

Answers

1 Problem solving

1 The basket was half full at 10:59.

2 \$3.50

Work backwards, use guess and check or algebra.

3 Let x be the number of consecutive shots.

$$\begin{aligned}\frac{3+x}{7+x} &> \frac{16+x}{25+x} \\ (3+x)(25+x) &> (16+x)(7+x) \\ 75+28x+x^2 &> 112+23x+x^2 \\ 5x &> 37 \\ x &> 7.4\end{aligned}$$

So he needs 8 consecutive successful shots.

$$\begin{aligned}4 \quad \frac{1}{x+5} = 4 &\Rightarrow x+5 = \frac{1}{4} \\ x+8 &= 3\frac{1}{4} \\ x+8 &= \frac{13}{4} \\ \frac{1}{x+8} &= \frac{4}{13}\end{aligned}$$

$$5 \quad \frac{pd}{p+r}$$

6 The difference in weight represents half the contents

$$\text{so } \frac{1}{2}c = 24 - 13 \Rightarrow c = 22 \text{ kg.}$$

So the suitcase weighs 2 kg.

7 The area of the cross-section of the water will be the same.

$$\text{Area } \triangle ADC = \frac{1}{2} \times 50 \times 40 = 1000 \text{ cm}^2$$

Let the height be h .

$$100h = 1000$$

$$h = 10 \text{ cm}$$

8 Clearly the number of correct answers must be even, and greater than 10 or the total would be less than 48.

Correct	Wrong	Unanswered	Total
10	1	9	48
12	6	2	48
14	11	X	X

From the table we see that the greatest number of questions answered correctly is 12.

9 Let the even number be $2k$, where k is an integer.

$$(2k)^2 = 4k^2$$

Since any multiple of 4 is even, the square of any even number is even.

10 Let the odd number be $2k + 1$, where k is an integer.

$$\begin{aligned}(2k+1)^2 &= 4k^2 + 4k + 1 \\ &= 4(k^2 + k) + 1\end{aligned}$$

This is an even number + 1, which is odd.

Further practice (Page 4)

1 One friend has x marbles.

The second has $x - 2$ marbles.

$$x + 1 = 2(x - 3)$$

$$x = 7$$

$$x - 2 = 5$$

The friends have 12 marbles in total.

2 Looking at all of the possible sets of three ages to give a product of 72:

Ages			Sum
1	2	36	39
1	3	24	28
1	4	18	23
1	6	12	19
1	8	9	18
2	2	18	22
2	3	12	17
2	4	9	15
2	6	6	14
3	3	8	14
3	4	6	13

Since there is some doubt about the answer after the clue about Alma's house number, the sum of the ages must be 14.

The fact that Peta says she has an oldest child means the answer must be 3, 3, 8.

- 3 There is no reason to add the \$270 and \$20 here. You can do the calculations this way:

Actual cost of suite: \$250

Money returned: \$30 to men
\$20 to lift attendant

Total \$300

- 4 Speed = $\frac{\text{distance}}{\text{time}}$

For the 'against the wind' journey:

Distance = $300 \times 2.5 = 750$ km

For the 'with the wind' journey:

Time = $\frac{750}{400} = 1.875$ hours = 112.5 minutes

- 5 Sanil's watch is moving at $\frac{5}{6}$ of the speed of a normal watch.

Hence a normal watch is going $\frac{6}{5}$ times faster than Sanil's watch.

Sunil's watch moves through 8 hours between 9 a.m. and 5 p.m.

Hence a normal watch will move through

$\frac{6}{5} \times 8 = 9\frac{3}{5}$ hours = 9 hours 36 minutes

Hence the actual time is 6.36 p.m.

- 6 If Kyle, Leon or Ryan committed the crime then more than one suspect would be telling the truth.

If Dave committed the crime then Dave is lying, Kyle is lying, Ryan is telling the truth and Leon is lying.

Answer: Dave

- 7 Let the total volume of the bath be b .

Hot tap fills $\frac{b}{12}$ of the bath per minute.

Cold tap fills $\frac{b}{6}$ of the bath per minute.

Together they fill $\frac{b}{12} + \frac{b}{6} = \frac{b}{4}$ per minute.

Together they fill $\frac{1}{4}$ of the bath in one minute so it takes 4 minutes to fill the bath.

OR

The hot tap will fill 1 bath in 12 minutes.

The cold tap will fill 2 baths in 12 minutes.

Together they will fill 3 baths in 12 minutes.

So 1 bath in 4 minutes.

- 8 Let the distance be d .

$$\frac{d}{4} = \frac{d}{6} + \frac{3\frac{3}{4}}{60}$$

$$15d = 10d + 3\frac{3}{4}$$

$$5d = 3\frac{3}{4}$$

$$d = \frac{3}{4} \text{ km}$$

- 9 After the first refilling there is 9 litres of water and 36 litres of juice.

The fraction that is juice is $\frac{36}{45} = \frac{4}{5}$.

When 9 litres are removed, the amount of juice removed is $\frac{4}{5} \times 9 = \frac{36}{5} = 7.2$ litres

Volume of juice in the final mix is

$$45 - 9 - 7.2 = 28.8 \text{ litres}$$

Volume of water in the final mix is

$$45 - 28.8 = 16.2 \text{ litres}$$

Ratio is $16.2 : 28.8 = 9 : 16$

- 10 Let $a = 2k + 1$ and $b = 2m + 1$ where k and m are integers.

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ &= (2k+1+2m+1)(2k+1-(2m+1)) \\ &= (2k+2m+2)(2k-2m) \\ &= 2(k+m+1)2(k-m) \\ &= 4(k+m+1)(k-m) \end{aligned}$$

If k and m are either both even or both odd, $k - m$ is even and if one of these numbers is odd and the other is even, then $k + m + 1$ is even. In both cases, the number $(k+m+1)(k-m)$ is even.

