
4
2*ln(2*Ans)
4.158883083
4.236787458
4.273904975
4.291350208
4.299497196
4.30329053

```

The root is 4.3 (1 d.p.)

Check $4.25 \rightarrow -0.127$, $4.35 \rightarrow 0.102$, so the root of 4.3 is confirmed.

(iv)



(v) Starting with 4 for the rearrangement in part (a) gives:

```

4
0.5*e^(0.5*Ans)
3.694528049
3.171221498
2.441136155
1.694556233
1.166643632
.8959905935

```

Showing the iterations converging to the root between 0 and 1.

Starting with 5 for the rearrangement in part (a) gives:

```

5
0.5*e^(0.5*Ans)
6.09124698
10.51156742
95.83582089
3.231871571E20

```

Showing the iterations diverging.

Starting with 1 for the rearrangement in part (b) gives:

```

1
2*ln(2*Ans)
1.386294361
2.039562881
2.811765383
3.453919434
3.865313669

```

Showing the iterations converging to the root in [4,5].

Starting with 0 for the rearrangement in part (b) gives:

```

ERR:DOMAIN
1: Quit
2: Goto

```

As the function $g(x)$ is undefined.

$$16 \quad f(x) = x^2 e^{-x} - 0.5 \quad f'(x) = 2xe^{-x} - x^2 e^{-x}$$

$$x_{n+1} = x_n - \frac{x_n^2 e^{-x_n} - 0.5}{2x_n e^{-x_n} - x_n^2 e^{-x_n}}$$

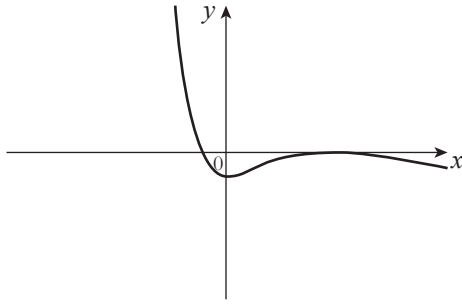
There is little to be gained from rearranging this, so use it as it is.

Do an integer search to find a starting point:

X	Y ₁	
-3	180.27	
-2	29.056	
-1	2.2183	
0	-0.5	
1	-.1321	
2	.04134	
3	-.0519	

X = -3

So, there are roots in the intervals $[-1, 0]$, $[1, 2]$ and $[2, 3]$.



A quick sketch confirms there are unlikely to be any others.

For the root in $[-1, 0]$

Start with -1

-1

Ans $- ((\text{Ans}^2 * e^{(-\text{Ans})}) - 0.5) / (2 * \text{Ans} * e^{(-\text{Ans})} - \text{Ans}^2 * e^{(-\text{Ans})})$

$-.7279799069$

$-.5827002391$

$-.5426020999$

$-.5398476398$

$-.5398352772$

$-.5398352769$

Giving a root of -0.54 (2 d.p.)

Check $-0.545 \rightarrow 0.012$, $-0.535 \rightarrow -0.011$ so the root of -0.54 is confirmed.

For the root in $[1, 2]$

Start with 1

1

Ans $- ((\text{Ans}^2 * e^{(-\text{Ans})}) - 0.5) / (2 * \text{Ans} * e^{(-\text{Ans})} - \text{Ans}^2 * e^{(-\text{Ans})})$

1.359140914

1.472984469

1.487713708

1.487961995

Giving a root of 1.49 (2 d.p.)

Check $1.485 \rightarrow -0.00051$, $1.495 \rightarrow 0.0012$, so the root of 1.49 is confirmed.

For the root in $[2, 3]$

Start with 3

3

Ans – ((Ans²*e[^](–Ans)) – 0.5) / (2*Ans*e[^](–Ans) – Ans²*e[^](–Ans))

2.652410513
2.618420938
2.617866767
2.617866613

Giving a root of 2.62 (2 d.p.)

Check $2.615 \rightarrow 0.0034$, $2.625 \rightarrow -0.00084$, so the root of 2.62 is confirmed.

17 (i)

Time, t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
Velocity, v	0	2.0111	2.3562	1.0352	0

$$\text{Upper bound} = \frac{\pi}{4}(2.0111 + 2 \times 2.3562 + 1.0352) = 6.0937$$

$$\text{Lower bound} = \frac{\pi}{4}(0 + 2.0111 + 1.0352 + 0) = 2.3926$$

(ii) Distance travelled $\approx \frac{1}{2} \times \frac{\pi}{4}(0 + 0 + 2(2.0111 + 2.3562 + 1.0352)) = 4.2431 \text{ m}$

(iii) Using more strips always improves the accuracy.

18 (i) $x^4 - 3x^2 - x + 2 = 0$

$$3x^2 = x^4 - x + 2$$

$$x = \sqrt{\frac{1}{3}(x^4 - x + 2)}$$

$$\text{So, } x_{n+1} = \sqrt{\frac{1}{3}(x_n^4 - x_n + 2)}$$

(ii) Using a graphing calculator:

Start with $x_0 = 1$

1

$\sqrt{(1/3*(\text{Ans}^4 - \text{Ans} + 2))}$

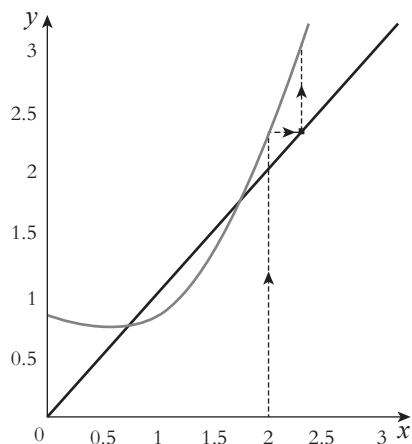
.8164965809
.7366473294
.7206064391
.7185723518
.7183409293

So, the root is 0.72 (2 d.p.)

Check $0.715 \rightarrow 0.0127$, $0.725 \rightarrow -0.02559$, so the root of 0.72 is confirmed.

(iii) Starting with 1 gives the root in $[0, 1]$

Starting with 2 gives:



The iterations do not converge, as shown by the staircase moving away from the intersection.

(iv) Using a graphing calculator:

Start with $x_0 = 2$

```

      2
√(( 3*Ans²+Ans-2)
/Ans²)
1.732050808
1.706072567
1.702650208
1.70218414
1.702120388

```

So, the root is 1.70 (2 d.p.)

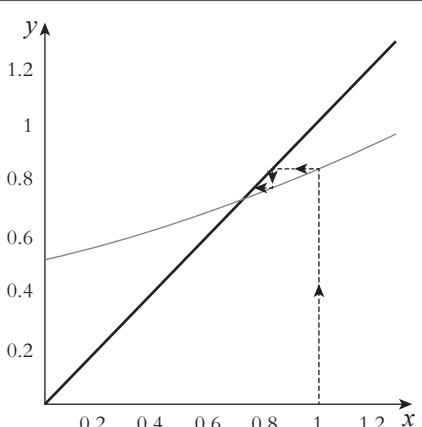
Check $1.695 \rightarrow -0.0598$, $1.705 \rightarrow 0.02472$, so the root of 1.70 is confirmed.

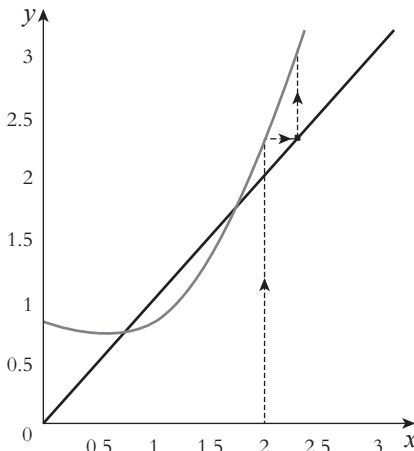
Mark scheme

	Answer	Marks	Guidance												
1	$1^5 - 1^2 - 3 = -3$	B1	Establishes one end is negative												
	$2^5 - 2^2 - 3 = 25$	B1	Establishes one end is positive												
2	$e^{0.9^2} + 0.9^3 - 4 = -1.02\dots$	B1	Establishes one end is negative												
	$e^{1.1^2} + 1.1^3 - 4 = 0.684\dots$	B1	Establishes one end is positive												
	1 ± 0.1	B1	Centre of the interval $\pm$ max error												
3	Discontinuity at $\sqrt{2}$ masks root	B1	Must be more than ‘discontinuity’												
4	Overestimate, concave upwards	B1	oe												
5	<table border="1"><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>y</td><td>1</td><td>2.6458</td><td>3.8730</td><td>5</td><td>6.0828</td></tr></table>	x	0	1	2	3	4	y	1	2.6458	3.8730	5	6.0828	M1	Calculates five ordinates
	x	0	1	2	3	4									
	y	1	2.6458	3.8730	5	6.0828									
12.52	A1	Lower bound													
17.60	A1	Upper bound													

	Answer	Marks	Guidance
6	Negative root cannot be found from an integer starting value	E1	oe
7 (i)	$x^5 = 4 - x^2$	B1	Answer given; interim step shown
(ii)(a)	$x = 1$	B1	cao
(ii)(b)	$= B2$	B1	oe
(ii)(c)	$1.20555^5 + 1.20555^2 - 4 = -0.000252$ $1.20565^5 + 1.20565^2 - 4 = 0.00105$	M1	Tests both bounds
	1.2056	A1	cao
8 (i)	0.5	B1	oe
(ii)	$= 0.5 \times (C4 + C5)$	B1	Or $= 0.5 \times 0.5 \times (C4 + C5)$
(iii)	7.4375	B1	cao
9	$1.31105^3 - 4 \times 1.31105^2$ $+ 2 \times 1.31105 + 2$ $= 0.0001926$	M1	By calculator (BC) and checks for change of sign
	$1.31115^3 - 4 \times 1.31115^2$ $+ 2 \times 1.31115 + 2$ $= -0.0001405\dots$		
	Correct	A1	
	Not justified	A1	
10 (i)	$\ln(0.1^2 + 1) - 0.1^3 - 0.01$ $= -0.001049\dots < 0$	M1	Evaluates at both end points
	$\ln(0.2^2 + 1) - 0.2^3 - 0.01$ $= 0.021220\dots > 0$ Change of sign $\Rightarrow$ root	A1	
			States conclusion
(ii)	Root lies between 0.106 and 0.107	M1	Conducts decimal search BC
	Tests 0.1065	B1	Midpoint correct
	0.106 (3 d.p.)	A1	Checks fourth decimal place
(iii)	$[-0.1, 0]$	B1	cao
	$[0.7, 0.8]$	B1	cao
(iv)	Identifies two roots in $[0, 1]$	M1	Realises there must be another in $[0, 1]$
	States integer search fails	A1	States conclusion
11 (i)	$0e^0 - 1 = -1$ $1e^1 - 1 = 1.718$	M1	Checks for change in sign
	Change in sign implies root	A1	Draws conclusion
(ii)	$xe^x = e^{-x}e^x = 1$	M1	Multiplies by e^x
	$xe^x - 1 = 0$	A1	Answer given

	Answer	Marks	Guidance
(iii)	0	B1	Sequence of values with at least 4 d.p. and first two correct
	1	B1	All correct
	0.367879		
	0.692201		
	0.500474		
	0.606244		
	0.545396		
	0.579612		
	0.560115		
	0.571143		
	0.564879		
	0.568429		
	0.566415		
	0.565 $\rightarrow$ -0.0059	M1	BC and checks the root
	0.575 $\rightarrow$ 0.0219		
	0.57	A1	cao
(iv)		B1	Sketches two graphs
		B1	Shows two iterations
		B1	Shows four iterations
(v)	Cobweb is slower, small gradient makes it slower	B1	Appropriate reason
(vi)	Only by one step as 1 is the second value	B1	Appropriate reason
12 (i)	1, 0.5, 0.3333, 0.25	B1	Calculates appropriate ordinates
	$\frac{1}{2} \times 1 \times (1 + 0.25 + 2(0.5 + 0.3333 \dots))$	M1	Uses trapezium rule
	1.458	A1	awrt 1.46
(ii)		B1	Sketches graph
		B1	Decision plus references graph

	Answer	Marks	Guidance
13 (i)	$-2 \rightarrow -25$	M1	Checks for change in sign
	$-1 \rightarrow 3$ Change in sign	A1	States conclusion
(ii)	$f(x) = 5x^4 - 3$	B1	Differentiates correctly
	$x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$	M1	Correct use of Newton-Raphson formula
	$x_{n+1} = \frac{4x_n^5 - 1}{5x_n^4 - 3}$	A1	Answer given; one correct line of working must be seen
(iii)	$f(-1.395) = -0.098, f(-1.385) = 0.0588$	M1	BC and checks the root
	-1.39 (2 d.p.)	A1	cao
	$f(0.325) = 0.0286, f(0.335) = -0.00078$	M1	BC and checks the root
	0.33 (2 d.p.)	A1	cao
	$f(1.205) = -0.0744, f(1.215) = 0.00278$	M1	BC and checks the root
	1.21 (2 d.p.)	A1	cao
14 (i)	$\frac{1}{2} \times 8 \times (3 + 3 + 2 \times 10.7)$	M1	Uses trapezium rule
	$= 109.6 \text{ m}^2$	A1	cao
(ii)	$\frac{1}{2} \times 2 \times (3 + 3 + 2(7.8 + 9.6$	M1	Uses trapezium rule
	$+ 10.4 + 10.7 + 10.4 + 9.6 + 7.8))$	A1	cao
(iii)	$= 138.6 \text{ m}^2$	A1	cao
	Second as more strips	E1	oe
15 (i)	$f(0) = 1, f(1) = -0.3513$	M1	Evaluates at endpoints
	Change of sign indicates root	A1	States conclusion
	$[4, 5]$	B1	cao
(ii)(a)	$e^{\frac{1}{2}x} = 2x$	B1	Answer given; must see step
(ii)(b)	$e^{\frac{1}{2}x} = 2x \Rightarrow \frac{1}{2}x = \ln 2x$	B1	Answer given; must see step
(iii)	$0.65 \rightarrow 0.084, 0.75 \rightarrow -0.045$	M1	BC and checks the root
	0.7	A1	cao
	$4.25 \rightarrow -0.127, 4.35 \rightarrow 0.102$	M1	BC and checks the root
	4.3	A1	cao
(iv)		B1 B1	Sketches graphs Staircase diagram

	Answer	Marks	Guidance
(v)	Starting with 4 for the rearrangement in part (a) results in the root in $[0,1]$ Starting with 5 for the rearrangement in part (a) results in divergent iterations Starting with 1 for the rearrangement in part (b) results in the root in $[4,5]$ Starting with 1 for the rearrangement in part (b) results in an error as $\ln$ is undefined at 0.	B1 B1 B1 B1	All four cases need to be examined for full marks
16	$x_{n+1} = x_n - \frac{x_n^2 e^{-x} - 0.5}{2x_n e^{-x_n} - x_n^2 e^{-x_n}}$ $-0.545 \rightarrow 0.012, -0.535 \rightarrow -0.011$ -0.54 $1.485 \rightarrow -0.00051, 1.495 \rightarrow 0.0012$ 1.49 $2.615 \rightarrow 0.0034, 2.625 \rightarrow -0.00084$ 2.62	M1 M1 A1 M1 A1 M1 A1	Derives NR formula BC and checks the root cao BC and checks the root cao BC and checks the root cao
17 (i)	0, 2.0111, 2.3562, 1.0352, 0 LB = 2.3926 UB = 6.0937	B1 B1 B1	Correct velocities Correct to 4 d.p. Correct to 4 d.p.
(ii)	$\frac{1}{2} \times \frac{\pi}{4} (0 + 0 + 2(2.0111 + 2.3562 + 1.0352))$ 4.2431 m	M1 A1	Uses trapezium rule awrt 5.42
(iii)	Use more strips	E1	oe
18 (i)	$3x^2 = x^4 - x + 2$ $x = \sqrt{\frac{1}{3}(x^4 - x + 2)}$	M1 A1	Selects term in x^2 to isolate Answer given must show this before version given
(ii)	$0.715 \rightarrow 0.0127, 0.725 \rightarrow -0.02559$ 0.72	M1 A1	BC and checks the root cao
(iii)	 Iterations are moving away from the root	B1 B1	Graph with staircase States conclusion
(iv)	$1.695 \rightarrow -0.0598, 1.705 \rightarrow 0.02472$ 1.70	M1 A1	BC and checks the root cao

15 Probability

1 Either

$$D = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$$S = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

So, $P(D \cap S) = 0$ and the events are mutually exclusive.

Or

D occurs when the scores are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6) with totals of 2, 4, 6, 8, 10 and 12 respectively.

S does not happen when D happens, so $P(D \cap S) = 0$ and the events are mutually exclusive.

2 A and B are independent, so $P(A \cap B) = P(A) \times P(B) = 0.6 \times 0.7 = 0.42$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.7 - 0.42 = 0.88$$

3 For Zac to win, both others must not win first and then Zac must win.

$$P(Z) = 0.5 \times (1 - 0.4) \times p = 0.195$$

$$p = \frac{0.195}{0.5 \times 0.6} = 0.65$$

4 (i) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$\text{Rearrange to give } P(A \cap B) = P(A | B) \times P(B)$$

$$= \frac{p}{3} \times 2p$$

$$= \frac{2p^2}{3}$$

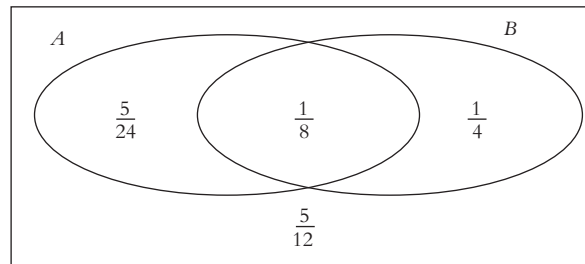
(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= p + 2p - \frac{2p^2}{3}$$

$$= 3p - \frac{2p^2}{3}$$

$$= \frac{p}{3}(9 - 2p)$$

5



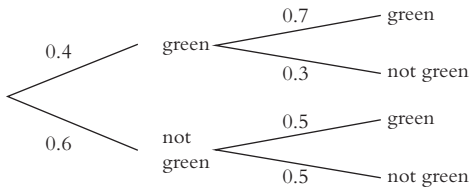
$$P(A) = \frac{5}{24} + \frac{1}{8} = \frac{1}{3} \text{ and } P(B) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

$$P(A \cap B) = \frac{1}{8} \text{ and } P(A) \times P(B) = \frac{1}{3} \times \frac{3}{8} = \frac{1}{8}$$

$P(A) \times P(B) = P(A \cap B)$, so the events are independent.