So $(k+m+1)(k-m) = 2p$, where p is an integer.

So $4(k+m+1)(k-m) = 4 \times 2p = 8p$,

which is a multiple of 8.

2 Algebra

2.1 Background algebra (Page 5)

$$\begin{aligned} 1 \quad (i) \quad & 2(a-3b) - 3(b-3a) \\ &= 2a - 6b - 3b + 9a \\ &= 11a - 9b \end{aligned}$$

$$\begin{aligned} (ii) \quad & 7cd(d^2 - 2) - 3cd^2(8d + 5c^3) \\ &= 7cd^3 - 14cd - 24cd^3 - 15c^4d^2 \\ &= -17cd^3 - 14cd - 15c^4d^2 \end{aligned}$$

$$(iii) \frac{8f^4}{3g} \times \frac{9g^3}{12fg}$$

$$= 2f^3g$$

$$(iv) \frac{x-1}{4} + \frac{5-x}{5}$$

$$= \frac{5(x-1)}{20} + \frac{4(5-x)}{20}$$

$$= \frac{5x-5+20-4x}{20}$$

$$= \frac{x+15}{20}$$

$$2 \quad (i) \quad 12mn^2 + 9mn^3 = 3mn^2(4 + 3n)$$

$$(ii) \quad p^2 - p - 12 = (p-4)(p+3)$$

$$3 \quad (i) \quad 2(x+5) = x-7$$

$$2x+10 = x-7$$

$$x = -17$$

$$(ii) \quad \frac{1}{2}(6x+8) - 3 = 9 - \frac{3}{2}(4-10x)$$

$$3x+4-3 = 9-6+15x$$

$$3x+1 = 3+15x$$

$$-12x = 2$$

$$x = \frac{2}{-12} = -\frac{1}{6}$$

4 Let the distance from Auckland to Hamilton be x .

Hakim's speed is $\frac{x}{2}$ km/h

Ravi's speed is $\left(\frac{x}{2} - 4\right)$ km/h

Distance = speed $\times$ time so

$$\frac{x}{2} \times 2 = \left(\frac{x}{2} - 4\right) \times 2 \frac{1}{10}$$

$$x = 1.05x - 8.4$$

$$8.4 = 0.05x$$

$$x = 168 \text{ km}$$

$$5 \quad (i) \quad \frac{v}{b} - c = \frac{d}{e}$$

$$\frac{v-bc}{b} = \frac{d}{e}$$

$$\frac{b}{v-bc} = \frac{e}{d}$$

$$e = \frac{bd}{v-bc}$$

$$(ii) \quad km^2 + n = p - wk$$

$$km^2 + wk = p - n$$

$$k(m^2 + w) = p - n$$

$$k = \frac{p-n}{m^2+w}$$

$$6 \quad (i) \quad \sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$$

$$(ii) \quad \sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$$

$$(iii) \quad \sqrt{1\frac{7}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3}$$

$$7 \quad (i) \quad \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$(ii) \quad \frac{3}{\sqrt{7}-1} = \frac{3}{\sqrt{7}-1} \times \frac{\sqrt{7}+1}{\sqrt{7}+1}$$

$$= \frac{3\sqrt{7}+3}{7-1}$$

$$= \frac{3\sqrt{7}+3}{6}$$

$$= \frac{\sqrt{7}+1}{2}$$

$$(iii) \quad \frac{1+\sqrt{3}}{1-\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{1+2\sqrt{3}+3}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= -2-\sqrt{3}$$

$$8 \quad (i) \quad (\sqrt{2}+1)(\sqrt{2}-1) = 2-1=1$$

$$(ii) \quad (1-\sqrt{b})(2+\sqrt{b}) = 2-\sqrt{b}-b$$

2.2 Quadratic equations (Page 7)

$$1 \quad (i) \quad x^2 + 5x = 0$$

$$x(x+5) = 0$$

$$x = 0 \text{ or } -5$$

$$(ii) \quad x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4 \text{ or } -2$$

$$(iii) \quad 2x^2 - x - 3 = 0$$

$$(2x-3)(x+1) = 0$$

$$x = \frac{3}{2} \text{ or } -1$$

$$(iv) \quad 3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$$x = 0 \text{ or } 2$$

$$(v) \quad x^2 - 3x - 40 = 0$$

$$(x+5)(x-8) = 0$$

$$x = -5 \text{ or } 8$$

$$(vi) \quad 6x^2 + 7x - 3 = 0$$

$$(3x-1)(2x+3) = 0$$

$$x = \frac{1}{3} \text{ or } -\frac{3}{2}$$

$$2 \quad (i) \quad \text{Let } t = x^2.$$

$$t^2 + 3t - 4 = 0$$

$$(t+4)(t-1) = 0$$

$$t = -4 \text{ or } 1$$

$$x^2 = -4 \text{ or } x^2 = 1$$

$$x = \pm 1$$

(ii) Multiply by x .

$$5x - 2 = 2x^2$$

$$0 = 2x^2 - 5x + 2$$

$$0 = (2x - 1)(x - 2)$$

$$x = \frac{1}{2} \text{ or } 2$$

(iii) Let $t = \sqrt{x}$.

$$t^2 + 2t = 8$$

$$t^2 + 2t - 8 = 0$$

$$(t + 4)(t - 2) = 0$$

$$t = -4 \text{ or } 2$$

$$\sqrt{x} = -4 \text{ or } \sqrt{x} = 2$$

$$x = 4$$

(iv) Let $t = x^3$.

$$t^2 + 8 = 9t$$

$$t^2 - 9t + 8 = 0$$

$$(t - 1)(t - 8) = 0$$

$$t = 1 \text{ or } 8$$

$$x^3 = 1 \text{ or } x^3 = 8$$

$$x = 1 \text{ or } x = 2$$

$$(ii) \quad 3x^2 + 12x - 4$$

$$= 3(x^2 + 4x) - 4$$

$$= 3[(x + 2)^2 - 4] - 4$$

$$= 3(x + 2)^2 - 12 - 4$$

$$= 3(x + 2)^2 - 16$$

$$(iii) \quad 4x^2 + 24x - 16$$

$$= 4(x^2 + 6x - 4)$$

$$= 4[(x + 3)^2 - 13]$$

$$= 4(x + 3)^2 - 52$$

$$(iv) \quad 9x^2 - 6x$$

$$= 9\left(x^2 - \frac{6}{9}x\right)$$

$$= 9\left(x^2 - \frac{2}{3}x\right)$$

$$= 9\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9}\right]$$

$$= 9\left(x - \frac{1}{3}\right)^2 - 1$$

$$4 \quad 3 - 8x - x^2$$

$$= -(x^2 + 8x - 3)$$

$$= -[(x + 4)^2 - 19]$$

$$= -(x + 4)^2 + 19$$

$$= 19 - (x + 4)^2$$

$$a = 19, b = 4$$

2.3 Completing the square (Page 8)

$$1 \quad (i) \quad (x - 3)^2 - 8$$

$$(ii) \quad (x + 2)^2 - 4$$

$$2 \quad (i) \quad x^2 - 6x + 1 = 0$$

$$(x - 3)^2 - 8 = 0$$

$$(x - 3)^2 = 8$$

$$x - 3 = \pm\sqrt{8}$$

$$x = 3 \pm \sqrt{8}$$

$$(ii) \quad x^2 + 4x = 0$$

$$(x + 2)^2 - 4 = 0$$

$$(x + 2)^2 = 4$$

$$x + 2 = \pm 2$$

$$x = 0 \text{ or } -4$$

$$3 \quad (i) \quad 2x^2 - 4x + 7$$

$$= 2(x^2 - 2x) + 7$$

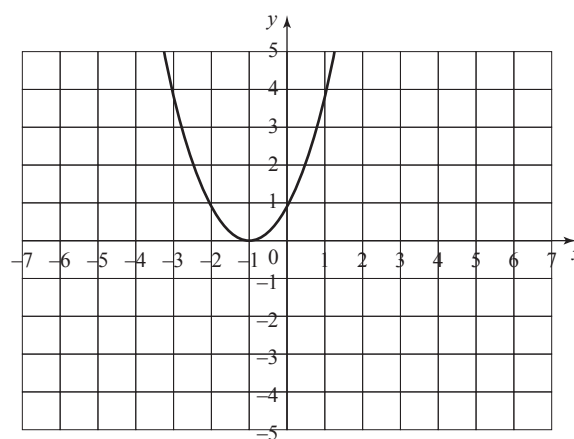
$$= 2[(x - 1)^2 - 1] + 7$$

$$= 2(x - 1)^2 - 2 + 7$$

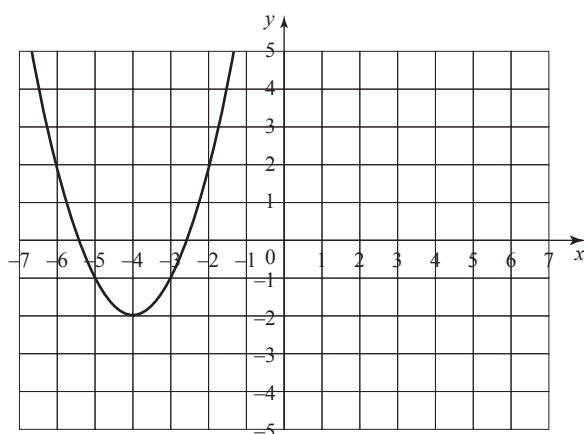
$$= 2(x - 1)^2 + 5$$

2.4 The graphs of quadratic functions (Page 9)

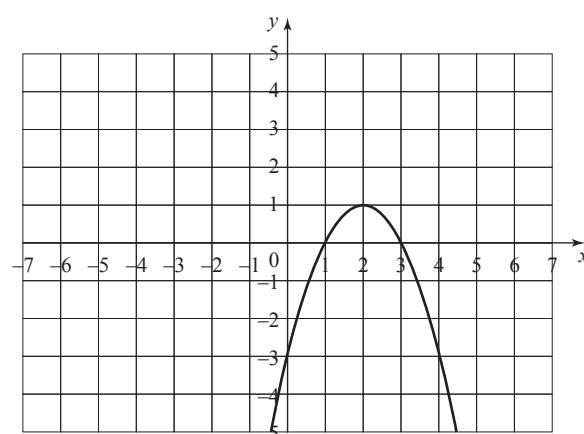
1 (i)

Vertex $(-1, 0)$

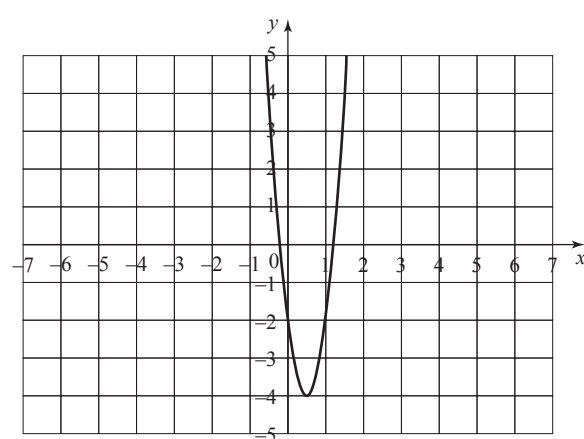
(ii)

Vertex $(-4, -2)$

(iii)