$P(A \cap B) = \frac{1}{8} \neq 0$, so the events are not mutually exclusive.

6



$$P(\text{green at the second set}) = 0.4 \times 0.7 + 0.6 \times 0.5 = 0.58$$

7 Complete rows and columns which already have two figures:

	French	German	Total
Male			170
Female		83	150
Total	165	155	320

Complete the table:

	French	German	Total
Male	98	72	170
Female	67	83	150
Total	165	155	320

There are 165 students studying French.

$$P(\text{French student being male}) = \frac{98}{165} = 0.5939$$

There are 170 male students.

$$P(\text{male student studying French}) = \frac{98}{170} = 0.5765$$

$0.5939 > 0.5765$, so French student being male is more likely.

8 B and C are independent.

$$P(B \cap C) = P(B) \times P(C)$$

$$= \frac{1}{2} \times \frac{1}{3}$$

$$= \frac{1}{6}$$

$$P(B \cap C') = P(B \text{ only})$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$= \frac{1}{3}$$

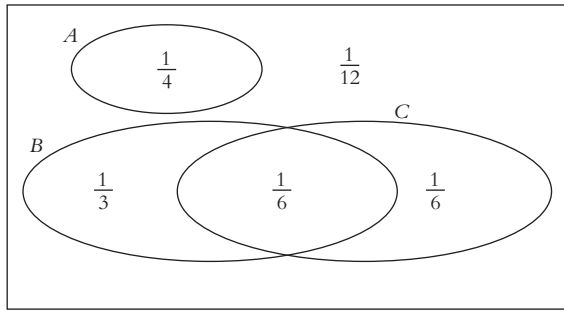
$$P(B' \cap C) = P(C \text{ only})$$

$$= \frac{1}{3} - \frac{1}{6}$$

$$= \frac{1}{6}$$

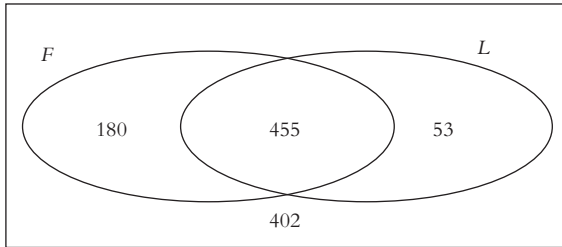
A does not overlap B or C .

$$\text{The border of the Venn diagram is } P(A \cup B \cup C)' = \frac{1}{12}$$



$$P(A) = 1 - \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12} \right) = \frac{1}{4}$$

9 (i)



$$(ii)(a) P(F^c) = \frac{402 + 53}{1090} = \frac{455}{1090}$$

$$(b) P(F \cap L) = \frac{455}{1090}$$

$$(c) P(F|L) = \frac{455}{455 + 53} = \frac{455}{508}$$

$$(d) P(L|F) = \frac{455}{635}$$

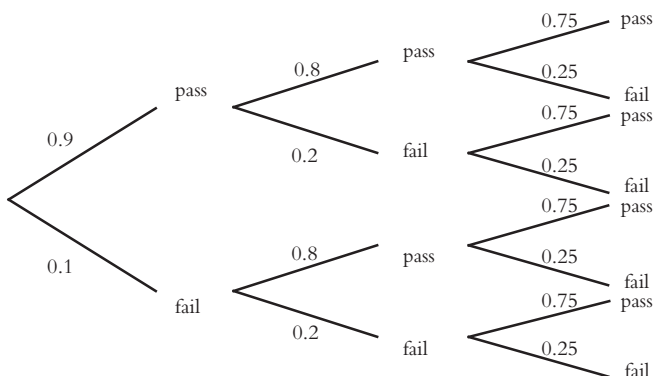
(ii) If the events are independent, $P(F \cap L) = P(F) \times P(L)$.

$$P(F \cap L) = \frac{455}{1090} \\ = 0.4174$$

$$P(F) \times P(L) = \frac{635}{1090} \times \frac{455 + 53}{1090} \\ = 0.2715$$

So, $P(F \cap L) \neq P(F) \times P(L)$, so the events are not independent.

10



$$(i) P(2 \text{ passes}) = 0.9 \times 0.8 \times 0.25 + 0.9 \times 0.2 \times 0.75 + 0.1 \times 0.8 \times 0.75 \\ = 0.18 + 0.135 + 0.06 = 0.375$$

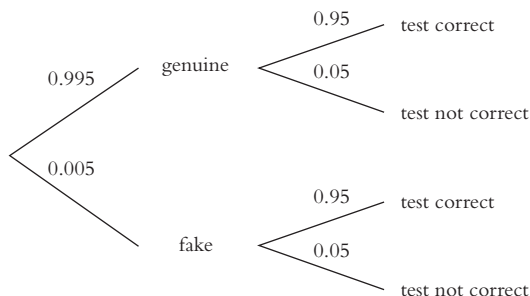
$$(ii) P(3 \text{ passes}) = 0.9 \times 0.8 \times 0.75 = 0.54$$

$$P(\text{success}) = P(2 \text{ or } 3 \text{ passes}) = 0.54 + 0.375 = 0.915$$

$$\text{Expected number} = np = 0.915 \times 50 = 45.75$$

$$\begin{aligned}
 \text{(iii)} \quad P(3 \text{ passes} | \text{success}) &= \frac{P(\text{success with 3 passes})}{P(\text{success})} \\
 &= \frac{0.54}{0.915} \\
 &= \frac{36}{61}
 \end{aligned}$$

11 (i)

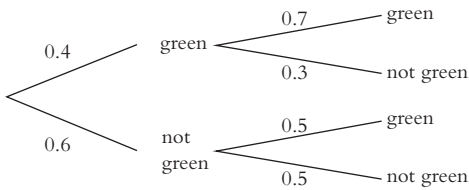


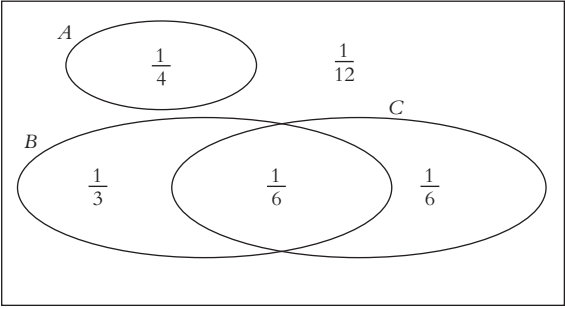
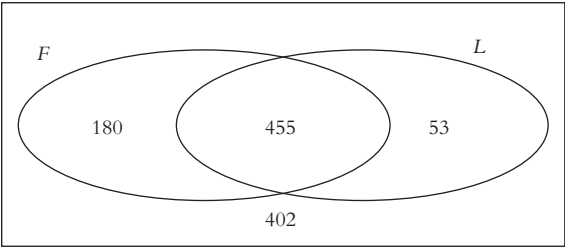
$$\begin{aligned}
 \text{(ii)} \quad P(\text{recognised as fake}) &= P(\text{genuine and test not correct or fake and test correct}) \\
 &= 0.995 \times 0.05 + 0.005 \times 0.95 = 0.0545
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad P(\text{fake} | \text{recognised as fake}) &= \frac{P(\text{fake and recognised as fake})}{P(\text{recognised as fake})} \\
 &= \frac{0.005 \times 0.95}{0.0545} \\
 &= \frac{19}{218}
 \end{aligned}$$

Mark scheme

	Answer	Marks	Guidance
1	Either $D = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$ $S = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ So, $P(D \cap S) = 0$ and the events are mutually exclusive Or D occurs when the scores are (1,1), (2,2), (3,3), (4,4), (5,5) and (6,6) with totals of 2, 4, 6, 8, 10 and 12 respectively S does not happen when D happens, so $P(D \cap S) = 0$ and the events are mutually exclusive	M1 E1 M1 E1	Finds lists for each set and shows no overlap Clear explanation Attempts to show that all the elements of one set do not belong to the other Allow for any argument which shows that every element of one set cannot be in the other
2	A and B are independent so, $P(A \cap B) = P(A) \times P(B)$ $= 0.6 \times 0.7$ $= 0.42$ $P(A \cup B) = 0.6 + 0.7 - 0.42$ $= 0.88$	M1 M1 A1	Uses product of probabilities Uses addition rule

	Answer	Marks	Guidance																
3	$P(Z) = 0.5 \times 0.6 \times p = 0.195$ $p = \frac{0.195}{0.5 \times 0.6}$ $= 0.65$	M1 A1 A1	Product of three probabilities = 0.195 0.6 used (may be implied) cao																
4 (i)	$P(A \cap B) = P(A B) \times P(B)$ $\frac{p}{3} \times 2p$ $= \frac{2p^2}{3}$	M1 A1	Uses the formula for conditional probability cao																
(ii)	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $p + 2p - \frac{2p^2}{3} = 3p - \frac{2p^2}{3}$ $= \frac{p}{3}(9 - 2p)$	M1 A1	Uses the addition formula Any form																
5	$P(A) = \frac{5}{24} + \frac{1}{8} = \frac{1}{3}$ $P(B) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$ $P(A \cap B) = \frac{1}{8}$ and $P(A) \times P(B) = \frac{1}{3} \times \frac{3}{8} = \frac{1}{8}$ $P(A) \times P(B) = P(A \cap B)$, so the events are independent $P(A \cap B) = \frac{1}{8} \neq 0$, so the events are not mutually exclusive	M1 M1 E1 B1	Finds either $P(A)$ or $P(B)$ Finds the product of their $P(A)$ and $P(B)$ Clear statement that these are equal Clear statement that probability is not zero, or should be zero for mutually exclusive																
6	 $P(\text{green at the second set}) = 0.4 \times 0.7 + 0.6 \times 0.5 = 0.58$	M1 M1 M1 A1	First branches correct 0.3 and 0.5 correct branches Selects correct branches cao																
7	<table border="1" data-bbox="304 1563 668 1709"> <thead> <tr> <th></th><th>French</th><th>German</th><th>Total</th></tr> </thead> <tbody> <tr> <th>Male</th><td>98</td><td>72</td><td>170</td></tr> <tr> <th>Female</th><td>67</td><td>83</td><td>150</td></tr> <tr> <th>Total</th><td>165</td><td>155</td><td>320</td></tr> </tbody> </table> $P(\text{French student being male}) = \frac{98}{165} = 0.5939$ $P(\text{male student studying French}) = \frac{98}{170} = 0.5765$ $0.5939 > 0.5765$, so French student being male is more likely		French	German	Total	Male	98	72	170	Female	67	83	150	Total	165	155	320	M1 A1 B1 B1 E1	Attempts to complete the table 98 male students allow SC2 for 98 seen without working ft their 98 if clear ft their 98 States conclusion from seen probabilities; ft their values
	French	German	Total																
Male	98	72	170																
Female	67	83	150																
Total	165	155	320																

	Answer	Marks	Guidance
8	$P(B \cap C) = P(B) \times P(C) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ $P(B \cap C') = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$ $P(B' \cap C) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$	B1 M1	Uses definition of independent Attempts to find probabilities equal to $P(B \text{ only})$ or $P(C \text{ only})$ to put on the Venn diagram
	 $P(A) = 1 - \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12} \right)$ $= \frac{1}{4}$	B1 B1 M1 A1	$\frac{1}{12}$ correctly put onto the diagram A not overlapping the other two sets Allow overlapping if all relevant areas marked with probability of zero Uses total probability = 1; ft their values provided $0 < P(A) < 1$
9 (i)		B1 B1 B1	455 and 53 correctly placed For 180 For 402; ft their 180 (total 1090)
(ii)(a)	$P(F') = \frac{402 + 455}{1090} = \frac{508}{1090}$	M1 A1	Adds 402 and 53 Allow decimal equivalent if to at least 4 d.p.
(b)	$P(F \cap L) = \frac{455}{1090}$	B1	cao
(c)	$P(F L) = \frac{455}{455 + 53} = \frac{455}{508}$	B1 B1	Numerator Denominator
(d)	$P(L F) = \frac{455}{635}$	B1 B1	Numerator Denominator
(iii)	If the events are independent, $P(F \cap L) = P(F) \times P(L)$ $P(F \cap L) = \frac{455}{1090}$ $= 0.4174$ $P(F) \times P(L) = \frac{635}{1090} \times \frac{455 + 53}{1090}$ $= 0.2715$ So, $P(F \cap L) \neq P(F) \times P(L)$, so the events are not independent	M1 M1 E1	Finds probability of the intersection Finds the product of two probabilities Clear statement

	Answer	Marks	Guidance
10 (i)	$0.9 \times 0.8 \times 0.25$ $+ 0.9 \times 0.2 \times 0.75$ $+ 0.1 \times 0.8 \times 0.75$ $= 0.18 + 0.135 + 0.06$ $= 0.375$	M1 A1 A1	Also allow for completing tree diagram and selecting relevant branches At least two correct terms cao
(ii)	$P(3 \text{ passes}) = 0.9 \times 0.8 \times 0.75 = 0.54$ $P(\text{success}) = 0.54 + 0.375 = 0.915$ Expected number = np $= 0.915 \times 50 = 45.75$	B1 M1 M1 A1	ft their part (i) ft their probability Do not award if rounded to integer
(iii)	$P(3 \text{ passes} \text{success}) = \frac{0.54}{0.915} = \frac{36}{61}$	B1 B1	Numerator Denominator
11 (i)		B1 B1	0.995 and 0.005 oe For 0.05 twice
(ii)	$0.995 \times 0.05 + 0.005 \times 0.95$ $= 0.0545$	M1 M1 A1	Selects correct branches Multiplies and then adds from their branches
(iii)	$= \frac{0.005 \times 0.95}{0.0545}$ $= \frac{19}{218}$	M1 M1 A1	Probability of the intersection Uses conditional probability formula cao

16 Statistical distributions

1 (i) standard deviation = $\sqrt{\text{variance}} = \sqrt{10}$

(ii) (a) $P(X > 85) = P\left(Z > \frac{85 - 75}{\sqrt{10}}\right)$
 $= P(Z > 3.16227766)$
 $= 0.0007827$

(b) $P(X < 80) = P\left(Z < \frac{80 - 75}{\sqrt{10}}\right)$
 $= P(Z < 1.58113883)$
 $= 0.9431$

(c) $P(70 < X < 78) = P\left(\frac{70 - 75}{\sqrt{10}} < Z < \frac{78 - 75}{\sqrt{10}}\right)$
 $= P(-1.5811388 < Z < 0.94868329)$
 $= 0.7717$

- 2 Let X be the weight of soap powder in a box.

$$\begin{aligned} \text{(i)} \quad P(X > 1005) &= P\left(Z > \frac{1005 - 1002}{1.2}\right) \\ &= P(Z > 2.5) \\ &= 0.006210 \end{aligned}$$

- (ii) Box is underweight if $X < 1000$

$$\begin{aligned} P(X < 1000) &= P\left(Z < \frac{1000 - 1002}{1.2}\right) \\ &= P(Z < -1.666667) \\ &= 0.04779 \end{aligned}$$

- 3 $P(X > k) = 0.7$

$$\begin{aligned} P\left(Z > \frac{k - 45}{10}\right) &= 0.7 \\ \frac{k - 45}{10} &= -0.5244005 \\ k &= -0.5244005 \times 10 + 45 \\ k &= 39.756 \\ k &= 39.8 \text{ (3 s.f.)} \end{aligned}$$

- 4 Let X be the speed of a car $X \sim N(68, 5^2)$.
Let k be the speed at which the police issue fines.

$$\begin{aligned} P(X > k) &= 0.08 \\ P\left(Z > \frac{k - 68}{5}\right) &= 0.08 \\ \frac{k - 68}{5} &= 1.40507156 \\ k &= 68 + 5 \times 1.40507156 \\ k &= 75.025 \end{aligned}$$

- 5 $P(X < 100) = 0.25$

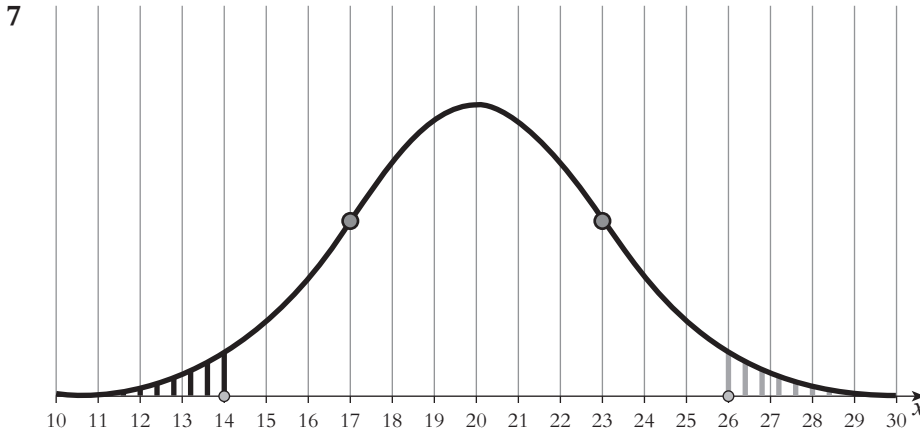
$$\begin{aligned} P\left(Z < \frac{100 - \mu}{17}\right) &= 0.25 \\ \frac{100 - \mu}{17} &= -0.6744897 \\ 100 - \mu &= -0.6744897 \times 17 \\ \mu &= 100 + 0.6744897 \times 17 \\ \mu &= 111.5 \text{ (4 s.f.)} \end{aligned}$$

- 6 $P(X > 36) = 0.4$

$$P\left(Z > \frac{36 - 35}{\sigma}\right) = 0.4$$

Using inverse Normal facility on the calculator gives:

$$\begin{aligned} \frac{36 - 35}{\sigma} &= 0.2533471 \\ \sigma &= \frac{1}{0.2533471} \\ &= 3.947 \text{ (4 s.f.)} \end{aligned}$$



Bell-shaped curve with the maximum point where $X = 20$.