Vertex $(2, 1)$

(iv)

Vertex $(\frac{1}{2}, -4)$

2 (i) $y = (x-4)^2 - 2$

(ii) $y = -(x-1)^2 + 5$

2.5 The quadratic formula (Page 10)

1 (i) $x^2 + x - 5 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -5}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{21}}{2}$$

$$= 1.79 \text{ or } -2.79 \text{ (3 s.f.)}$$

(ii) $2x^2 - 5x = 6$

$2x^2 - 5x - 6 = 0$

$$x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times -6}}{2 \times 2}$$

$$= \frac{5 \pm \sqrt{73}}{4}$$

$$= -0.886 \text{ or } 3.39 \text{ (3 s.f.)}$$

2 (i) Discriminant $= (-2)^2 - 4 \times 1 \times 4 = -12$

No solutions

(ii) Discriminant $= 4^2 - 4 \times 4 \times 1 = 0$

One solution

3 (i) (a) $4^2 - 4 \times k \times -1 = 0$

$16 + 4k = 0$

$k = -4$

(b) $k^2 - 4 \times 1 \times (k-1) = 0$

$k^2 - 4k + 4 = 0$

$(k-2)(k-2) = 0$

$k = 2$

(ii) (a) $(-2)^2 - 4 \times 1 \times k > 0$

$4 - 4k > 0$

$4 > 4k$

$1 > k$

$k < 1$

(b) $(-k)^2 - 4 \times k \times 1 > 0$

$k^2 - 4k > 0$

$k(k-4) > 0$

$k < 0 \text{ or } k > 4$

(iii) (a) $(-2k)^2 - 4 \times 2 \times 1 < 0$

$4k^2 - 8 < 0$

$4k^2 < 8$

$k^2 < 2$

$-\sqrt{2} < k < \sqrt{2}$

(b) $(-4)^2 - 4 \times k \times 2k < 0$

$16 - 8k^2 < 0$

$16 < 8k^2$

$2 < k^2$

$k > \sqrt{2} \text{ or } k < -\sqrt{2}$

4 (i) Roots of 6 and $-2 \Rightarrow (x-6)(x+2) = 0$

$x^2 - 4x - 12 = 0 \Rightarrow m = -4, n = -12$

$$\begin{aligned}
 \text{(ii)} \quad x^2 - 4x - 12 &= p \Rightarrow x^2 - 4x - 12 - p = 0 \\
 b^2 - 4ac &= 0 \\
 (-4)^2 - 4 \times 1 \times (-12 - p) &= 0 \\
 16 + 48 + 4p &= 0 \\
 64 + 4p &= 0 \\
 p &= -16
 \end{aligned}$$

2.6 Simultaneous equations (Page 12)

1 (i) $x = 3, y = -1$

(ii) $x = -2, y = 5$

2 (i) $x^2 + 4x = x - 2$
 $x^2 + 3x + 2 = 0$
 $(x + 2)(x + 1) = 0$
 $x = -1$ or -2
 $y = -3$ or -4
 $(-1, -3)$ or $(-2, -4)$

(ii) $x^2 + 4x - 2 = -2x - 2$
 $x^2 + 6x = 0$
 $x(x + 6) = 0$
 $x = 0$ or -6
 $y = -2$ or 10
 $(0, -2)$ or $(-6, 10)$

(iii) $x^2 + (-x - 1)^2 = 25$
 $x^2 + x^2 + 2x + 1 = 25$
 $2x^2 + 2x - 24 = 0$
 $x^2 + x - 12 = 0$
 $(x + 4)(x - 3) = 0$
 $x = -4$ or 3
 $y = 3$ or -4
 $(-4, 3)$ or $(3, -4)$

(iv) $x(7 - 2x) = 3$
 $7x - 2x^2 = 3$
 $0 = 2x^2 - 7x + 3$
 $0 = (2x - 1)(x - 3)$
 $x = \frac{1}{2}$ or 3
 $y = 6$ or 1
 $(\frac{1}{2}, 6)$ or $(3, 1)$

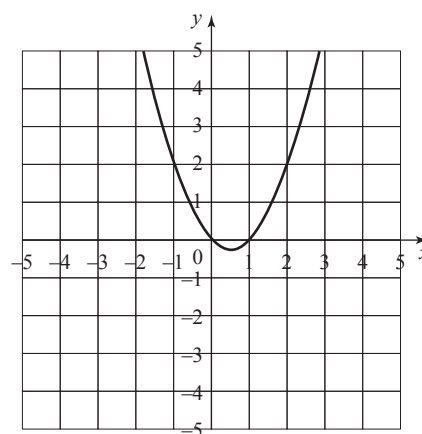
2.7 Inequalities (Page 13)

1 (i) $3(x + 4) \leq -15$
 $x + 4 \leq -5$
 $x \leq -9$

(ii) $1 - \frac{3x}{4} > 7$
 $4 - 3x > 28$
 $-3x > 24$
 $x < -8$

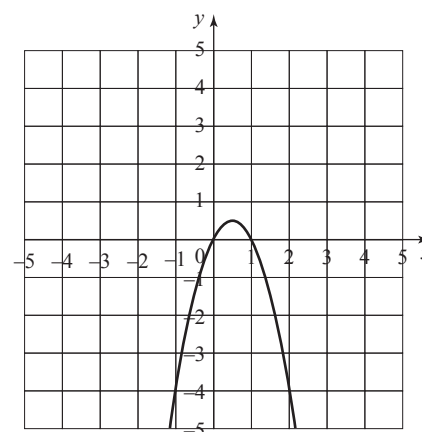
(iii) $-4x - 1 < -x + 5$ $-4x - 1 < -x + 5$
 $-3x < 6$ or $6 < 3x$
 $x > -2$ $-2 < x$

2 (i)



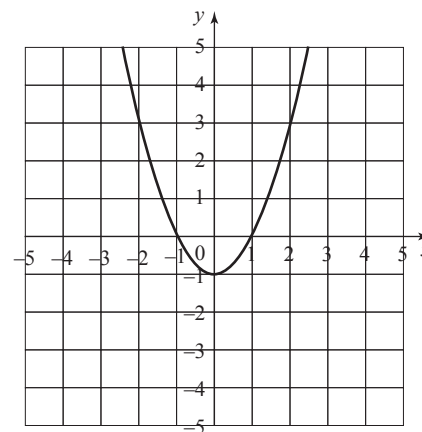
$x < 0$ or $x > 1$

(ii)



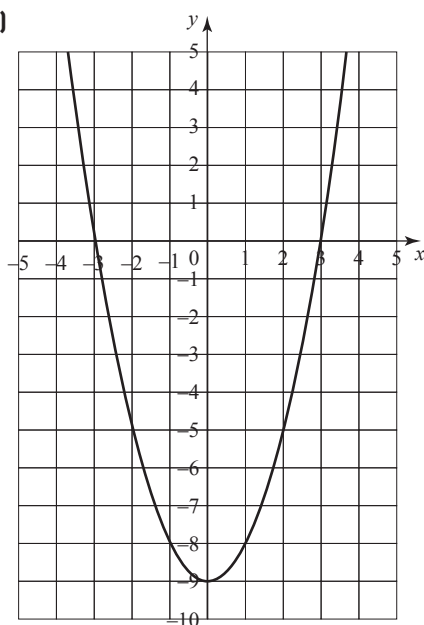
$0 \leq x \leq 1$

(iii)



$x < -1$ or $x > 1$

(iv)



$$x^2 < 9$$

$$x^2 - 9 < 0$$

$$-3 < x < 3$$

Further practice (Page 14)

1 (i) $(3q-1)(q+2)$

(ii) $t(s+p) - 2u(s+p)$

$$= (t-2u)(s+p)$$

2 (i) $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

$$\frac{1}{f} = \frac{v+u}{uv}$$

$$uv = f(v+u)$$

$$uv = fv + fu$$

$$uv - fv = fu$$

$$v(u-f) = fu$$

$$v = \frac{fu}{u-f}$$

(ii) $\sqrt{d-3e} = \frac{1}{2\pi} \sqrt{\frac{p}{w}}$

$$d-3e = \frac{1}{4\pi^2} \left(\frac{p}{w} \right)$$

$$d - \frac{1}{4\pi^2} \left(\frac{p}{w} \right) = 3e$$

$$e = \frac{d}{3} - \frac{p}{12\pi^2 w}$$

3 (i) $x^2 - 25 = 0$

$$x^2 = 25$$

$$x = \pm\sqrt{25}$$

$$x = 5 \text{ or } -5$$

(ii) $x^2 + 5x - 14 = 0$

$$(x+7)(x-2) = 0$$

$$x = -7 \text{ or } 2$$

(iii) $3x^2 - 5x - 2 = 0$

$$(3x+1)(x-2) = 0$$

$$x = -\frac{1}{3} \text{ or } 2$$

(iv) $5x^2 + 13x - 6 = 0$

$$(5x-2)(x+3) = 0$$

$$x = \frac{2}{5} \text{ or } -3$$

(v) Multiply by x^4 :

$$3 - 11x^2 = 4x^4$$

$$0 = 4x^4 + 11x^2 - 3$$

$$0 = (4x^2 - 1)(x^2 + 3)$$

$$x^2 = \frac{1}{4} \text{ or } x^2 = -3$$

$$x = \pm\frac{1}{2}$$

(vi) Multiply by x :

$$2 - x = 4x - 9\sqrt{x}$$

$$0 = 5x - 9\sqrt{x} - 2$$

Let $t = \sqrt{x}$.

$$0 = 5t^2 - 9t - 2$$

$$0 = (5t+1)(t-2)$$

$$t = -\frac{1}{5} \text{ or } t = 2$$

$$\sqrt{x} = -\frac{1}{5} \text{ or } \sqrt{x} = 2$$

$$x = 4$$

4 (i) $2x^2 + 12x + 11$

$$= 2(x^2 + 6x) + 11$$

$$= 2[(x+3)^2 - 9] + 11$$

$$= 2(x+3)^2 - 18 + 11$$

$$= 2(x+3)^2 - 7$$

(ii) $5x^2 - 40x + 72$

$$= 5(x^2 - 8x) + 72$$

$$= 5[(x-4)^2 - 16] + 72$$

$$= 5(x-4)^2 - 80 + 72$$

$$= 5(x-4)^2 - 8$$

5 $7 + 8x - 2x^2$

$$= -2x^2 + 8x + 7$$

$$= -2(x^2 - 4x) + 7$$

$$= -2[(x-2)^2 - 4] + 7$$

$$= -2(x-2)^2 + 8 + 7$$

$$= -2(x-2)^2 + 15$$

$$= 15 - 2(x-2)^2$$

$$a = 15, b = 2, c = 2$$

6 (i) $y = x(x+2) = (x+1)^2 - 1$

(ii) $y = -(x-3)^2 + 4$

7 (i) $1 - 3x^2 = 5x$

$$0 = 3x^2 + 5x - 1$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times -1}}{2 \times 3}$$

$$= \frac{-5 \pm \sqrt{37}}{6}$$

$$= 0.180 \text{ or } -1.85 \text{ (3 s.f.)}$$

(ii) $12x = 6x^2 - 5$

$$0 = 6x^2 - 12x - 5$$

$$x = \frac{12 \pm \sqrt{12^2 - 4 \times 6 \times -5}}{2 \times 6}$$

$$= \frac{12 \pm \sqrt{264}}{12}$$

$$= -0.354 \text{ or } 2.35$$

8 $\sqrt{90} = \sqrt{9 \times 10} = \sqrt{9} \times \sqrt{10} = 3\sqrt{10}$
so $a = 3$