Points of inflection are one standard deviation from the mean at 17 and 23.

Outliers are more than two standard deviations from the mean, so $X < 14$ and $X > 26$.

- 8 (i) Y has a Normal distribution.

Using $Y = 3X - 1$:

$$\begin{aligned}\bar{Y} &= 3\bar{X} - 1 \\ &= 3 \times 5 - 1 \\ &= 14\end{aligned}$$

$$\begin{aligned}\sigma_Y &= 3\sigma_X \\ &= 3 \times 1.2 \\ &= 3.6\end{aligned}$$

So, $Y \sim N(14, 3.6^2)$

$$\begin{aligned}\text{(ii) } P(Y > 10) &= P\left(Z > \frac{10 - 14}{3.6}\right) \\ &= P\left(Z > -\frac{10}{9}\right) \\ &= 0.8667\end{aligned}$$

- 9 Let X be the distance the child throws.

$$X \sim N(4.8, 0.45^2)$$

(If your calculator has the facility, you can just give the answers here; you could quote the z -value(s) also.)

$$\begin{aligned}\text{For a gold star, } P(X > 5.4) &= P\left(Z > \frac{5.4 - 4.8}{0.45}\right) \\ &= P\left(Z > \frac{4}{3}\right) \\ &= 0.09121121\end{aligned}$$

$$\begin{aligned}\text{Number of children with gold star} &= np \\ &= 30 \times 0.09121121 \\ &= 2.7 \text{ children}\end{aligned}$$

So, three gold stars are estimated to be awarded.

$$\begin{aligned}\text{For a silver star, } P(4 < X < 5.4) &= P\left(\frac{4 - 4.8}{0.45} < Z < \frac{5.4 - 4.8}{0.45}\right) \\ &= P\left(\frac{16}{9} < Z < \frac{4}{3}\right) = 0.96227982 - 0.09121121 \\ &= 0.87106861\end{aligned}$$

$$\begin{aligned}
 \text{Number of children with silver star} &= np \\
 &= 30 \times 0.8710686 \\
 &= 26.1 \text{ children}
 \end{aligned}$$

So, 26 silver stars are estimated to be awarded.

For 'I tried' sticker, $30 - 2.7 - 26.1 = 1.2$ children.

So, one 'I tried' sticker is estimated to be awarded.

10 Let X be the mass of a lettuce.

$$\begin{aligned}
 \text{(i)} \quad P(X > 595) &= P\left(Z > \frac{595 - 560}{20}\right) \\
 &= P(Z > 1.75) \\
 &= 0.04006
 \end{aligned}$$

$$X \sim N(560, 20^2)$$

$$\begin{aligned}
 \text{(ii)} \quad P(\text{all three more than 595 g}) &= 0.04005915^3 \\
 &= 6.428 \times 10^{-5}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad P(X < 530) &= P\left(Z < \frac{530 - 560}{20}\right) \\
 &= P(Z < -1.5) \\
 &= 0.0668072
 \end{aligned}$$

$$\begin{aligned}
 \text{Expected number in a box of 24} &= np \\
 &= 24 \times 0.0668072 \\
 &= 1.6 \text{ (2 s.f.)}
 \end{aligned}$$

11 (i) Substitute $\bar{x} = 108$ into the formula $\bar{w} = 1.1\bar{x} - 100$

$$\begin{aligned}
 \bar{w} &= 1.1 \times 108 - 100 \\
 &= 18.8 \text{ kg}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_w &= 1.1\sigma_x \\
 &= 1.1 \times 2.5 \\
 &= 2.75 \text{ kg}
 \end{aligned}$$

(ii) Let W be the weight of a child.

$$\begin{aligned}
 P(W < k) &= 0.9 \\
 P\left(Z < \frac{k - 18.8}{2.75}\right) &= 0.9 \\
 \frac{k - 18.8}{2.75} &= 1.2815157 \\
 \frac{k - 18.8}{2.75} &= 1.2815157 \\
 k &= 1.2815157 \times 2.75 + 18.8 \\
 k &= 22.32 \text{ kg}
 \end{aligned}$$

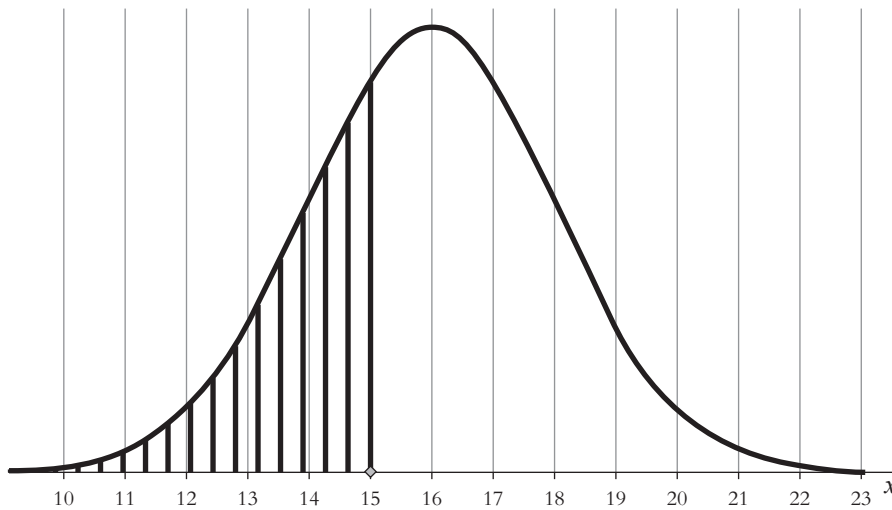
12 (i) Most of the population lies within three standard deviations from the mean, so the range is approximately six standard deviations.

$$22 - 10 = 12, \text{ so standard deviation is } \frac{12}{6} = 2 \text{ cm.}$$

(ii) A Normal distribution is symmetric, so the midrange is the same as the mean.

$$\mu = \frac{10 + 22}{2} = 16$$

(iii)

(iv) Let X be the length of a leaf.

$$\begin{aligned}
 P(X < 15) &= P\left(Z < \frac{15 - 16}{2}\right) \\
 &= P(Z < -0.5) \\
 &= 0.3085 \text{ (4 d.p.)}
 \end{aligned}$$

13 (i) Let X be the length of a piece of cable.

$$X \sim N(\mu, 0.5^2)$$

$$P(X > 200) = 0.95$$

$$P\left(Z > \frac{200 - \mu}{0.5}\right) = 0.95$$

$$\frac{200 - \mu}{0.5} = -1.6448536$$

$$\mu = 200 + 0.5 \times 1.6448536$$

$$\mu = 200.822 \text{ cm}$$

(ii) $X \sim N(200.822, 0.5^2)$ $P(X > 202) = 0.0092365$ using calculator.If only $N(0,1)$ on calculator:

$$P(X > 202) = P\left(Z > \frac{202 - 200.822}{0.5}\right)$$

$$= P(Z > 2.355)$$

$$= 0.0092365$$

Expected number = np

$$= 100 \times 0.0092365$$

$$= 0.92365$$

$$= 0.92 \text{ (2 d.p.)}$$

The answer is not rounded because the expected number is a statistic.

14 (i) Let X be the life of the tyre.

$$X \sim N(28500, \sigma^2)$$

$$P(X > 23000) = 0.90$$

$$P\left(Z > \frac{23000 - 28500}{\sigma}\right) = 0.90$$

$$\frac{-5500}{\sigma} = -1.2815516$$

$$\sigma = 4292 \text{ km}$$

$$\begin{aligned}
 \text{(ii)} \quad P(X > 35000) &= P\left(Z > \frac{35000 - 28500}{4291.673}\right) \\
 &= P(Z > 1.514561) \\
 &= 0.0649
 \end{aligned}$$

15 (i) Let X be the weight of an egg.

50% of eggs are over 63 g (argument from symmetry), so the mean $\mu = 63$ g

25% of eggs are more than 73 g

$$P(X > 73) = 0.25$$

$$P\left(Z > \frac{73 - 63}{\sigma}\right) = 0.25$$

$$\frac{10}{\sigma} = 0.67448975$$

$$\sigma = \frac{10}{0.67448975}$$

$$\sigma = 14.83 \text{ g}$$

(ii) Outliers are more than two standard deviations from the mean.

$$\mu + 2\sigma = 63 + 2 \times 14.83 = 92.66$$

$$\mu - 2\sigma = 63 - 2 \times 14.83 = 33.34$$

Ranges $X > 92.66$ or $X < 33.34$

16 Let X be the mark obtained.

(i) She uses Grade 9 for more than 248 marks.

Examination marks are discrete data, so a continuity correction is applied.

$$P(X > 248.5) = 0.03$$

$$P\left(Z > \frac{248.5 - \mu}{\sigma}\right) = 0.03$$

$$\frac{248.5 - \mu}{\sigma} = 1.8807361$$

$$248.5 = \mu + 1.881\sigma \quad (4 \text{ s.f.})$$

(ii) Grade 7 for more than 159 marks.

Examination marks are discrete data so a continuity correction is applied.

$$P(X > 159.5) = 0.20$$

$$P\left(Z > \frac{159.5 - \mu}{\sigma}\right) = 0.20$$

$$\frac{159.5 - \mu}{\sigma} = 0.84162123$$

$$159.5 = \mu + 0.8416\sigma \quad (4 \text{ s.f.})$$

(iii) Solve the simultaneous equations:

$$248.5 = \mu + 1.881\sigma$$

$$159.5 = \mu + 0.8416\sigma$$

Gives $\mu = 87.44$ and $\sigma = 85.63$ (4 s.f.)

(iv) If $X \sim N(87.44, 85.63^2)$

Using the Normal facility on the calculator:

$$P(X < 0) = 0.1623287, \text{ which is about } 16\%$$

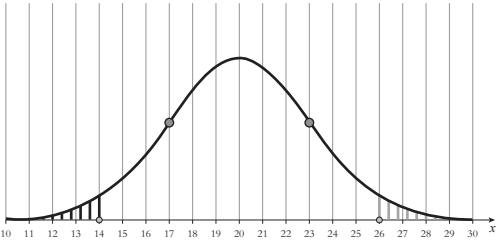
Or, if only $N(0, 1)$ available on the calculator:

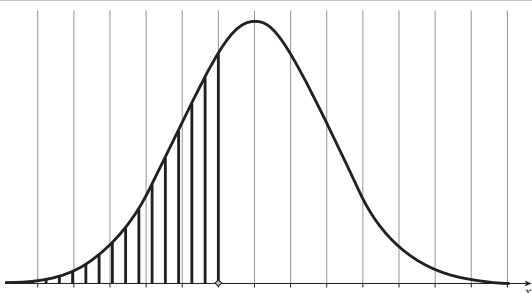
$$\begin{aligned}
 P\left(Z < \frac{0 - 87.44}{85.63}\right) &= P(Z < -1.0211) \\
 &= 0.15359467 \\
 &\approx 15\%
 \end{aligned}$$

- (v) This is impossible as all the marks will be positive, so the data does not seem to fit a Normal distribution which is the teacher's incorrect assumption.

Mark scheme

	Answer	Marks	Guidance
1 (i)	$\sqrt{10}$	B1	
(ii)(a)	$P(X > 85) = P\left(Z > \frac{85 - 75}{\sqrt{10}}\right)$ $= P(Z > 3.16227766) = 0.0007827$	M1 A1	Uses Z or the calculator facility Allow www
(b)	$P(X < 80) = P\left(Z < \frac{80 - 75}{\sqrt{10}}\right)$ $= P(Z < 1.58113883) = 0.9431$	M1 A1	Uses Z or the calculator facility Allow www
(c)	$P(70 < X < 78)$ $= P\left(\frac{70 - 75}{\sqrt{10}} < Z < \frac{78 - 75}{\sqrt{10}}\right)$ $= P(-1.5811388 < Z < 0.94868329)$ $= 0.7717$	M1 A1	Uses Z or the calculator facility Allow www
2 (i)	$P(X > 1005) = P\left(Z > \frac{1005 - 1002}{1.2}\right)$ $= P(Z > 2.5) = 0.006210$	M1 A1	Uses Z or the calculator facility Allow www
(ii)	$P(X < 1000) = P\left(Z < \frac{1000 - 1002}{1.2}\right)$ $= P(Z < -1.666667) = 0.04779$	M1 A1	Uses Z or the calculator facility Allow www
3	$P\left(Z > \frac{k - 45}{10}\right) = 0.7$ $\frac{k - 45}{10} = -0.5244005$ $k = -0.5244005 \times 10 + 45$ $= 39.756 = 39.8 \text{ (3 s.f.)}$	M1 M1 B1 A1	Uses Z Uses inverse Normal facility Correct boundary value seen Must be to at least 3 s.f. Allow trial and improvement only if the answer is shown to be correct to 3 s.f.
4	$P(X > k) = 0.08$ $P\left(Z > \frac{k - 68}{5}\right) = 0.08$ $\frac{k - 68}{5} = 1.40507156$ $k = 68 + 5 \times 1.40507156$ $= 75.025$	M1 B1 M1 A1	Uses Z Correct boundary value seen Uses boundary in an equation for k Must be to at least 2 s.f. Allow trial and improvement only if the answer is shown to be correct to at least 2 s.f.

	Answer	Marks	Guidance
5	$P\left(Z < \frac{100 - \mu}{17}\right) = 0.25$ $\frac{100 - \mu}{17} = -0.6744897$ $100 - \mu = -0.6744897 \times 17$ $\mu = 100 + 0.6744897 \times 17$ $= 111.5 \text{ (4 s.f.)}$	M1 B1 M1 A1	Uses Z Correct boundary value (must be negative) Attempt to solve an equation for μ Answer to 4 s.f. Allow mark for trial and improvement if the answer is correct to 4 s.f. Allow sc for trial and improvement leading to a correct value to 2 or 3 s.f.
6	$P\left(Z > \frac{36 - 35}{\sigma}\right) = 0.4$ $\frac{36 - 35}{\sigma} = 0.2533471$ $\sigma = \frac{1}{0.2533471} = 3.947 \text{ (4 s.f.)}$	M1 M1 B1 A1	Uses Z Forms an equation with σ and their positive boundary value 0.2533 or better seen Answer to 4 s.f. Allow mark for trial and improvement if the answer is correct to 4 s.f. Allow sc for trial and improvement leading to a correct value to 2 or 3 s.f.
7		B1 B1 B1 B1	Correct shape Mean labelled 20 Points of inflection at 17 and 23 Shaded regions $X > 26$ and $X < 14$
8 (i)	$Y \text{ has a Normal distribution}$ $\text{Using } Y = 3X - 1$ $\bar{Y} = 3\bar{X} - 1 = 3 \times 5 - 1 = 14$ $\sigma_Y = 3\sigma_X = 3 \times 1.2 = 3.6$ $\text{So, } Y \sim N(14, 3.6^2)$	B1 B1 B1	Normal distribution stated
(ii)	$P(Y > 10) = P\left(Z > \frac{10 - 14}{3.6}\right)$ $= P\left(Z > -\frac{10}{9}\right) = 0.8667$	M1 A1	Uses Z or calculator facility Accept 4 s.f. or better
9	$\text{For a gold star } P(X > 5.4)$ $= P\left(Z > \frac{5.4 - 4.8}{0.45}\right)$ $= P\left(Z > \frac{4}{3}\right) = 0.09121121$	M1	Uses Z or calculator facility

	Answer	Marks	Guidance
	Number of children with gold star = $np = 30 \times 0.09121121 = 3$ children For a silver star $P(4 < X < 5.4)$ $= P\left(\frac{4 - 4.8}{0.45} < Z < \frac{5.4 - 4.8}{0.45}\right)$ $= P\left(\frac{50}{27} < Z < \frac{4}{3}\right)$ $= 0.96227982 - 0.09121121 = 0.87106861$ Number of children with silver star = $np = 30 \times 0.8710686 = 26$ children For 'I tried' sticker = $30 - 2.7 - 26.1 = 1$ child	A1 A1 M1 A1 B1	Allow www ft their probability Probability between values Allow for below 4 if done in a different order ft their other two answers (total 30)
10 (i)	$P(X > 595) = P\left(Z > \frac{595 - 560}{20}\right)$ $= P(Z > 1.75) = 0.04006$	M1 A1	Uses Z or calculator facility To at least 4 d.p.
(ii)	0.04005915^3 $= 6.428 \times 10^{-5}$	M1 A1	ft their answer from part (i)
(iii)	$P(X < 530) = P\left(Z < \frac{530 - 560}{20}\right)$ $= P(Z < -1.5) = 0.0668072$ Expected number in a box of 24 = $np = 24 \times 0.0668072 = 1.6$ (2 s.f.)	M1 A1 A1	Uses Z or calculator facility To at least 4 d.p. Do not award if rounded to integer
11 (i)	Substitute $\bar{x} = 108$ into the formula $\bar{w} = 1.1\bar{x} - 100$ $\bar{w} = 1.1 \times 108 - 100 = 18.8$ kg $\sigma_w = 1.1\sigma_x = 1.1 \times 2.5 = 2.75$ kg	M1 E1 A1	Method clear Answer given
(ii)	Let W be the weight of a child $P(W < k) = 0.9$ $P\left(Z < \frac{k - 18.8}{2.75}\right) = 0.9$ $\frac{k - 18.8}{2.75} = 1.2815157$ $k = 1.2815157 \times 2.75 + 18.8$ $= 22.3$ kg (1 d.p.)	M1 B1 A1	Attempts to solve an equation linking k and a boundary value 1.28 or better seen Must be to 1 d.p.