9 (i) Discriminant $= (-3)^2 - 4 \times -1 \times -2 = 1$
Two solutions

(ii) Discriminant $= 0^2 - 4 \times 1 \times 4 = -16$
No solutions

(iii) Discriminant $= (-1)^2 - 4 \times 5 \times 2 = -39$
No solutions

(iv) Discriminant $= 3^2 - 4 \times -4 \times 0 = 9$
Two solutions

10 (i) (a) $k^2 - 4 \times 1 \times 4 = 0$

$$k^2 - 16 = 0$$

$$k^2 = 16$$

$$k = 4 \text{ or } -4$$

(b) $kx^2 - kx - 1 = 0$

$$(-k)^2 - 4 \times k \times -1 = 0$$

$$k^2 + 4k = 0$$

$$k(k+4) = 0$$

$$k = 0 \text{ or } -4$$

(ii) (a) $(-k)^2 - 4 \times 3 \times 3 > 0$

$$k^2 - 36 > 0$$

$$k^2 > 36$$

$$k < -6 \text{ or } k > 6$$

(b) $3 - 2kx^2 = 6x$

$$2kx^2 + 6x - 3 = 0$$

$$6^2 - 4 \times 2k \times -3 > 0$$

$$36 + 24k > 0$$

$$24k > -36$$

$$k > \frac{-36}{24}$$

$$k > -\frac{3}{2}$$

(iii) (a) $(-1)^2 - 4 \times 9 \times k < 0$

$$1 - 36k < 0$$

$$1 < 36k$$

$$\frac{1}{36} < k$$

$$k > \frac{1}{36}$$

(b) $kx^2 + 2x - 1 = 0$

$$2^2 - 4 \times k \times -1 < 0$$

$$4 + 4k < 0$$

$$4k < -4$$

$$k < -1$$

11 (i) $x = 3, y = 12$

(ii) $x = 6, y = -3$

12 (i) $x^2 + 9 = 4x + 5$

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2) = 0$$

$$x = 2, y = 13$$

$$(2, 13)$$

(ii) $(x-2)^2 + (9-2x)^2 = 5$

$$x^2 - 4x + 4 + (81 - 36x + 4x^2) = 5$$

$$5x^2 - 40x + 80 = 0$$

$$x^2 - 8x + 16 = 0$$

$$(x-4)^2 = 0$$

$$x = 4, y = 1$$

$$(4, 1)$$

(iii) $x^2 + 2(1-x)^2 = 9$

$$x^2 + 2(1-2x+x^2) = 9$$

$$x^2 + 2 - 4x + 2x^2 = 9$$

$$3x^2 - 4x - 7 = 0$$

$$(3x-7)(x+1) = 0$$

$$x = \frac{7}{3} \text{ or } -1$$

$$y = -\frac{4}{3} \text{ or } 2$$

$$\left(\frac{7}{3}, -\frac{4}{3}\right) \text{ or } (-1, 2)$$

(iv) $2x + 3y = 7 \Rightarrow y = \frac{7-2x}{3}$

$$3x^2 = 4 + 4x\left(\frac{7-2x}{3}\right)$$

$$3x^2 = 4 + \frac{28x}{3} - \frac{8x^2}{3}$$

$$9x^2 = 12 + 28x - 8x^2$$

$$17x^2 - 28x - 12 = 0$$

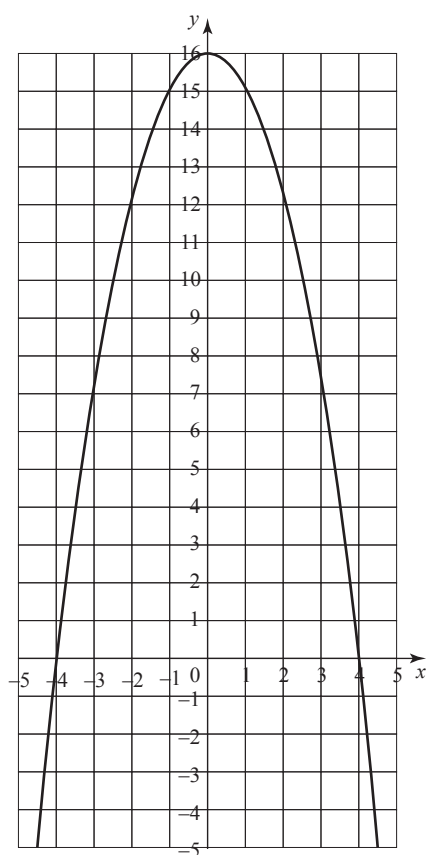
$$(17x+6)(x-2) = 0$$

$$x = -\frac{6}{17} \text{ or } 2$$

$$y = \frac{131}{51} \text{ or } 1$$

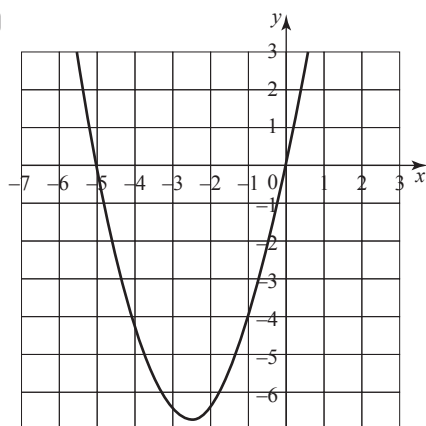
$$\left(-\frac{6}{17}, \frac{131}{51}\right) \text{ or } (2, 1)$$

13 (i)



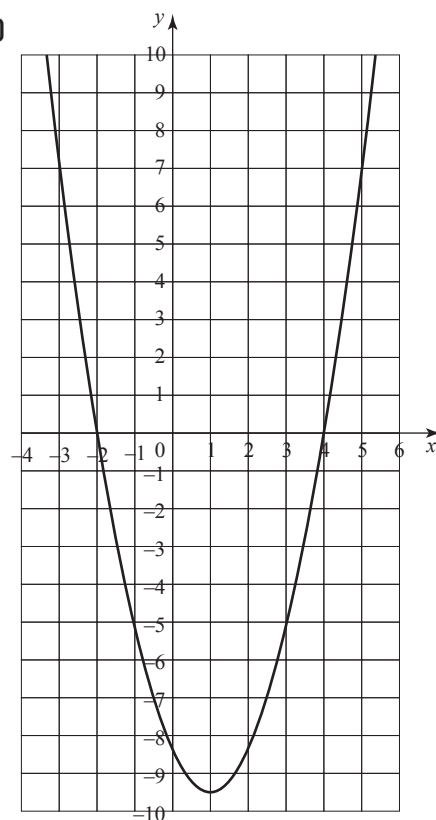
$$-4 \leq x \leq 4$$

(ii)



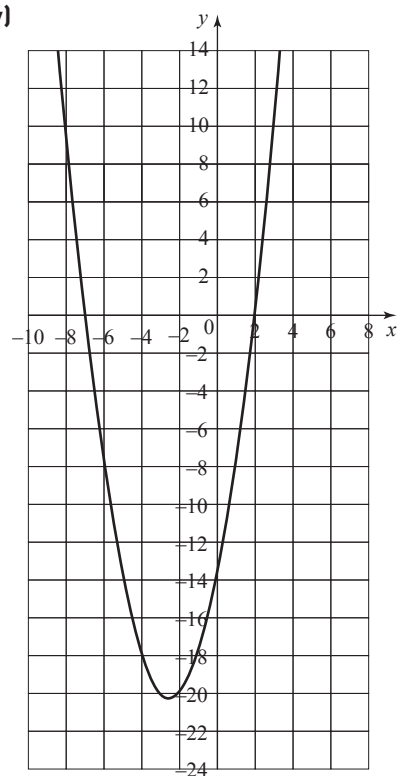
$$x < -5 \text{ or } x > 0$$

(iii)



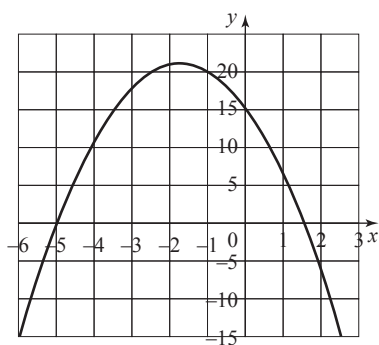
$$-2 \leq x \leq 4$$

(iv)



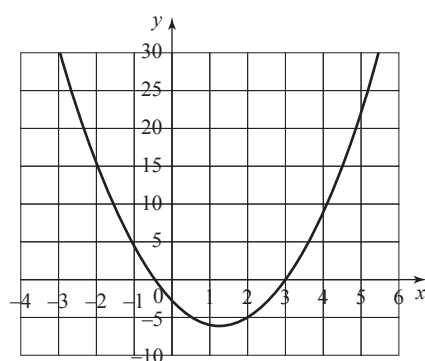
$$x < -7 \text{ or } x > 2$$

(v)



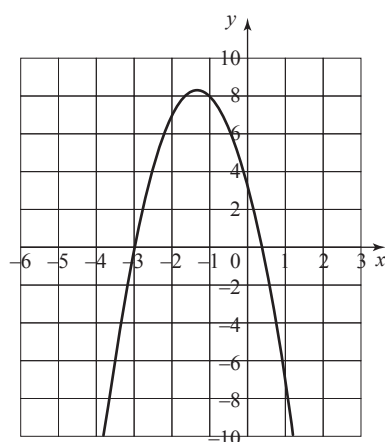
$$x < -5 \text{ or } x > \frac{3}{2}$$

(vi)



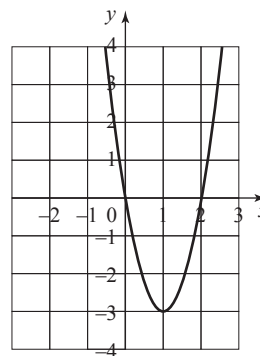
$$-\frac{1}{2} < x < 3$$

(vii)



$$-3 \leq x \leq \frac{1}{3}$$

$$\text{(viii)} \quad 3x^2 > 6x \Rightarrow 3x^2 - 6x > 0$$



$$x < 0 \text{ or } x > 2$$

$$\begin{aligned} \text{14 (i)} \quad & 4x^2 + 32x + 70 \\ &= 4(x^2 + 8x) + 70 \\ &= 4[(x+4)^2 - 16] + 70 \\ &= 4(x+4)^2 - 64 + 70 \\ &= 4(x+4)^2 + 6 \\ &\text{Vertex is } (-4, 6) \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & 4(x+4)^2 + 6 < 22 \\ & 4(x+4)^2 < 16 \\ & (x+4)^2 < 4 \\ & x^2 + 8x + 12 < 0 \\ & (x+6)(x+2) < 0 \\ & -6 < x < -2 \end{aligned}$$