12 (i)	Range ≈ 6 standard deviations $\frac{22 - 10}{6} = 2$	M1 A1	Clear explanation Answer given
(ii)	$\mu = \frac{10 + 22}{2} = 16$	B1	
(iii)		B1 B1 B1	Correct shape with mean at about 16 Minimum and Maximum at 10 and 22 Shaded area

	Answer	Marks	Guidance
(iv)	$P(X < 15) = P\left(Z < \frac{15 - 16}{2}\right)$ $= P(Z < -0.5) = 0.3085$ Probability 0.31 (2 s.f.)	M1 A1	Uses Z or parameters (may be implied) Must be to 2 s.f.
13 (i)	$P(X > 200) = 0.95$ $P\left(Z > \frac{200 - \mu}{0.5}\right) = 0.95$ $\frac{200 - \mu}{0.5} = -1.6448536$ $\mu = 200 + 0.5 \times 1.6448536 = 200.822 \text{ cm}$	M1 B1 A1 A1	Uses Z Correct equation including signs 1.645 or better seen Must be to 3 d.p.
(ii)	$P(X > 202) = P\left(Z > \frac{202 - 200.822}{0.5}\right)$ $= P(Z > 2.355) = 0.0092365$ Expected number = $np = 100 \times 0.0092365 = 0.92365$	M1 A1 A1	Uses their mean and given standard deviation Do not award if rounded to a whole number
14 (i)	$P(X > 23000) = 0.90$ $P\left(Z > \frac{23000 - 28500}{\sigma}\right) = 0.90$ $\frac{-5500}{\sigma} = -1.2815516$ $\sigma = 4292 \text{ km}$	M1 B1 A1 A1	Uses Z 1.28 or better seen Boundary sign correct At least 3 s.f.
(ii)	$P(X > 35000) = P\left(Z > \frac{35000 - 28500}{4291.673}\right)$ $= P(Z > 1.514561) = 0.0649$	M1 A1	Z need not be seen At least 3 d.p.
15 (i)	By symmetry $\mu = 63$ $P(X > 73) = 0.25$ $P\left(Z > \frac{73 - 63}{\sigma}\right) = 0.25$ $\frac{10}{\sigma} = 0.67448975$ $\sigma = \frac{10}{0.67448975} = 14.83 \text{ g}$	B1 M1 M1 A1	Allow without evidence Allow for 53 g used Uses Z At least 3 s.f.
(ii)	Outliers are more than two standard deviations from the mean $\mu + 2\sigma = 63 + 2 \times 14.83 = 92.66$ $\mu - 2\sigma = 63 - 2 \times 14.83 = 33.34$ Ranges $X > 92.66$ or $X < 33.34$	B1 B1 B1	Award if seen Award if seen Both correct inequalities; ft their values if consistent
16 (i)	$P(X > 248.5) = 0.03$ $P\left(Z > \frac{248.5 - \mu}{\sigma}\right) = 0.03$ $\frac{248.5 - \mu}{\sigma} = 1.8807361$ Rounding to 4 s.f gives $248.5 = \mu + 1.881\sigma$	M1 M1 E1	Uses the Grade 9 information Uses Z Answer given

	Answer	Marks	Guidance
(ii)	Grade 7 for more than 159 marks $P(X > 159.5) = 0.20$ $P\left(Z > \frac{159.5 - \mu}{\sigma}\right) = 0.2$ $\frac{159.5 - \mu}{\sigma} = 0.84162123$ Rounding to 4 s.f. gives $159.5 = \mu + 0.8416\sigma$	M1 A1	Uses Z for the Grade 7 information To at least 4 s.f.
(iii)	Solve the simultaneous equations $248.5 = \mu + 1.881\sigma$ $159.5 = \mu + 0.8416\sigma$ Gives $\mu = 87.44$ and $\sigma = 85.63$ (4 s.f.)	M1 A1	Calculator method acceptable
(iv)	If $X \sim N(87.44, 85.63^2)$ Using the Normal facility on the calculator: $P(X < 0) = 0.1623287$ Or If only $N(0, 1)$ available on the calculator: $P\left(Z < \frac{0 - 87.44}{85.63}\right) = P(Z < -1.0211)$ $= 0.15359467$, which is about 15%	M1 A1	Uses Normal distribution To more than 2 d.p.
(v)	This is impossible as all the marks will be positive, so the data does not seem to fit a Normal distribution which is the teacher's incorrect assumption	M1 E1	Negative numbers impossible Distribution not Normal

17 Hypothesis testing

- The points do not suggest a linear relationship.
- Rank correlation is always linked to association and not correlation.
- The points are close to a curve.
 - H_0 : There is no association between BSA and weight.
 H_1 : There is some association between BSA and weight.
The actual value for rank correlation is 0.9863.
 $0.9863 > 0.3977$ (the critical value) so is more extreme.
So, there is enough evidence at the 5% level to reject the null hypothesis in favour of the alternative hypothesis.
The evidence supports the alternative hypothesis that there is association between BSA and weight.
- H_0 : There is no association between age and time spent watching television.
 H_1 : There is some association between age and time spent watching television.
 - Actual value $-0.2727 > -0.5874$, so is less extreme.
So, there is not enough evidence at the 5% level to reject the null hypothesis in favour of the alternative hypothesis.
The evidence supports the null hypothesis that there is no association between age and time spent watching television.

- 5 (i) The ages are all integers, so there are no points between the integer values.
 (ii) The graph shows a non-linear relationship.
 (iii) H_0 : There is no association between height and age of the tree.
 H_1 : There is positive association between height and age of the tree.
 Actual value $0.8603 > 0.3139$, so is more extreme.
 So, there is enough evidence at the 1% significance level to reject the null hypothesis in favour of the alternative hypothesis.
 The evidence supports the alternative hypothesis that there is positive association between age and height of the trees.
- 6 (i) Weekdays and weekends
 (ii) $H_0 : \rho = 0$ There is no correlation between ice cream sales and maximum air temperature.
 $H_1 : \rho \neq 0$ There is some correlation between ice cream sales and maximum air temperature.
 Actual value $0.3814 > 0.3610$, so is more extreme.
 So, there is enough evidence at the 5% level to reject the null hypothesis in favour of the alternative hypothesis.
 The evidence supports the alternative hypothesis that there is correlation between ice cream sales and maximum air temperature.
 (iii) Maria uses association and not correlation which is not correct when correlation coefficient is used.
 Maria uses the word proof; however, she has made an *inference*, not a proof.
- 7 $H_1 : \mu \neq 14$ 2-tail
 $H_1 : \mu < 14$ 1-tail
 $H_1 : \mu > 14$ 1-tail
- 8 Normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$
 You could also write $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
- 9 (i) Normal with mean 50 and standard deviation $\frac{2}{\sqrt{10}}$
 which can be written $\bar{X} \sim N\left(50, \frac{2^2}{10}\right) = N(50, 0.4)$
- (ii) (a) Critical region is $\bar{X} > k_1$ where k_1 is the value for which $P(\bar{X} > k_1) = 10\%$
- $$P(\bar{X} > k_1) = P\left(Z > \frac{k_1 - 50}{\sqrt{0.4}}\right)$$
- $$= 10\%$$
- $$\frac{k_1 - 50}{\sqrt{0.4}} = 1.2815516$$
- $$k_1 = 50 + 1.2815516 \times \sqrt{0.4}$$
- $$= 50.8105244$$
- Critical region is $\bar{X} > 50.8105244$

- (b) Critical region is $\bar{X} < k_2$, where k_2 is the value for which $P(\bar{X} < k_2) = 10\%$

$$P(\bar{X} < k_2) = P\left(Z < \frac{k_2 - 50}{\sqrt{0.4}}\right)$$

$$= 10\%$$

$$\frac{k_2 - 50}{\sqrt{0.4}} = -1.2815516$$

$$k_2 = 50 - 1.2815516 \times \sqrt{0.4}$$

$$= 49.1894756$$

Critical region is $\bar{X} < 49.1894756$

- (c) Either $\bar{X} < k_3$ or $\bar{X} > k_4$

$$P(\bar{X} < k_3) = P\left(Z < \frac{k_3 - 50}{\sqrt{0.4}}\right)$$

$$= 5\%$$

$$\frac{k_3 - 50}{\sqrt{0.4}} = -1.6448536$$

$$k_3 = 50 - 1.6448536 \times \sqrt{0.4}$$

$$= 48.9597032$$

$$P(\bar{X} > k_4) = P\left(Z > \frac{k_4 - 50}{\sqrt{0.4}}\right)$$

$$= 5\%$$

$$\frac{k_4 - 50}{\sqrt{0.4}} = 1.6448536$$

$$k_4 = 50 + 1.6448536 \times \sqrt{0.4}$$

$$= 51.0402968$$

So, critical region is $\bar{X} < 48.9597032$ or $\bar{X} > 51.0402968$

$$10 \quad \bar{X} \sim N\left(25, \frac{8^2}{100}\right) = N(25, 0.64)$$

$$(i) \quad P(\bar{X} < 24) = P(Z < -1.25) \\ = 0.10564977$$

$$(ii) \quad P(\bar{X} < 23) = P(Z < -2.5) \\ = 0.0062097$$

$$(iii) \quad P(\bar{X} > 26.5) = P(Z > 1.875) \\ = 0.03039636$$

$$11 \quad (i) \quad \text{Normally distributed with mean 13.5 and standard deviation } \frac{1.5}{\sqrt{5}} = 0.67082$$

$$\bar{X} \sim N\left(13.5, \frac{1.5^2}{5}\right) = N(13.5, 0.45)$$

$$(ii) \quad H_0 : \mu = 13.5$$

$$H_1 : \mu < 13.5$$

$$(iii) \quad P(\bar{X} < 12.5) = P\left(Z < \frac{12.5 - 13.5}{\left(\frac{1.5}{\sqrt{5}}\right)}\right) \\ = P(Z < -1.490712) \\ = 0.0680186$$

(iv) $P(\bar{X} < 12.5) = 0.0680186 > 5\%$

So, there is not enough evidence to reject the null hypothesis.

There is not enough evidence at the 5% level that the sprint times have reduced.

12 (i) $H_0 : \mu = 130$

$H_1 : \mu < 130$

(ii) $\bar{X} \sim N\left(130, \frac{10^2}{20}\right) = N(130, 5)$

Critical region is $\bar{X} < k$, where k is the value for which $P(\bar{X} < k) = 1\%$

$$P(\bar{X} < k) = P\left(Z < \frac{k - 130}{\sqrt{5}}\right) = 1\%$$

$$\frac{k - 130}{\sqrt{5}} = -2.3264379$$

$$k = 130 - 2.3264379 \times \sqrt{5}$$

$$= 124.798$$

Critical region is $\bar{X} < 124.798$

(iii) Actual value 125.5 cm is not in the critical region.

There is not enough evidence to reject H_0 .

There is not enough evidence that growing wheat near a road reduces the height of wheat plants.

13 (i) $H_1 : \mu \neq 17$

(ii) Sample mean $\bar{X} \sim N\left(17, \frac{1^2}{8}\right)$ under H_0

2-tail test, so critical region in two parts,

$\bar{X} < k_1$, where $P(\bar{X} < k_1) = 2.5\%$, and $\bar{X} > k_2$, where $P(\bar{X} > k_2) = 2.5\%$

$$P(\bar{X} < k_1) = P\left(Z < \frac{k_1 - 17}{\left(\frac{1}{\sqrt{8}}\right)}\right) = 2.5\%$$

$$\sqrt{8}(k_1 - 17) = -1.959964$$

$$k_1 = 17 - \frac{1.959964}{\sqrt{8}}$$

$$= 16.30704808$$

$$k_2 = 17 + \frac{1.959964}{\sqrt{8}}$$

$$= 17.69295192$$

So, critical region is $\bar{X} < 16.30704808$ or $\bar{X} > 17.69295192$

(iii) Actual sample mean = $\bar{x}$

$$= \frac{1}{8}(17 + 18 + 19.5 + 15.5 + 16 + 16 + 19 + 18) \\ = 17.375$$

$17.375 < 17.69295192$ and is not in the critical region.

There is not enough evidence to reject H_0 .

There is not enough evidence to suggest that the weights of these goldfinches are significantly different from European goldfinches.

14 (i) $H_0 : \mu = 21$

$H_1 : \mu \neq 21$

(ii) Normal distribution with mean 21 and standard deviation $\frac{3}{\sqrt{10}}$

$$\bar{X} \sim N\left(21, \frac{3^2}{10}\right) = N(21, 0.9)$$

(iii) Either

$$P(\bar{X} > 23) = P\left(Z > \frac{23 - 21}{\left(\frac{3}{\sqrt{10}}\right)}\right) \\ = P(Z > 2.10818511) \\ = 0.01750749 < 2.5\%$$

So, result is significant; reject the null hypothesis.

There is enough evidence at the 5% level to suggest that playing classical music affects concentration span.

Or

The critical region for $\bar{X} \sim N(21, 0.9)$ at 5% level 2-tail test is:

$$\bar{X} < 19.1406149 \text{ or } \bar{X} > 22.8592851$$

Actual value $23 > 22.8592851$, so lies in the critical region; reject the null hypothesis.

There is enough evidence at the 5% level to suggest that playing classical music affects concentration span.

15 $H_0 : \mu = 87.3$

$H_1 : \mu < 87.3$

$$\text{Under } H_0, \bar{X} \sim N\left(87.3, \frac{9^2}{20}\right) = N(87.3, 4.05)$$

Either

$$P(\bar{X} < 85.4) = P\left(Z < \frac{85.4 - 87.3}{\sqrt{4.05}}\right) \\ = P(Z < -0.94411759) \\ = 0.17255478 > 5\%$$

There is not enough evidence to reject the null hypothesis.

There is not enough evidence to suggest that the new golf club has reduced scores.

Or

$$\text{Critical region 1-tail test } \bar{X} < k, \text{ where } P\left(Z < \frac{k - 87.3}{\sqrt{4.05}}\right) = 5\%$$

$$\frac{k - 87.3}{\sqrt{4.05}} = -1.6448536$$

$$k = 87.3 - 1.6448536 \times \sqrt{4.05}$$

$$= 83.98979598$$

Actual value = 85.4 > 83.98979598, so is not in the critical region.

Not enough evidence to reject H_0 .

There is not enough evidence to suggest that the new golf club has reduced scores.

Mark scheme

	Answer	Marks	Guidance
1	Non-linear graph	B1	Allow points near curve, etc.
2	Association	B1	
3 (i)	The points are close to a curve	B1	
(ii)	H_0 : There is no association between BSA and weight	B1	Do not award for correlation
	H_1 : There is some association between BSA and weight	B1	Must be 2-tail
	The actual value for rank correlation is 0.9863 > 0.3977, the critical value, and so is more extreme	M1	Comparison must be explicit
	So, there is enough evidence at the 5% level to reject the null hypothesis in favour of the alternative hypothesis	E1	In terms of null and alternative hypothesis
	The evidence supports the alternative hypothesis that there is association between BSA and weight	E1	Must be in context
4 (i)	H_0 : There is no association between age and time spent watching television	B1	
	H_1 : There is some association between age and time spent watching television	B1	Must be 2-tail
(ii)	Actual value $-0.2727 > -0.5874$ and is less extreme	M1	Comparison must be explicit between two negative numbers; allow for two positive numbers if clear which is more extreme
	So, there is not enough evidence at the 5% level to reject the null hypothesis in favour of the alternative hypothesis	M1	In terms of null and alternative hypothesis
	The evidence supports the null hypothesis that there is no association between age and time spent watching television	E1	Must be in context
5 (i)	Ages all integers	B1	
(ii)	The graph shows a non-linear relationship	B1	Allow 'fits a curve'
(iii)	H_0 : There is no association between height and age of the tree	B1	
	H_1 : There is positive association between height and age of the tree	B1	Must be 1-tail
	Actual value 0.8603 > 0.3139 which is more extreme	M1	Comparison must be explicit

	Answer	Marks	Guidance
	So, there is enough evidence at the 1% significance level to reject the null hypothesis in favour of the alternative hypothesis	E1	In terms of null and alternative hypothesis
	The evidence supports the alternative hypothesis that there is positive association between age and height of the trees	E1	Must be in context
6 (i)	Weekdays and weekends	B1	
(ii)	$H_0: \rho = 0$ There is no correlation between ice cream sales and maximum air temperature $H_1: \rho \neq 0$ There is some correlation between ice cream sales and maximum air temperature Actual value $0.3814 > 0.3610$ and is more extreme So, there is enough evidence at the 5% level to reject the null hypothesis in favour of the alternative hypothesis The evidence supports the alternative hypothesis that there is correlation between ice cream sales and maximum air temperature	B1 B1 M1 E1 E1	Must be 2-tail Comparison must be explicit In terms of null and alternative hypothesis Must be in context
(iii)	Maria uses association and not correlation which is not correct when correlation coefficient is used Maria uses the word proof; however, she has made an <i>inference</i> , not a proof	B1 B1	oe oe
7	$H_1: \mu \neq 14$, 2-tail $H_1: \mu < 14$, 1-tail $H_1: \mu > 14$, 1-tail	B1 B1 B1	cao cao cao
8	Normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$	B1 B1 B1	States distribution Allow $N\mu$ Allow $N\left(\mu, \frac{\sigma^2}{n}\right)$
9 (i)	$\bar{X} \sim N\left(50, \frac{2^2}{10}\right)$	B1 B1	Normal with mean 50 Variance 0.4 or standard deviation $\frac{2}{\sqrt{10}}$
(ii)(a)	$\bar{X} > 50.8105244$	B1 B1	Boundary value seen Correct inequality
(ii)(b)	$\bar{X} < 49.1894756$	B1 B1	Boundary value seen Correct inequality
(ii)(c)	Either $\bar{X} < 48.9597032$ or $\bar{X} > 51.0402968$	B2	B1 for each inequality

	Answer	Marks	Guidance
10 (i)	$\bar{X} \sim N\left(25, \frac{8^2}{100}\right) = N(25, 0.64)$	M1	Or by calculator
	$P(\bar{X} < 24) = P(Z < -1.25)$ $= 0.10564977$	A1	Allow without working
(ii)	$P(\bar{X} < 23) = P(Z < -2.5)$ $= 0.0062097$	M1	Or by calculator
		A1	Allow without working
(iii)	$P(\bar{X} > 26.5) = P(Z > 1.875)$ $= 0.03039636$	M1	Or by calculator
		A1	Allow without working
11 (i)	$\bar{X} \sim N\left(13.5, \frac{1.5^2}{5}\right)$ $= N(13.5, 0.45)$	B1	Allow Normal with mean 13.5 and standard deviation $\frac{1.5}{\sqrt{5}} = 0.67082$
(ii)	$H_0 : \mu = 13.5$ $H_1 : \mu < 13.5$	B1 B1	Do not allow 'the mean has not changed' unless a figure is quoted
(iii)	$P(\bar{X} < 12.5) = P(Z < -1.490712)$ $= 0.0680186$	M1 A1	Allow calculator method if answer to 4 d.p. or better
(iv)	$P(\bar{X} < 12.5) = 0.0680186 > 5\%$	M1	Explicit comparison with 5%
	So, there is not enough evidence to reject the null hypothesis	A1	Accept H_0
	There is not enough evidence at the 5% level that the sprint times have reduced	A1	Must be in context
12 (i)	$H_0 : \mu = 130$ $H_1 : \mu < 130$	B1 B1	
(ii)	$\bar{X} \sim N\left(130, \frac{10^2}{20}\right) = N(130, 5)$	M1	Allow for mean 130 and standard deviation $\frac{10}{\sqrt{20}} = 2.236$ or better used
	Critical region is $\bar{X} < k$ where k is the value for which $P(\bar{X} < k) = 1\%$ $P(\bar{X} < k) = P\left(Z < \frac{k - 130}{\sqrt{5}}\right) = 1\%$ $\frac{k - 130}{\sqrt{5}} = -2.326438$ $k = 130 - 2.326438 \times \sqrt{5} = 124.798$	M1	Uses Z and inverse Normal
	Critical region is $\bar{X} < 124.798$	A1	Must be clearly stated as inequality
(iii)	Actual value 125.5 cm is not in the critical region since $125.5 > 124.798$	M1	Clear statement needed
	There is not enough evidence to reject H_0	E1	Must be in context
	There is not enough evidence that growing wheat near a road reduces the height of wheat plants	E1	Must be in context

	Answer	Marks	Guidance
13 (i)	$H_1 : \mu \neq 17$	B1	2-tail test needed
(ii)	Sample mean $\bar{X} \sim N\left(17, \frac{1^2}{8}\right)$ under H_0 2-tail test so critical region in two parts $\bar{X} < k_1$ where $P(\bar{X} < k_1) = 2.5\%$ and $\bar{X} > k_2$ where $P(\bar{X} > k_2) = 2.5\%$ $P(\bar{X} < k_1) = P\left(Z < \frac{k_1 - 17}{\left(\frac{1}{\sqrt{8}}\right)}\right) = 2.5\%$ $\sqrt{8}(k_1 - 17) = -1.959964$ $k_1 = 17 - \frac{1.959964}{\sqrt{8}} = 16.30704808$ $k_2 = 17 + \frac{1.959964}{\sqrt{8}} = 17.69295192$ So, critical region is $\bar{X} < 16.30704808$ or $\bar{X} > 17.69295192$	M1	Uses 2.5% for both tails
		M1	Attempts to solve equation one for k_1 and 1.96 or better
		A1	Must be inequalities and boundary value to at least 3 s.f.
(iii)	Actual sample mean = $\bar{x} = 17.375$ $17.375 < 17.69295192$ and is not in the critical region There is not enough evidence to reject H_0 There is not enough evidence to suggest that the weights of these goldfinch are significantly different from European goldfinches	B1	
		M1	Explicit comparison
		E1	Conclusion in terms of H_0
		E1	Conclusion in context
14 (i)	$H_0 : \mu = 21$	B1	
	$H_1 : \mu \neq 21$	B1	Must be 2-tail
(ii)	$\bar{X} \sim N\left(21, \frac{3^2}{10}\right)$	B1	Normal with mean 21
		B1	Also allow standard deviation $\frac{3}{\sqrt{10}}$
(iii)	Either $P(\bar{X} > 23) = P(Z > 2.10818511)$ $= 0.01750749$ $< 2.5\%$ So result is significant – reject the null hypothesis There is enough evidence at the 5% level to suggest that playing classical music affects concentration span Or The critical region for $\bar{X} \sim N(21, 0.9)$ at 5% level 2-tail test is ...	M1	Finds the correct probability
		A1	0.0175 or better
		M1	Compares with 5% to reach conclusion
		E1	States conclusion in context
		M1	Attempts to find two regions with 2.5% probability

	Answer	Marks	Guidance
	$\bar{X} < 19.1406149$ or $\bar{X} > 22.8592851$ Actual value $23 > 22.8592851$ so lies in the critical region – reject the null hypothesis There is enough evidence at the 5% level to suggest that playing classical music affects concentration span	A1 B1 B1	Condone missing left tail Comparison explicit Conclusion in context
15	$H_0 : \mu = 87.3$ $H_1 : \mu < 87.3$ Under H_0 $\bar{X} \sim N\left(87.3, \frac{9^2}{20}\right) = N(87.3, 4.05)$ Either $P(\bar{X} < 85.4) = P\left(Z < \frac{85.4 - 87.3}{\sqrt{4.05}}\right)$ $= P(Z < -0.94411759)$ $= 0.17255478 > 5\%$ There is not enough evidence to reject the null hypothesis There is not enough evidence to suggest that the new golf club has reduced scores Or Critical region 1-tail test $\bar{X} < k$ where $P\left(Z < \frac{k - 87.3}{\sqrt{4.05}}\right) = 5\%$ $\frac{k - 87.3}{\sqrt{4.05}} = -1.6448536$ Actual value $= 85.4 > 83.98979598$ and so is not in the critical region Not enough evidence to reject null hypothesis There is not enough evidence to suggest that the new golf club has reduced scores	B1 B1 B1 M1 A1 M1 E1 E1 M1 A1 M1 E1 E1	Must be 1-tail soi Accept calculator method Explicit comparison with 5% seen In terms of null hypothesis Must be in context Solves equation for k in terms of -1.64 or better Explicit comparison seen In terms of null hypothesis Must be in context

18 Kinematics

- 1 For the ball, list the variables involved:

$$s = 2.5, u = u, v = 0, a = -9.8 \Rightarrow v^2 = u^2 + 2as$$

Substituting gives:

$$0 = u^2 + 2 \times -9.8 \times 2.5 \Rightarrow u^2 = 49 \Rightarrow u = 7 \text{ ms}^{-1}$$

- 2 First find the acceleration.

The relevant variables are:

$$s = 21, u = 0, a = a, t = 5 \Rightarrow s = ut + \frac{1}{2}at^2$$

Substituting gives:

$$21 = 0 + \frac{1}{2} \times a \times 5^2 \Rightarrow a = 1.68 \text{ m s}^{-2}$$

Up to when the speed reaches 28 m s^{-1} , the variables are:

$$u = 0, v = 28, a = 1.68, s = s \Rightarrow v^2 = u^2 + 2as \Rightarrow s = 233\frac{1}{3} > 9 \times 21$$

So, the train is still accelerating when it has finished passing Rob.

The relevant variables are:

$$s = 189, u = 0, a = 1.68, t = t \Rightarrow s = ut + \frac{1}{2}at^2$$

Substituting gives:

$$189 = 0 + \frac{1}{2} \times 1.68 \times t^2 \Rightarrow t = 15 \text{ s}$$

- 3 The expression for the velocity includes t^2 , so the acceleration is not constant.

Integrating the expression for the velocity with respect to time gives the displacement:

$$s = \int (4 - t^2) dt = 4t - \frac{t^3}{3}$$

The displacement is zero, so:

$$4t - \frac{t^3}{3} = 0 \Rightarrow t(12 - t^2) = 0 \Rightarrow t = \sqrt{12} = 3.46 \text{ seconds.}$$

- 4 (i) position after 1.635 seconds = original position + displacement

$$\begin{aligned} \begin{pmatrix} X \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ Y \end{pmatrix} + \begin{pmatrix} 2 \\ 0.3 \end{pmatrix} 1.635 + \begin{pmatrix} 0 \\ -4.9 \end{pmatrix} 1.635^2 \\ &= \begin{pmatrix} 0 \\ Y \end{pmatrix} + \begin{pmatrix} 3.27 \\ 0.4905 \end{pmatrix} + \begin{pmatrix} 0 \\ -13.0988025 \end{pmatrix} \\ &= \begin{pmatrix} 3.27 \\ Y - 12.608325 \end{pmatrix} \end{aligned}$$

$$X = 3.27$$

$$Y = 12.61$$

$$(ii) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 12.61 \end{pmatrix} + \begin{pmatrix} 2 \\ 0.3 \end{pmatrix} t + \begin{pmatrix} 0 \\ -4.9 \end{pmatrix} t^2$$

$$x = 2t \Rightarrow t = 0.5x$$

$$y = 12.61 + 0.3t - 4.9t^2$$

$$\text{Hence, } y = 12.61 + 0.15x - 1.225x^2$$

$$(iii) y = 12.61 + 0.15x - 1.225x^2$$

$$\frac{dy}{dx} = 0.15 - 2.45x$$

$$\frac{dy}{dx} = \pm 1 \Rightarrow x = 0.469 \text{ (} x > 0 \text{)} \Rightarrow y = 12.4088$$

Stone is 12.4 m above the sea.

Or

$$\left(\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt} \right)$$

$$x = 2t \Rightarrow \dot{x} = 2$$

$$y = 12.61 + 0.3t - 4.9t^2 \Rightarrow \dot{y} = 0.3 - 9.8t$$

$$|y| = |x| \Rightarrow 0.3 - 9.8t = -2 \Rightarrow t = 0.2347 \text{ and } y = 12.4$$

Stone is 12.4 m above the sea.

5 (i) $\mathbf{r} = \begin{pmatrix} 3 \\ 11 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ -3 \end{pmatrix} = 11\mathbf{i} + 5\mathbf{j}$

(ii) At time T hours after midday, ship has position $x\mathbf{i} + y\mathbf{j}$, where $x = 3 + 4T$ and $y = 11 - 3T$
 Square of distance from coastguard station $= (3 + 4T)^2 + (11 - 3T)^2 = 25T^2 - 42T + 130$

This is minimum when $50T - 42 = 0$

$T = 0.84$ hours after midday (or 50 minutes and 24 seconds after midday).

Or

Ship travels on line $y - 11 = -0.75(x - 3) \Rightarrow 4y = 53 - 3x$

Perpendicular line through coastguard station is $3y = 4x$

These meet when $x = 6.36$, $y = 8.48$ and $T = 0.84$

6 Integrating acceleration with respect to time to obtain the velocity:

$$\mathbf{v} = \int \mathbf{a} \, dt$$

$$= \int \left(6(t^2 - 1)\mathbf{i} + \frac{1}{25}t^4\mathbf{j} - 4t\mathbf{k} \right) dt$$

$$= (2t^3 - 6t + c_1)\mathbf{i} + \left(\frac{1}{125}t^5 + c_2 \right)\mathbf{j} + (-2t^2 + c_3)\mathbf{k}$$

$$\mathbf{v} = 0 \text{ when } t = 0; c_1 = c_2 = c_3 = 0$$

$$\text{Hence, } \mathbf{v} = (2t^3 - 6t)\mathbf{i} + \left(\frac{1}{125}t^5 \right)\mathbf{j} + (-2t^2)\mathbf{k}$$

Integrating velocity with respect to time to obtain the displacement:

$$\mathbf{r} = \int \mathbf{v} \, dt$$

$$= \int \left((2t^3 - 6t)\mathbf{i} + \left(\frac{1}{125}t^5 \right)\mathbf{j} + (-2t^2)\mathbf{k} \right) dt$$

$$= \left(\frac{1}{2}t^4 - 3t^2 + c_4 \right)\mathbf{i} + \left(\frac{1}{750}t^6 + c_5 \right)\mathbf{j} + \left(-\frac{2}{3}t^3 + c_6 \right)\mathbf{k}$$

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \text{ when } t = 0 \Rightarrow c_4 = 2, c_5 = 3 \text{ and } c_6 = 5$$

$$\text{Hence, } \mathbf{r} = \left(\frac{1}{2}t^4 - 3t^2 + 2 \right)\mathbf{i} + \left(\frac{1}{750}t^6 + 3 \right)\mathbf{j} + \left(-\frac{2}{3}t^3 + 5 \right)\mathbf{k}$$

Solving the equation to find the time when the third component of the position is -13 :

$$-13 = -\frac{2}{3}t^3 + 5 \Rightarrow t^3 = 27 \Rightarrow t = 3$$

$$t = 3 \Rightarrow \mathbf{r} = 15.5\mathbf{i} + 3.972\mathbf{j} - 13\mathbf{k}$$

$$x = 15.5 \text{ and } y = 3.972$$

Mark scheme

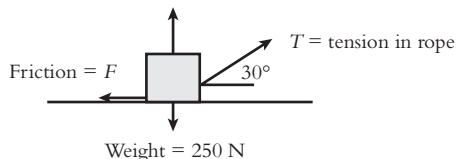
	Answer	Marks	Guidance
1	$0 = u^2 + 2 \times -9.8 \times 2.5$ 7 m s^{-1}	M1 A1	Uses kinematics equation cao
2	$21 = 0 + \frac{1}{2} \times a \times 5^2$ $a = 1.68 \text{ m s}^{-2}$ $233 \frac{1}{3} > 9 \times 21$ Train is still accelerating when it has finished passing Rob $189 = 0 + \frac{1}{2} \times 1.68 \times t^2$ $t = 15 \text{ seconds}$	M1 A1 M1 A1 M1 A1	Completes method to find a cao Checks whether still accelerating Interprets result Completes method to find time cao

	Answer	Marks	Guidance
3	$s = \int(4 - t^2)dt$	M1	Integrates velocity
	$= 4t - \frac{t^3}{3}$	A1	Correct
	$4t - \frac{t^3}{3} = 0 \Rightarrow t(12 - t^2) = 0$	M1	Solves equation
	$\Rightarrow t = \sqrt{12} = 3.46 \text{ seconds}$	A1	cao
4 (i)	$\begin{pmatrix} 2 \\ 0.3 \end{pmatrix} 1.635 + \begin{pmatrix} 0 \\ -4.9 \end{pmatrix} 1.635^2$	M1	Displacement = $ut + 0.5at^2$ for $t = 1.635$ or general t , in vector form or as components
	$= \begin{pmatrix} 3.27 \\ -12.61 \end{pmatrix}$		
	$X = 3.27$	A1	3.27
	$Y = 12.61$	A1	12.6 or better (12.6083025)
(ii)	$x = 2t$	M1	Or in vector form
	$y = 12.61 + 0.3t - 4.9t^2$		x and y as functions of t
	Substitute $t = 0.5x$	M1	Eliminates t for their expressions
	$y = 12.61 + 0.15x - 1.225x^2$	A1	$y = 12.6$ (or better) $+ 0.15x - 1.225x^2$
(iii)	$\frac{dy}{dx} = 0.15 - 2.45x$	M1	Or $\frac{dy}{dt} = 0.3 - 9.8t$ and $\frac{dx}{dt} = 2$
	45° to the horizontal when $\frac{dy}{dx} = -1$	M1	Or $t = 0.235$ (or better)
	$x = 0.469$		
	$y = 12.4$	A1	12.4 as final answer
5 (i)	$\mathbf{r} = \begin{pmatrix} 3 \\ 11 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ -3 \end{pmatrix}$	M1	$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$
	$= 11\mathbf{i} + 5\mathbf{j}$	A1	
(ii)	distance ² = $(3 + 4T)^2 + (11 - 3T)^2$	M1	distance ² oe valid method
	At minimum distance		
	$50T - 42 = 0$	M1	Or an equation that leads to when $x = 6.36$ or $y = 8.48$ or distance = 10.6, oe
	So, $T = 0.84$	A1	0.84 oe
6	$\mathbf{v} = \int \left(6(t^2 - 1)\mathbf{i} + \frac{1}{25}t^4\mathbf{j} - 4t\mathbf{k} \right) dt$	M1	Integrates acceleration wrt time
	$= (2t^3 - 6t)\mathbf{i} + \left(\frac{1}{125}t^5 \right)\mathbf{j} + (-2t^2)\mathbf{k}$	A1	Correct expression
	$\mathbf{r} = \int \left((2t^3 - 6t)\mathbf{i} + \left(\frac{1}{125}t^5 \right)\mathbf{j} + (-2t^2)\mathbf{k} \right) dt$	M1	Integrates velocity wrt time
	$= \left(\frac{1}{2}t^4 - 3t^2 + 2 \right)\mathbf{i} + \left(\frac{1}{750}t^6 + 3 \right)\mathbf{j} + \left(-\frac{2}{3}t^3 + 5 \right)\mathbf{k}$	M1	$\frac{1}{2}t^4 - 3t^2, \frac{1}{750}t^6, -\frac{2}{3}t^3$
		A1	Constants correct
	$-13 = -\frac{2}{3}t^3 + 5$	M1	Their coefficient of $k = -13$ and solving
	$t = 3$	A1	cao
	$x = 15.5$ and $y = 3.972$	B1	Both correct

19 Forces and motion

1 The component of the weight down the slope $= 20 \cos(90 - 10)^\circ = 20 \sin 10^\circ = 3.47 \text{ N}$

2 Normal reaction from floor $= R$



3 The four forces are in equilibrium, so their resultant is zero.

$$\mathbf{F}_1 = 9\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{F}_2 = -2\mathbf{i}$$

$$\mathbf{F}_3 = \mathbf{i} - 7\mathbf{j}$$

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 0\mathbf{i} + 0\mathbf{j}$$

$$(9\mathbf{i} + 3\mathbf{j}) + (-2\mathbf{i}) + (\mathbf{i} - 7\mathbf{j}) + \mathbf{F}_4 = 0\mathbf{i} + 0\mathbf{j}$$

$$\text{So, } \mathbf{F}_4 = -8\mathbf{i} + 4\mathbf{j}$$

4 Magnitude $= \sqrt{65} = 8.06 \text{ N}$

$$\text{Angle} = 180^\circ - \arctan(1.75) = 119.7^\circ$$

5 Using Newton's second law:

$$\begin{pmatrix} 13 \\ 4 \end{pmatrix} = 0.5\mathbf{a} \text{ so } \mathbf{a} = \begin{pmatrix} 26 \\ 8 \end{pmatrix}$$

6 Finding the resultant force using components and vector addition:

$$\mathbf{P} + \mathbf{Q} = \begin{pmatrix} 6 \cos 10 \\ 6 \sin 10 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 5.91 \\ -2.96 \end{pmatrix}$$

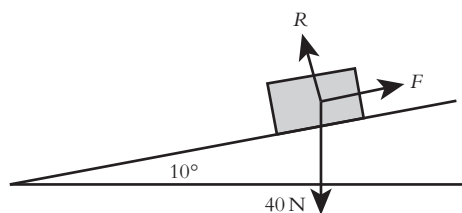
$$\begin{aligned} \text{magnitude} &= \sqrt{(5.91)^2 + (-2.96)^2} \\ &= 6.61 \text{ N} \end{aligned}$$

7 Applying Newton's second law to each box:

$$\text{A: } T = 2a$$

$$\text{B: } g - T = a$$

8 (i)



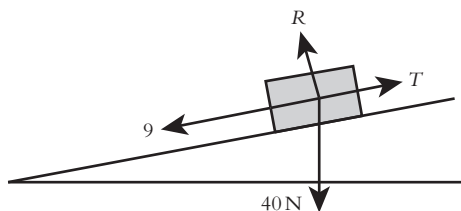
(ii) Frictional force acts up the slope, opposing the motion.

Resolving parallel to the slope:

$$F = 40 \sin 10^\circ$$

$$= 6.95 \text{ N}$$

(iii)



Equation of motion parallel for crate:

$$m = \frac{40}{g}$$

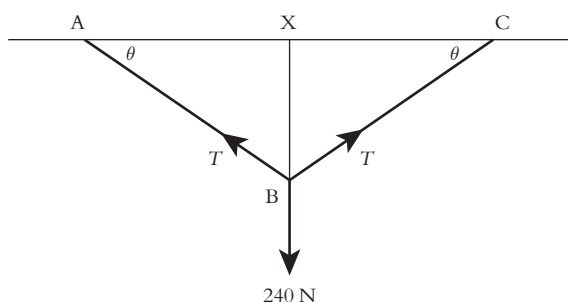
$$m = 4.08$$

$$\text{So, } T - 9 - 6.95 = 4.08$$

$$T = 20.03$$

Tension = 20.0 N (3 s.f.)

- 9 (i)** Tension in AB = tension in BC, since pulley is smooth.
Midpoint of AC = X.



Using trigonometry:

$$\begin{aligned}\cos \theta &= \frac{AX}{AB} \\ &= \frac{\frac{1}{2}AXC}{\frac{1}{2}ABC} \\ &= \frac{3.6}{4.5} \\ &= 0.8\end{aligned}$$

- (ii)** Resolving vertically at B:

$$2T \sin \theta = 240$$

$$\sin \theta = 0.6$$

Using $\sin \theta = \sqrt{1 - \cos^2 \theta}$

$$T = \frac{240}{1.2}$$
$$= 200$$

- 10 (i)** Using Pythagoras:

$2.16^2 + 2.88^2 = 3.6^2$, so the angle between the strings is 90°

$$\begin{aligned} \text{(ii) (a) } \cos \text{BAC} &= \frac{\text{AB}}{\text{AC}} \\ &= 0.6 \text{ (and } \sin \text{BAC} = 0.8) \end{aligned}$$

$$\begin{aligned} \text{(b) } \cos \text{BCA} &= \frac{\text{BC}}{\text{AC}} \\ &= 0.8 \text{ (and } \sin \text{BCA} = 0.6) \end{aligned}$$

- (iii) (a)** Resolving horizontally at B:

$$T_{AB} \cos BAC = T_{BC} \cos BCA$$

$$0.6 \ T_{AB} = 0.8 \ T_{BC}$$

Resolving vertically at B:

$$T_{AB} \sin BAC + T_{BC} \sin BCA = 240$$

$$0.8 T_{AB} + 0.6 T_{BC} = 240$$

$$T_{BC} = 0.75 T_{AB}$$

$$\text{So, } 0.8 T_{AB} + 0.45 T_{AB} = 240$$

$$1.25 T_{AB} = 240$$

$$T_{AB} = 192 \text{ N}$$

$$\text{(b) } T_{BC} = 0.75 \times 192$$

$$= 144 \text{ N}$$

- 11 (i) Applying Newton's second law:

$$T - 2000 = 1500 \times 0$$

$$T = 2000 \text{ N}$$

- (ii) Applying Newton's second law:

$$F - T - 800 = 3000 \times 0$$

$$F = 2800 \text{ N}$$

- (iii) Component of weight down slope = $mg \sin \theta$

Drive force needs to overcome resistances and produce acceleration.

$$F - 800 - 3000 \times 9.8 \times 0.02 - 2000 - 1500 \times 9.8 \times 0.02 = 4500 \times 0.01$$

$$F = 45 + 2800 + 882$$

$$= 3727 \text{ N}$$

- 12 (i) Finding resultant force gives:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (6\mathbf{i} + 7\mathbf{j}) + (x\mathbf{i} - 5\mathbf{j}) + (4\mathbf{j})$$

$$= (6 + x)\mathbf{i} + (7 - 5 + 4)\mathbf{j}$$

$$= (6 + x)\mathbf{i} + 6\mathbf{j}$$

Using Newton's second law:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = m \times 2\mathbf{j}$$

$$\text{So, } (6 + x)\mathbf{i} + (6)\mathbf{j} = 2m\mathbf{j}$$

$$\mathbf{i} \text{ terms: } 6 + x = 0 \Rightarrow x = -6$$

- (ii) $\mathbf{j}$ terms: $6 = 2m \Rightarrow m = 3$

- 13 (i) Using Newton's second law on the train:

Resultant force = 500 N (previously there was no resultant force, now there is 500 N resulting from the loss of a truck)

$$500 = 150\,000a$$

$$a = 0.00333 \text{ m s}^{-2}$$

- (ii) After 3 seconds, speed is given by $v = u + at = 12.01 \text{ m s}^{-1}$

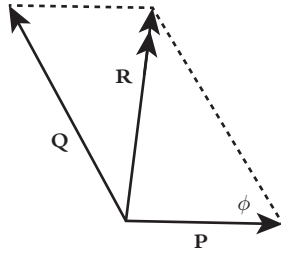
Using Newton's second law:

$$-2000 = 80\,000a$$

$$a = -0.025 \text{ m s}^{-2}$$

Then using $v^2 = u^2 + 2as$, trucks come to rest after travelling a further $\frac{12.01^2}{0.05} = 2880 \text{ m}$ (3 s.f.)

- 14 (i) Angle required is $\theta = 180^\circ - \phi$



Using cosine rule on triangle of forces:

$$(1.4\mathbf{P})^2 = (\mathbf{P})^2 + (2\mathbf{P})^2 - 2 \times \mathbf{P} \times 2\mathbf{P} \times \cos \phi$$

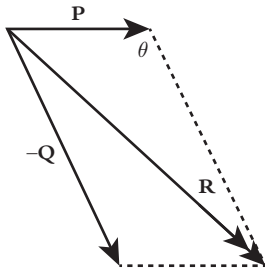
$$\cos \phi = \frac{1 + 4 - 1.96}{4}$$

$$= 0.76$$

$$\phi = 40.54^\circ$$

Angle between $\mathbf{P}$ and $\mathbf{Q} = 139.5^\circ$

- (ii) Using cosine rule on triangle of forces:



$$(\mathbf{R})^2 = (\mathbf{P})^2 + (2\mathbf{P})^2 - 2 \times \mathbf{P} \times 2\mathbf{P} \times \cos \theta$$

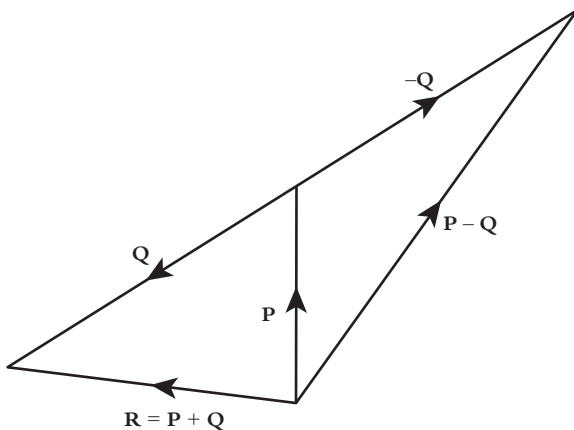
$$\cos \theta = -0.76$$

$$\mathbf{R}^2 = \mathbf{P}^2 + 4\mathbf{P}^2 + 3.04\mathbf{P}^2$$

$$= 8.04\mathbf{P}^2$$

Magnitude of resultant of $\mathbf{P}$ and $-\mathbf{Q} = 2.84\mathbf{P}$

- (iii)



$\mathbf{Q}$ has magnitude $2\mathbf{P}$, $\mathbf{P} + \mathbf{Q}$ has magnitude $1.4\mathbf{P}$, $\mathbf{P} - \mathbf{Q}$ has magnitude $2.84\mathbf{P}$.

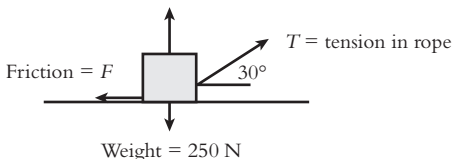
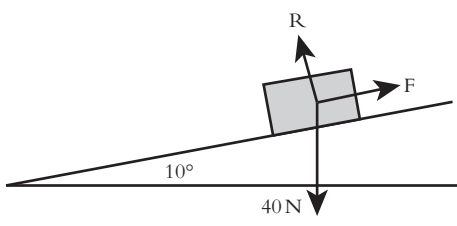
Using cosine rule on the triangle with sides:

$\mathbf{P} + \mathbf{Q}$, $\mathbf{P} - \mathbf{Q}$ and $2\mathbf{Q}$

$$(4\mathbf{P})^2 = (2.84\mathbf{P})^2 + (1.4\mathbf{P})^2 - 2 \times 2.84\mathbf{P} \times 1.4\mathbf{P} \times \cos \theta$$

$$16 = 8.0656 + 1.96 - 7.952 \cos \theta \Rightarrow \cos \theta = 0.75775 \Rightarrow \theta = 138.7^\circ$$

Mark scheme

	Answer	Marks	Guidance
1	$20 \sin 10^\circ$ $= 3.47 \text{ N}$	M1 A1	Uses trigonometry Correct to 3 s.f.
2	Normal reaction from floor = R  Friction = F T = tension in rope 30° Weight = 250 N	B1 B1 B1	The correct four forces Labelled correctly Arrows in the correct direction
3	$(9\mathbf{i} + 3\mathbf{j}) + (-2\mathbf{i}) + (\mathbf{i} - 7\mathbf{j})$ $+ \mathbf{F}_4 = 0\mathbf{i} + 0\mathbf{j}$ so $\mathbf{F}_4 = -8\mathbf{i} + 4\mathbf{j}$	M1 A1	Forms equation = 0 Fully correct
4	Magnitude = $\sqrt{65} = 8.06 \text{ N}$ Angle = $180^\circ - \arctan(1.75) = 119.7^\circ$	B1 B1	
5	$\begin{pmatrix} 13 \\ 4 \end{pmatrix} = 0.5\mathbf{a}$, so $\mathbf{a} = \begin{pmatrix} 26 \\ 8 \end{pmatrix}$	M1 A1	Uses Newton's second law Must be in vector form
6	$\mathbf{P} + \mathbf{Q} = \begin{pmatrix} 6 \cos 10 \\ 6 \sin 10 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ $\begin{pmatrix} 5.91 \\ -2.96 \end{pmatrix}$ $\sqrt{(5.91)^2 + (-2.96)^2}$ 6.61 N	M1 A1 M1 A1	Adds components Correct vector Uses Pythagoras cao
7	A: $T = 2a$ B: $g - T = a$	B1 B1	
8 (i)		B1	Force diagram shows three forces acting in the appropriate directions
(ii)	Frictional force acts up the slope $F = 40 \sin 10^\circ \text{ N} = 6.95 \text{ N}$	B1 B1	
(iii)	$T - 9 - 40 \sin 10^\circ = \frac{40}{g}$ $T = 9 + 6.95 + 4.08 = 20.03$ Tension = 20.0 N (3 s.f.)	M1 B1 A1	Forms equation of motion $m = \frac{40}{g} = 4.08$ 20.0 N or better
9 (i)	$\cos \theta = \frac{\frac{1}{2} \text{AXC}}{\frac{1}{2} \text{ABC}}$ $= 0.8$	M1 A1	Uses trigonometry 0.8
(ii)	$2T \sin \theta = 240$ $T = 200 \text{ N}$	M1 A1	Resolves and solves 200

	Answer	Marks	Guidance
10 (i)	$2.16^2 + 2.88^2 = 3.6^2$	B1	Pythagoras (or cosine rule) to show 90° given
(ii) (a)	$\frac{AB}{AC} = 0.6$	B1	
(b)	$\frac{BC}{AC} = 0.8$	B1	
(iii)(a)	$0.6 T_{AB} = 0.8 T_{BC}$	M1	Resolves horizontally at B; ft from part (ii)
	$0.8 T_{AB} + 0.6 T_{BC} = 240$	M1	Resolving vertically at B; ft from part (ii)
	$T_{AB} = 192 \text{ N}$	B1	192
(b)	$T_{BC} = 144 \text{ N}$	B1	144
11 (i)	$T - 2000 = 0$	B1	In this form, with resultant force on one side
	$T = 2000 \text{ N}$	B1	2000
(ii)	$F - T - 800 = 0$	M1	Forms equation of motion for tractor or for whole
	$F = 2800 \text{ N}$	A1	2800
(iii)	$F - 800 - 3000 \times 9.8 \times 0.02 - 2000 - 1500 \times 9.8 \times 0.02 = 4500 \times 0.01$	M1	Applies Newton's second law parallel to slope
	$= 3730 \text{ N (3 s.f.)}$	B1	Use of $mg \sin \theta$
		A1	3730 or better
12 (i)	$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (6 + x)\mathbf{i} + 6\mathbf{j}$	B1	$(6 + x)\mathbf{i} + 6\mathbf{j}$ oe unsimplified
	$(6 + x)\mathbf{i} + 6\mathbf{j} = m \times 2\mathbf{j}$	M1	Finds resultant force = $2m\mathbf{j}$
	$x = -6$	A1	-6
(ii)	3	B1	cao
13 (i)	$500 = 150\,000a$	M1	Uses Newton's second law
	$a = 0.00333 \text{ ms}^{-2}$	A1	oe
(ii)	$12 + 3 \times 0.00333 = 12.01 \text{ ms}^{-1}$	B1	12.01 as speed when other trucks uncouple
	$-2000 = 80000a$	M1	Uses Newton's second law to find acceleration
	$a = -0.025$	M1	Uses their 12.01 and their -0.025
	$0^2 = 12.01^2 + 2 \times -0.025 \times s$	M1	
	$s = 2880 \text{ m}$	A1	2880
14 (i)	$(1.4P)^2 = (P)^2 + (2P)^2$	M1	Uses triangle of forces and cosine rule, oe
	$-2 \times P \times 2P \times \cos(180 - \theta)$	M1	Sight of ± 0.76 as a cosine or 0.65 as a sine
	$\cos(180 - \theta) = 0.76$	A1	139°, 140°
	$\theta = 139.5^\circ$		
(ii)	$(\mathbf{R})^2 = (\mathbf{P})^2 + (2\mathbf{P})^2 - 2 \times P \times 2P \times \cos \theta$	M1	Uses triangle of forces and cosine rule, oe
	$\cos \theta = -0.76$		
	$\mathbf{R}^2 = \mathbf{P}^2 + 4\mathbf{P}^2 + 3.04\mathbf{P}^2 = 8.04\mathbf{P}^2$	M1	ft their $\cos \theta$
	Resultant of $\mathbf{P}$ and $-\mathbf{Q}$ has magnitude $2.84\mathbf{P}$	A1	$2.84\mathbf{P}$

	Answer	Marks	Guidance
(iii)	Using cosine rule on the triangle with sides, $P + Q$, $P - Q$ and $2Q$ $(4P)^2 = (2.84P)^2 + (1.4P)^2$ $-2 \times 2.84P \times 1.4P \times \cos \theta$	M1	Uses cosine rule Or use sine rule in the separate triangles
	$16 = 8.0656 + 1.96 - 7.952 \cos \theta$	M1	Substitutes correctly
	$\cos \theta = 0.75775 \Rightarrow \theta = 138.7^\circ$	A1	138.5° to 138.9°

20 Moments

- If we take moments about B, we obtain $dX + 2dZ$, which cannot equal zero.
Thus, the system cannot be in equilibrium.
- The total forces on the object in two perpendicular directions will be zero. This gives two equations.
The total moment of the forces (about any point) will be zero, giving a third equation.
- The net moment will be the same for all points.
Choose the most convenient point: one at which an unknown force is applied, or such that the least number of forces appears in the expression for the net moment.
- Vertical equilibrium $\Rightarrow X + 10 = 30 \Rightarrow X = 20$
Taking moments about the right-hand end (for example):
 $30 \times 0.5 - 20(1 - d) = 0 \Rightarrow -5 + 20d = 0 \Rightarrow d = 0.25$
- Taking moments about A:
 $3dY - dZ = 3dY - d(X + Y) = d(2Y - X)$
 Taking moments about B:
 $2dY - dX = d(2Y - X)$
 Taking moments about C:
 $2dZ - 3dX = 2d(X + Y) - 3dX = d(2Y - X)$
 So, all the moments are equal, as required.
- In order for there to be vertical equilibrium, the additional force must be of magnitude 150 N and act downwards.
Suppose that its line of action is at a distance d from A.
In order for there to be rotational equilibrium, the net moment about A (say) must be zero; i.e.
 $200 \times 1 + 50 \times 4 - 150d = 0 \Rightarrow d = \frac{400}{150} = \frac{8}{3} \text{ m}$

Mark scheme

	Answer	Marks	Guidance
1	$dX + 2dZ \dots$	M1	Takes moments about B
	$\dots$ which cannot equal zero $\dots$	A1	Identifies not zero
	$\dots$ so the system cannot be in equilibrium	E1	Convincing
2	The total forces on the object in two perpendicular directions will be zero; the total moment of the forces (about any point) will be zero	B1	Resolve in one direction
		B1	Resolve in a non-parallel direction
		B1	Take moments

	Answer	Marks	Guidance
3	The net moment will be the same for all points	B1	oe
	Choose the most convenient point: one at which an unknown force is applied, or such that the least number of forces appears in the expression for the net moment	E2	E1 for one valid point; E1 for a different valid point
4	$X + 10 = 30$	M1	Resolves vertically
	$\Rightarrow X = 20$	A1	cao
	$30 \times 0.5 - 20(1 - d) = 0$	M1	Takes moments about the right-hand end
	$d = 0.25$	A1	Correct equation cao
5	$3dY - dZ$	M1A1	Takes moments about A
	$= 3dY - d(X + Y)$	B1	Uses $X + Y = Z$
	$= d(2Y - X)$	A1	
	$2dY - dX$	M1	Takes moments about B
	$= d(2Y - X)$	A1	In the same form as for A
	$2dZ - 3dX$	M1	Takes moments about C
6	$= 2d(X + Y) - 3dX$		
	$= d(2Y - X)$	A1	In the same form as for A
	vertical equilibrium $\Rightarrow$ additional force is 150 N downwards	B1	Magnitude
	If at distance d from A,	B1	Direction
	net moment about, e.g., A is zero	B1	Takes moments
	$200 \times 1 + 50 \times 4 - 150d = 0$	M1	Correct equation
	$\Rightarrow d = \frac{8}{3}$	A1	Position of force

21 Projectiles

1 $\text{Range} = \frac{\sin 2\theta U^2}{g}$

This may be known, and so quoted, or worked out, as follows.

Vertically, the values of the variables are:

$$s = 0, u = U \sin \theta, a = -g, t = t \Rightarrow s = ut + \frac{1}{2}at^2$$

Substituting gives:

$$0 = U \sin \theta t - \frac{1}{2}gt^2 \Rightarrow t(2U \sin \theta - gt) = 0 \Rightarrow t = \frac{2U \sin \theta}{g} \text{ when the projectile lands.}$$

Horizontally, the speed is $U \cos \theta$ and the time is $\frac{2U \sin \theta}{g}$

$$\text{So, range} = U \cos \theta \times \frac{2U \sin \theta}{g} = \frac{U^2 \sin 2\theta}{g}$$

Let the other angle be ϕ .

Then $\sin 2\phi = \sin 2\theta$, so that $2\phi = 180 - 2\theta$, and $\phi = 90 - \theta$

- 2 (i) $v^2 = u^2 + 2as$ because t is not given or required.
 (ii) $v = u + at$ because s is not given or required.
- 3 The acceleration due to gravity is constant (the variation in vertical distance travelled is relatively small).
 The acceleration due to gravity has a constant direction (the horizontal distance travelled is relatively small).

4 A particle does not rotate, but a golf ball usually does.

5 Consider the vertical component of the motion.

Going up, taking upwards as the positive direction:

$$s = vt - \frac{1}{2}at^2 \Rightarrow H = 0 - \frac{1}{2}(-g)T^2 \Rightarrow T^2 = \frac{2H}{g}$$

Going down, taking downwards as the positive direction:

$$s = ut + \frac{1}{2}at^2 \Rightarrow H = 0 + \frac{1}{2}gT^2 \Rightarrow T^2 = \frac{2H}{g}$$

So, the two times are equal.

6 Consider the vertical component of the motion.

Going up, taking upwards as the positive direction:

$$v^2 = u^2 + 2as \Rightarrow 0 = u^2 + 2(-g)H \Rightarrow u^2 = 2gH$$

Going down, taking downwards as the positive direction:

$$v^2 = u^2 + 2as \Rightarrow v^2 = 0 + 2gH \Rightarrow v^2 = 2gH$$

So, the square of the vertical component of the speed is unchanged.

As the horizontal component of the speed is constant, the final speed of the particle, $\sqrt{v_x^2 + v_y^2}$, is unchanged.

7 For the vertical motion:

$$s_y = \frac{1}{2}(u_y + v_y)t \Rightarrow 10 = \frac{1}{2}(u_y + 0) \times 1 \Rightarrow u_y = 20$$

$$\text{So, the speed of projection is given by } u \sin 30^\circ = 20 \Rightarrow u = \frac{20}{\left(\frac{1}{2}\right)} = 40 \text{ m s}^{-1}$$

8 **Method 1:** Find the time in flight and multiply by the horizontal component of the initial velocity.

Method 2: Set the vertical displacement equal to zero in the cartesian equation of the trajectory.

9 (i) Differentiating $y = x - 0.1x^2$

$$\frac{dy}{dx} = 1 - 0.2x$$

$$\text{When } x = 0, \frac{dy}{dx} = 1$$

The gradient is 1, so $\theta = 45^\circ$

(ii) The particle hits the ground when $y = 0$

$$\text{So, } x - 0.1x^2 = 0 \Rightarrow x(1 - 0.1x) = 0 \Rightarrow x = 0 \text{ or } 10$$

So, the horizontal distance travelled is 10 m.

(iii) The function $y = x(1 - 0.1x)$ has its maximum when $x = 5$, using the answer from part (ii) and the symmetry of the motion, and $y = 5 \times 0.5 = 2.5$

So, the greatest height reached is 2.5 m.

(iv) Using the vertical motion for an equation to find u :

$$s = 2.5, u = u \sin 45^\circ = \frac{u}{\sqrt{2}}, v = 0, a = -9.8$$

$$v^2 = u^2 + 2as \Rightarrow 0 = \frac{u^2}{2} + 2 \times (-9.8) \times 2.5 \Rightarrow u^2 = 98$$

$$u = 9.90 \text{ m s}^{-1} \text{ (3 s.f.)}$$

- (v) By symmetry, the speed at which the particle hits the ground is 9.90 m s^{-1} .
(The horizontal component of the speed is constant and the vertical component is reversed in sign.)

- 10 (i) The height above the ground is given by:

$$s_y = s_0 + u \sin \theta \cdot t + \frac{1}{2} \times (-g) \times t^2 \text{ (where upwards is the positive direction).}$$

Time in flight, t , is given by:

$$20 + (-10) \times 0.5 \times t - \frac{1}{2} \times 9.8 \times t^2 = 0$$

Or

$$9.8t^2 + 10t - 40 = 0$$

$$\Rightarrow t = \frac{-10 + \sqrt{100 - 4 \times 9.8 \times (-40)}}{2 \times 9.8} \text{ (taking the positive root)}$$

$$= 1.5735 = 1.57 \text{ s (3 s.f.)}$$

- (ii) Vertical component of velocity is given by:

$$v_y^2 = u_y^2 + 2 \times (-g) \times s_y, \text{ where } s_y \text{ is the vertical displacement relative to the starting point.}$$

$$\Rightarrow v_y^2 = (-10 \sin 30^\circ)^2 + 2 \times (-9.8) \times (-20)$$

$$= 25 + 392 = 417$$

$$\Rightarrow v_y = -20.421$$

Horizontal component of velocity is:

$$v_x = 10 \cos 30^\circ = 8.6603$$

$$\text{Hence, speed} = \sqrt{(8.6603)^2 + (-20.421)^2}$$

$$= 22.181$$

$$= 22.2 \text{ m s}^{-1} \text{ (3 s.f.)}$$

- 11 (i) Using the horizontal motion:

$$\text{Speed} = u \cos \theta, \text{ time} = t, \text{ distance} = x \Rightarrow t = \frac{x}{u \cos \theta}$$

Using the vertical motion:

$$s = y, u = u \sin \theta, t = t, a = -10, s = ut + \frac{1}{2}at^2$$

$$\Rightarrow y = u \sin \theta t + \frac{1}{2} \times (-10) \times t^2 \Rightarrow y = u \sin \theta t - 5t^2$$

Eliminating t between the equations gives:

$$y = u \sin \theta \times \frac{x}{u \cos \theta} - 5 \left(\frac{x}{u \cos \theta} \right)^2$$

$$\text{So, } y = x \tan \theta - 5 \left(\frac{x}{u \cos \theta} \right)^2$$

- (ii) The ball is thrown from a height of 2 m, $y = 2 + x \tan \theta - 5 \left(\frac{x}{u \cos \theta} \right)^2$

$$x = 10, y = 6 \Rightarrow 6 = 2 + 10 \tan \theta - \frac{500 \sec^2 \theta}{u^2}$$

$$\Rightarrow \frac{500 \sec^2 \theta}{u^2} = 10 \tan \theta - 4$$

$$\Rightarrow u^2 = \frac{500 \sec^2 \theta}{10 \tan \theta - 4} = \frac{250(\tan^2 \theta + 1)}{5 \tan \theta - 2}$$

- (iii) When $\theta = 45^\circ$, $u^2 = \frac{250(1+1)}{5 \times 1 - 2} = \frac{500}{3}$, so that $u = 12.910 = 12.9 \text{ (3 s.f.) (as } u > 0)$

(iv) $u^2 = \frac{250(\tan^2\theta + 1)}{5\tan\theta - 2}$ is required to have a minimum value.

Differentiating u^2 with respect to θ to find the stationary value gives:

$$\frac{d(u^2)}{d\theta} = \frac{(5\tan\theta - 2) \times 250 \times 2\tan\theta \sec^2\theta - 250(1 + \tan^2\theta) \times 5\sec^2\theta}{(5\tan\theta - 2)^2}$$

This is zero for a stationary value, so the numerator needs to be zero.

$$(5\tan\theta - 2) \times 250 \times 2\tan\theta \sec^2\theta - 250(1 + \tan^2\theta) \times 5\sec^2\theta = 0$$

Simplifying gives:

$$5\tan^2\theta - 4\tan\theta - 5 = 0$$

Solving using the quadratic formula gives:

$$\tan\theta = \frac{4 \pm \sqrt{16 + 100}}{10} = 1.47703$$

$$\Rightarrow \theta = \tan^{-1}(1.47703) = 55.901 = 55.9^\circ \text{ (3 s.f.)}$$

$$\text{and } u^2 = \frac{250(\tan^2\theta + 1)}{5\tan\theta - 2} = 147.703$$

$$\Rightarrow u = 12.153 = 12.2 \text{ m s}^{-1} \text{ (3 s.f.)}$$

12 (i) Using the horizontal motion:

$$R = u \cos 30^\circ \times 2T \Rightarrow T = \frac{R}{u\sqrt{3}}$$

(ii) Using the vertical motion:

$$'v = u + at' \Rightarrow 0 = u \sin 30^\circ - gT$$

$$\Rightarrow T = \frac{u}{2g}$$

(iii) Eliminating u from parts (i) and (ii):

$$u = \frac{R}{T\sqrt{3}} \text{ and } u = 2Tg$$

$$\text{Hence, } \frac{R}{T\sqrt{3}} = 2Tg, \text{ so that } R = 2\sqrt{3} \times T^2g \text{ and } T = \sqrt{\frac{R}{2\sqrt{3}g}}$$

$$R = 100 \Rightarrow \text{time in the air} = 2T = 2\sqrt{\frac{100}{2\sqrt{3} \times 9.8}} = 3.4326 = 3.43 \text{ s (3 s.f.)}$$

Mark scheme

	Answer	Marks	Guidance
1	$\text{Range} = \frac{\sin 2\theta \cdot U^2}{g}$ Let the other angle be ϕ. $\sin 2\phi = \sin 2\theta$, so that $2\phi = 180 - 2\theta$, and $\phi = 90 - \theta$	M1 A1 E1	Derives expression for the range Correct Valid argument
2	(i) $v^2 = u^2 + 2as$ (ii) $v = u + at$	B1 B1	

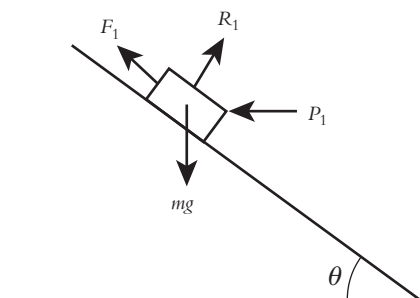
	Answer	Marks	Guidance
3	The acceleration due to gravity is constant in magnitude ...	B1	oe
	... and has a constant direction	B1	oe
4	A particle does not rotate, and a golf ball usually does	B1 B1	
5	$H = 0 - \frac{1}{2}(-g)T^2$	M1	Forms equations for upward motion
	$T^2 = \frac{2H}{g}$	A1	oe
	$H = 0 + \frac{1}{2}gT^2 \Rightarrow T^2 = \frac{2H}{g}$	A1	Finds expression for downward motion
	So, the two times are equal	A1	Draws conclusion
6	$0 = u^2 + 2(-g)H$	M1	Forms equations for upward motion
	$u^2 = 2gH$	A1	oe
	$v^2 = 0 + 2gH$		
	$v^2 = 2gH$	A1	Finds expression for downward motion
	So, the square of the vertical component of the speed is unchanged As the horizontal component of the speed is constant, the final speed of the particle, $\sqrt{v_x^2 + v_y^2}$ is unchanged	E1	Makes argument
7	$10 = \frac{1}{2}(u_y + 0) \times 1$	M1	Finds initial vertical velocity
	$u_y = 20$	A1	Correct vertical component
	$= 40 \text{ m s}^{-1}$	A1	cao
8	Find the time in flight, multiply by the horizontal component of the initial velocity	B2	oe
	Find the cartesian equation of the trajectory	B1	oe
	Set the vertical displacement equal to zero	B1	
9 (i)	$\frac{dy}{dx} = 1 - 0.2x, \frac{dy}{dx} = 1$	M1	Differentiates and evaluates at start
	$\theta = 45^\circ$	A1	cao
(ii)	$x - 0.1x^2 = 0$	M1	Uses equation with $y = 0$
	$\Rightarrow x = 0 \text{ or } 10$ 10 m	A1	cao
(iii)	$y = x(1 - 0.1x)$ has its maximum when $x = 5$	B1	Uses symmetry and part (ii)
	$y = 5(0.5) = 2.5 \text{ m}$	B1	cao
(iv)	$v^2 = u^2 + 2as$	M1	Uses vertical motion
	$\Rightarrow 0 = \frac{u^2}{2} + 2 \times (-9.8) \times 2.5 \Rightarrow u^2 = 98$	M1	Solves equation
	$u = 9.90 \text{ m s}^{-1} \text{ (3 s.f.)}$	A1	cao

	Answer	Marks	Guidance
(v)	By symmetry 9.90 m s^{-1}	B1 B1	Uses symmetry argument cao
10 (i)	$20 + (-10) \times 0.5 \times t - \frac{1}{2} \times 9.8 \times t^2 = 0$ $t = 1.5735$ $= 1.57 \text{ s (3 s.f.)}$	M1 A1 M1 A1	Substitutes into height equation Correct substitution Solves equation awrt 1.57
(ii)	$v_y^2 = (-10 \sin 30^\circ)^2 + 2(-9.8)(-20)$ $v_y = -20.421$ $v_x = 10 \cos 30^\circ = 8.6603$ $\text{speed} = \sqrt{(8.6603)^2 + (-20.421)^2}$ $= 22.181 = 22.2 \text{ m s}^{-1} \text{ (3 s.f.)}$	M1A1 A1 M1A1 M1 A1	Uses constant acceleration Accept 20.421 Finds horizontal component of velocity Uses Pythagoras
11 (i)	$t = \frac{x}{u \cos \theta}$ $y = u \sin \theta t + \frac{1}{2} \times (-10) \times t^2$ $(\Rightarrow y = u \sin \theta t - 5t^2)$ $y = u \sin \theta \times \frac{x}{u \cos \theta} - 5 \left(\frac{x}{u \cos \theta} \right)^2$ $\text{So, } y = x \tan \theta - 5 \left(\frac{x}{u \cos \theta} \right)^2$	B1 B1 M1 A1	Uses horizontal motion Uses the vertical motion Eliminates t All correct, AG
(ii)	$y = 2 + x \tan \theta - \frac{1}{2} g \left(\frac{x}{u \cos \theta} \right)^2$ $u^2 = \frac{250(\tan^2 \theta + 1)}{5 \tan \theta - 2}$	B1 M1 A1	Adapts equation of trajectory
(iii)	$u = 12.910 = 12.9 \text{ m s}^{-1} \text{ (3 s.f.)}$	B1	awrt 12.9
(iv)	u will be minimised when u^2 is minimised $\frac{(5t - 2) \times 250 \times 2t \sec^2 \theta - 250(1 + t^2) \times 5 \sec^2 \theta}{(5t - 2)^2}$ $\frac{d(u^2)}{d\theta} = 0$ $5t^2 - 4t - 5 = 0$ $t = \tan \theta = 1.47703$ $\Rightarrow \theta = \tan^{-1}(1.47703) = 55.901$ $= 55.9^\circ \text{ (3 s.f.)}$ $u = 12.153 = 12.2 \text{ (3 s.f.)}$	E1 M1 B1 A1 A1 M1 A1 M1 A1 A1 A1	Convincing explanation Attempts quotient rule ($t = \tan \theta$) $\sec^2 \theta$ as derivative of $\tan \theta$ Numerator correct Completely correct Uses condition for stationary value Correct quadratic Solves quadratic Correct value for t Evaluates θ

	Answer	Marks	Guidance
12 (i)	$R = u \cos 30^\circ \times 2T$ $\Rightarrow T = \frac{R}{u\sqrt{3}}$	M1 A1 A1	Uses horizontal motion Correct equation Correct expression
(ii)	$0 = u \sin 30^\circ - gT$ $\Rightarrow T = \frac{u}{2g}$	M1 A1 A1	Uses vertical motion Correct equation Correct expression
(iii)	$u = \frac{R}{T\sqrt{3}}$ $u = 2Tg$ $\Rightarrow T = \sqrt{\frac{R}{2\sqrt{3} \cdot g}}$ $2T = 3.4326 = 3.43 \text{ s (3 s.f.)}$	B1 B1 M1 A1 M1 A1	Correct expression Correct expression Solves simultaneous equations Correct expression

22 Friction

- Friction opposes motion or attempted motion.
- It could be in either direction, or the frictional force could be zero.
There are potentially three forces at work along the slope: the pushing force, the downhill component of the weight of the object, and the frictional force.
At one extreme, the object is on the point of moving uphill, and the frictional force will act downhill. At the other extreme, the object is on the point of moving downhill, and the frictional force will act uphill.
If the pushing force balances the downhill component of the weight of the object, then there will be no frictional force.
- $\mu > 0$ ($\mu = 0$ is only possible theoretically; usually, $\mu < 1$)
- As friction acts to oppose attempted motion, the frictional force acts up the slope in the first case, and down in the second case.



In the first case, resolving forces perpendicular to the slope:

The block is in equilibrium.

$$R_1 = mg \cos \theta + P_1 \sin \theta$$

$$\text{Limiting friction} \Rightarrow F_1 = \mu R_1$$

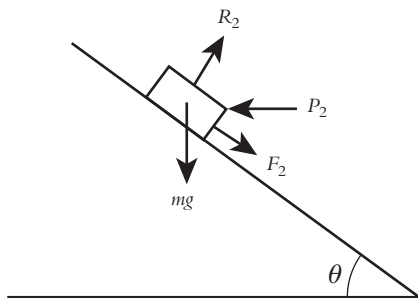
Resolving forces along the slope:

$$F_1 + P_1 \cos \theta = mg \sin \theta$$

Hence, $(mg \cos \theta + P_1 \sin \theta) + P_1 \cos \theta = mg \sin \theta$

So, $P_1 (\cos \theta + \mu \sin \theta) = mg (\sin \theta - \mu \cos \theta)$

Giving, $P_1 = \frac{mg(\sin \theta - \mu \cos \theta)}{\cos \theta + \mu \sin \theta}$



In the second case:

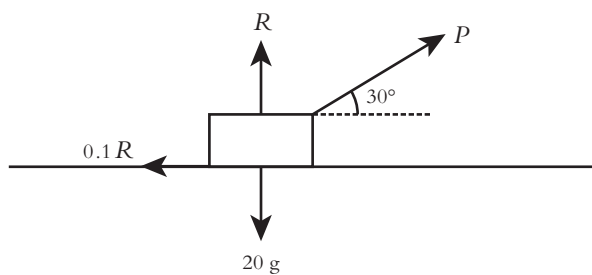
$R_2 = mg \cos \theta + P_2 \sin \theta$, $F_2 = \mu R_2$ and $F_2 + mg \sin \theta = P_2 \cos \theta$

Hence, $\mu(mg \cos \theta + P_2 \sin \theta) + mg \sin \theta = P_2 \cos \theta$

So, $P_2 (\cos \theta - \mu \sin \theta) = mg (\sin \theta + \mu \cos \theta)$

Giving, $P_2 = \frac{mg(\sin \theta + \mu \cos \theta)}{\cos \theta - \mu \sin \theta}$

5 (i)



(ii) Vertical equilibrium $\Rightarrow R + P \sin 30^\circ = 20g$

Horizontal equilibrium $\Rightarrow 0.1R = P \cos 30^\circ$

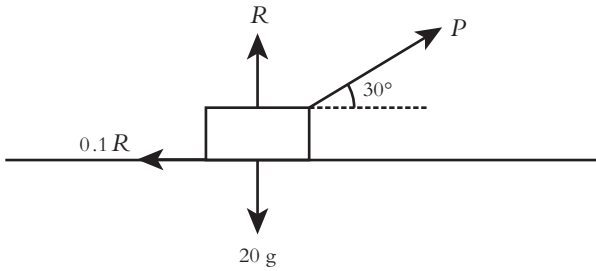
Substituting from ② into ①:

$$10P \cos 30^\circ + P \sin 30^\circ = 20g$$

$$\begin{aligned} P &= \frac{20 \times 9.8}{10\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}} \\ &= 21.397 \\ &= 21.4 \text{ N (3 s.f.)} \end{aligned}$$

Mark scheme

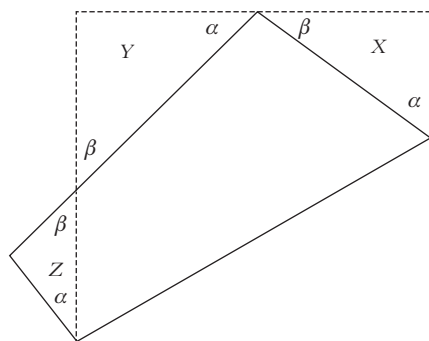
	Answer	Marks	Guidance
1	Friction opposes motion or attempted motion	B1 B1	Both ideas required for both marks
2	It could be in either direction, or the frictional force could be zero At one extreme, the object is on the point of moving uphill, and the frictional force will act downhill	B1 E1	Explains downhill option

	Answer	Marks	Guidance
	At the other extreme, the object is on the point of moving downhill, and the frictional force will act uphill	E1	Explains uphill option
	If the pushing force balances the downhill component of the weight of the object, then there will be no frictional force	E1	Explains zero option
3	$\mu > 0$	B1	
4	Frictional force acts up the slope in the first case, and down in the second case First case: $R_1 = mg \cos \theta + P_1 \sin \theta$ $F_1 = \mu R_1$ $F_1 + P \cos \theta = mg \sin \theta$ $\mu(mg \cos \theta + P_1 \sin \theta) + P_1 \cos \theta = mg \sin \theta$ $P_1(\cos \theta + \mu \sin \theta) = mg(\sin \theta - \mu \cos \theta)$ $\Rightarrow P_1 = \frac{mg(\sin \theta - \mu \cos \theta)}{\cos \theta + \mu \sin \theta}$ Second case: $F_2 + mg \sin \theta = P_2 \cos \theta$ $\mu(mg \cos \theta + P_2 \sin \theta) + mg \sin \theta = P_2 \cos \theta$ $\Rightarrow P_2(\cos \theta - \mu \sin \theta) = mg(\sin \theta + \mu \cos \theta)$ $\Rightarrow P_2 = \frac{mg(\sin \theta + \mu \cos \theta)}{\cos \theta - \mu \sin \theta}$	B1 B1 M1 A1 B1 M1A1 M1 M1 A1 M1 A1 M1 M1 A1	 Resolves perpendicular to the slope Resolves parallel to the slope Eliminates R_1 Simplifies Correct expression Resolves parallel to the slope Eliminates R_2 Simplifies Correct expression
5 (i)		B3	Deduct one mark for each error
(ii)	$R + P \sin 30^\circ = 20g$ $0.1R = P \cos 30^\circ$ $10P \cos 30^\circ + P \sin 30^\circ = 20g$ $P = \frac{20 \times 9.8}{10\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}}$ $= 21.397$ $= 21.4 \text{ N (3 s.f.)}$	M1A1 M1A1 M1A1 A1	Resolves vertically Resolves horizontally Eliminates R

23 Comprehension questions

Folding fractions

- 1 (i) Diagram shows angles that are equal in each triangle.



X is a right-angled triangle, so:

in triangle X , $\alpha + \beta = 90^\circ$, as the third angle is a right angle

in triangle Y , α (in Y) + β (in X) = 90° , as the third angle on the straight line is a right angle

β (in Y) + α (in Y) = 90° , as the third angle is a right angle

in triangle Z , β , in Z , is vertically opposite β (in Y), and α , in Z , makes up the angle sum of the triangle with the right angle.

- (ii) Side lengths of triangle X are $\frac{1}{2} = \frac{4}{8}, \frac{3}{8}, 1 - \frac{3}{8} = \frac{5}{8}$

So, the ratio of the sides is 3:4:5; as triangles are similar, they all have the same ratio of sides.

- 2 (i) $a^2 + \left(\frac{1}{5}\right)^2 = (1 - a)^2$

$$a^2 + \frac{1}{25} = 1 - 2a + a^2$$

$$2a = \frac{24}{25}$$

$$a = \frac{12}{25}$$

Using similar triangles:

$$\frac{b}{4} = \frac{\frac{1}{5}}{\frac{12}{25}}$$

$$b = \frac{4}{12} = \frac{1}{3}$$

Marking the midpoint gives $\frac{1}{6}$ as required.

- (ii) Side lengths are $\frac{1}{5} = \frac{5}{25}, \frac{12}{25}, 1 - \frac{12}{25} = \frac{13}{25}$

So, the ratio of the sides is 5:12:13; as triangles are similar, they all have the same ratio of sides.

- 3 $a^2 + \frac{1}{n^2} = 1 - 2a + a^2$

$$2a = 1 - \frac{1}{n^2} = \frac{n^2 - 1}{n^2}$$

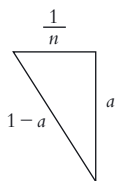
$$a = \frac{n^2 - 1}{2n^2}$$

$$\frac{b}{1 - \frac{1}{n}} = \frac{\frac{1}{n}}{a}$$

$$\frac{b}{\frac{n-1}{n}} = \frac{\frac{1}{n}}{\frac{n^2-1}{2n^2}} = \frac{\frac{1}{n}}{\frac{(n-1)(n+1)}{2n^2}}$$

$$b = \frac{\frac{1}{n}}{\frac{n+1}{2n}} = \frac{2}{n+1}$$

4 (i)



$$a = \frac{n^2 - 1}{2n^2}$$

$$1 - a = \frac{n^2 + 1}{2n^2}$$

$$\text{Ratio of sides } \frac{1}{n} : \frac{n^2 - 1}{2n^2} : \frac{n^2 + 1}{2n^2}$$

i.e. $2n : n^2 - 1 : n^2 + 1$, these are all integers assuming n is an integer.

$$\text{(ii)} \quad \lim_{n \rightarrow \infty} a = \lim_{n \rightarrow \infty} \left(\frac{n^2 - 1}{2n^2} \right) = \frac{1}{2} \text{ as required.}$$

5 Since any unit fraction can be obtained, multiples can be generated by repeating the unit.

Fermat's method of quadrature

1 From Figure 1, $k = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \dots$

$$\text{So, } kx_2 = x_1, kx_3 = x_2, \dots$$

$$x_2 - x_1 = x_1 \left(\frac{1}{k} - 1 \right),$$

$$x_3 - x_2 = x_2 \left(\frac{1}{k} - 1 \right), \dots$$

Hence,

$$\frac{x_2 - x_1}{x_3 - x_2} = \frac{x_1 \left(\frac{1}{k} - 1 \right)}{x_2 \left(\frac{1}{k} - 1 \right)} = \frac{x_1}{x_2}$$

2 From line ©:

$$\frac{x_2 - x_1}{x_3 - x_2} = \frac{x_1}{x_2}$$

$$\text{So, } \frac{y_1}{y_2} \times \frac{(x_2 - x_1)}{(x_3 - x_2)} = \frac{y_1 x_1}{y_2 x_2}$$

$$y = x^{-n} \text{ so } y_1 = x_1^{-n} \text{ and } y_2 = x_2^{-n}$$

Hence,

$$\frac{y_1 x_1}{y_2 x_2} = \frac{x_1^{-n} x_1}{x_2^{-n} x_2} = \frac{x_1^{1-n}}{x_2^{1-n}} = \left(\frac{x_2}{x_1} \right)^{n-1}$$

$$\frac{x_1}{x_2} = k$$

$$\text{So, } \frac{x_2}{x_1} = \frac{1}{k} \text{ and } \left(\frac{x_2}{x_1} \right)^{n-1} = \frac{1}{k^{n-1}}$$

- 3 Identify each area on the graph in Figure 1 to see it is the area between the curve and the x -axis from $x = x_1$ to ∞ .

4 (i) $\frac{x_1}{x_2} = \frac{x_2}{x_3} \Rightarrow x_3 = \frac{x_2^2}{x_1} = \frac{1}{0.9} = \frac{10}{9}$

(ii) $n = 2, k = \frac{0.9}{1} = 0.9$

$$\text{So, } K = k^{n-1} = 0.9$$

$$A = A_1 = y_1(x_2 - x_1) = 0.9^{-2} \times (1 - 0.9) = \frac{0.1}{0.81} = \frac{10}{81}$$

$$\text{So, } \frac{A}{1-K} = \frac{10}{81} \times \frac{1}{0.1} = \frac{100}{81}$$

5 $\int_{0.9}^{\infty} x^{-2} dx = \left[-x^{-1} \right]_{0.9}^{\infty}$

$$= 0 + \frac{1}{0.9}$$

$$= \frac{10}{9}$$

Mark scheme

	Answer	Marks	Guidance
Folding fractions			
1 (i)	Diagram showing that angles are identical in all the triangles together with statement that $\alpha + \beta = 90^\circ$ for each triangle	B1	oe
(ii)	Side lengths of triangle X are $\frac{1}{2} = \frac{4}{8}, \frac{3}{8}, 1 - \frac{3}{8} = \frac{5}{8}$ So, ratio of sides is 3:4:5; as triangles are similar, they all have the same ratio of sides	B1	
2 (i)	$a^2 + \frac{1}{25} = 1 - 2a + a^2$ $a = \frac{12}{25}$ $\frac{b}{\frac{4}{5}} = \frac{\frac{1}{5}}{\frac{12}{25}}$	M1 M1	Uses Pythagoras' theorem and finding a Uses similar triangles

	Answer	Marks	Guidance
	$b = \frac{4}{12} = \frac{1}{3}$ Marking the midpoint gives $\frac{1}{6}$ as required	A1	Finds b and deriving final fraction
(ii)	Side lengths of triangle X are $\frac{1}{5} = \frac{5}{25}, \frac{12}{25}, 1 - \frac{12}{25} = \frac{13}{25}$ So, ratio of sides is 5:12:13; as triangles are similar, they all have the same ratio of sides	B1	
3	$a = \frac{n^2 - 1}{2n^2} = \frac{(n-1)(n+1)}{2n^2}$	M1	Derives a
	$\frac{b}{n-1} = \frac{\frac{1}{n}}{\frac{(n-1)(n+1)}{2n^2}}$	M1	Substitutes for a
	$b = \frac{\frac{1}{n}}{\frac{n+1}{2n}} = \frac{2}{n+1}$	A1	Simplifies for b ; answer given
4	(i) $a = \frac{n^2 - 1}{2n^2}$	M1	Expression for a
	$1 - a = \frac{n^2 + 1}{2n^2}$	M1	Expression for hypotenuse
	Ratio of sides $\frac{1}{n} : \frac{n^2 - 1}{2n^2} : \frac{n^2 + 1}{2n^2}$ i.e. $2n : n^2 - 1 : n^2 + 1$, all whole numbers assuming n is a whole number	M1 A1	Forms the ratio States the integer form
	(ii) $\lim_{n \rightarrow \infty} a = \lim_{n \rightarrow \infty} \left(\frac{n^2 - 1}{2n^2} \right) = \frac{1}{2}$ as required	B1	
5	Explanation of finding multiples of any fraction that has been found	E1	
Fermat's method of quadrature			
1	$k = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \dots$		From lines (A) and (B)
	So, $kx_2 = x_1, kx_3 = x_2, \dots$	M1	$kx_2 = x_1$ seen, oe
	$x_2 - x_1 = x_1 \left(\frac{1}{k} - 1 \right),$		
	$x_3 - x_2 = x_2 \left(\frac{1}{k} - 1 \right), \dots$	M1	$x_2 - x_1 = x_1 \left(\frac{1}{k} - 1 \right)$ seen, oe

	Answer	Marks	Guidance
	Hence, $\frac{x_2 - x_1}{x_3 - x_2} = \frac{x_1 \left(\frac{1}{k} - 1 \right)}{x_2 \left(\frac{1}{k} - 1 \right)} = \frac{x_1}{x_2}$	A1	
2	$\frac{x_2 - x_1}{x_3 - x_2} = \frac{x_1}{x_2}$ So, $\frac{y_1}{y_2} \times \frac{(x_2 - x_1)}{(x_3 - x_2)} = \frac{y_1 x_1}{y_2 x_2}$ $y = x^{-n}$ So, $y_1 = x_1^{-n}$ and $y_2 = x_2^{-n}$ Hence, $\frac{y_1 x_1}{y_2 x_2} = \frac{x_1^{-n} x_1}{x_2^{-n} x_2} = \frac{x_1^{1-n}}{x_2^{1-n}} = \left(\frac{x_2}{x_1} \right)^{n-1}$ $\frac{x_1}{x_2} = k$ So, $\frac{x_2}{x_1} = \frac{1}{k}$ and $\left(\frac{x_2}{x_1} \right)^{n-1} = \frac{1}{k^{n-1}}$	E1 M1 E1 B1	From line © Uses line © Uses $y = x^{-n}$ Explains why Uses $\frac{x_1}{x_2} = k$ to conclude proof
3	The area between the curve and the x -axis from $x = x_1$ to ∞	B1 B1	Area under curve From ©
4 (i)	$\frac{10}{9}$	B1	1.11 or better
(ii)	$n = 2, k = \frac{0.9}{1} = 0.9$ So, $K = k^{n-1} = 0.9$ $A = A_1 = y_1(x_2 - x_1)$ $= 0.9^{-2} \times (1 - 0.9)$ $= \frac{0.1}{0.81} = \frac{10}{81}$ So, $\frac{A}{1-K} = \frac{10}{81} \times \frac{1}{0.1} = \frac{100}{81}$	B1 M1 A1	$k = 0.9$ $A = \frac{10}{81}$ $\frac{100}{81}$ or 1.23 or better
5	$\int_{0.9}^{\infty} x^{-2} dx = \left[-x^{-1} \right]_{0.9}^{\infty}$ $= 0 + \frac{1}{0.9} = \frac{10}{9}$	M1 A1	$\left[-x^{-1} \right]_{0.9}^{\infty}$ $\frac{10}{9}$ or 1.11 or better