Past exam questions (Page 15)

$$\begin{aligned} \text{1} \quad & (x-2)(x+1) > 0 \\ & x < -1 \text{ or } x > 2 \\ \text{2 (i)} \quad & x^2 + 6x + 2 = (x+3)^2 - 7 \\ \text{(ii)} \quad & (x+3)^2 - 7 > 9 \\ & (x+3)^2 > 16 \\ & x+3 > 4 \text{ or } x+3 < -4 \\ & x > 1 \text{ or } x < -7 \end{aligned}$$

$$\begin{aligned} \text{3 (i)} \quad & 2x^2 - 10x + 8 = 2(x^2 - 5x) + 8 \\ &= 2\left[\left(x - \frac{5}{2}\right)^2 - \frac{25}{4}\right] + 8 \\ &= 2\left(x - \frac{5}{2}\right)^2 - \frac{25}{2} + 8 \\ &= 2\left(x - \frac{5}{2}\right)^2 - \frac{9}{2} \end{aligned}$$

(ii) $2x^2 - 10x + 8 = kx$

$$2x^2 - 10x - kx + 8 = 0$$

$$2x^2 + (-10 - k)x + 8 = 0$$

$$(-10 - k)^2 - 4 \times 2 \times 8 < 0$$

$$100 + 20k + k^2 - 64 < 0$$

$$k^2 + 20k + 36 < 0$$

$$(k + 18)(k + 2) < 0$$

$$-18 < k < -2$$

Stretch and challenge (Page 16)

1 (i) $ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Given the roots, you can write the equation as

$$(x - \alpha)(x - \beta) = 0$$

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Equating, you get

$$-(\alpha + \beta) = \frac{b}{a} \Rightarrow \alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

(ii) $4x^2 + (k + 2)x + 72 = 0$

The roots are α and 2α .

$$\alpha + 2\alpha = -\frac{k + 2}{4} \text{ and } \alpha \times 2\alpha = \frac{72}{4}$$

Solving the second equation,

$$2\alpha^2 = 18$$

$$\alpha^2 = 9$$

$$\alpha = \pm 3$$

Substituting into the first equation you get

$$3 + 2 \times 3 = -\frac{k + 2}{4} \Rightarrow k = -38$$

$$\text{or } (-3) + 2 \times (-3) = -\frac{k + 2}{4} \Rightarrow k = 34$$

(iii) $\alpha + \beta = \frac{4}{3}$ and $\alpha\beta = \frac{7}{3}$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{4}{3}}{\frac{7}{3}} = \frac{4}{7} \text{ and } \frac{1}{\alpha\beta} = \frac{3}{7}$$

So the equation is

$$x^2 - \frac{4}{7}x + \frac{3}{7} = 0 \text{ or } 7x^2 - 4x + 3 = 0$$

(iv) $\alpha + \beta = 2$ and $\alpha\beta = 3$

$$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$

$$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$$

Rearranging, you get

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= 2^3 - 3(3)(2) = -10$$

$$(\alpha\beta)^3 = 3^3 = 27$$

The equation is $x^2 + 10x + 27 = 0$

2 $9^x - 3^{x+1} - 54 = 0$

$$(3^x)^2 - 3 \times 3^x - 54 = 0$$

Let $t = 3^x$.

$$t^2 - 3t - 54 = 0$$

$$(t + 6)(t - 9) = 0$$

$$t = -6 \text{ or } 9$$

$$3^x = -6 \text{ or } 9$$

$$3^x = -6 \text{ has no solutions so } 3^x = 9 \Rightarrow x = 2$$

3 $k2^x + 2^{-x} = 8$

Let $t = 2^x$; the equation can then be written as

$$kt + \frac{1}{t} = 8$$

$$kt^2 + 1 = 8t$$

$$kt^2 - 8t + 1 = 0$$

For a single solution, $b^2 - 4ac = 0$

$$(-8)^2 - 4 \times k \times 1 = 0 \Rightarrow k = 16$$

$$16t^2 - 8t + 1 = 0$$

$$(4t - 1)^2 = 0 \Rightarrow t = \frac{1}{4}$$

$$2^x = \frac{1}{4} \Rightarrow x = -2$$

3 Coordinate geometry

3.1 The length, gradient and midpoint of a line

(Page 17)

1 (i) $m = 3$

(ii) $m = 0$

(iii) $m = -\frac{1}{4}$

2 (i) Length = $AB = \sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47$

Midpoint = $(1, 0)$

Gradient = $\frac{1}{2}$

(ii) Length = $CD = \sqrt{4^2 + 10^2} = \sqrt{116} \approx 10.8$

Midpoint = $(10, 2)$

Gradient = $-\frac{10}{4} = -\frac{5}{2} = -2.5$

3 $\frac{-1 + m}{2} = 2 \Rightarrow m = 5$

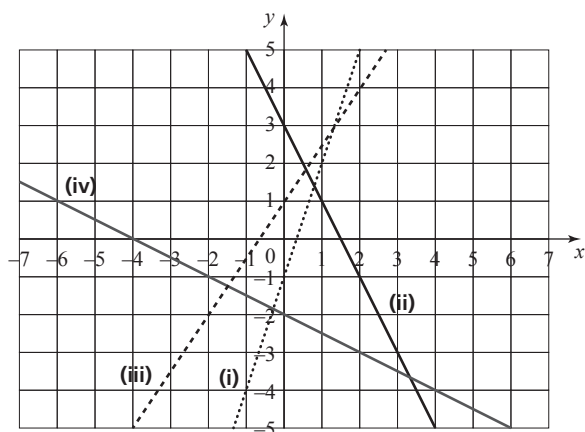
$$\frac{5 + n}{2} = 5 \Rightarrow n = 5$$

- 4 (i) $x = 0$
 (ii) $x = -4$
 (iii) $x = 1$
- 5 (i) $m(\text{Line 1}) = \frac{1}{2} \Rightarrow m_1 = \tan^{-1}\left(\frac{1}{2}\right) = 26.6^\circ$
 $m(\text{Line 2}) = 2 \Rightarrow m_2 = \tan^{-1}(2) = 63.4^\circ$
 $m(\text{Line 3}) = -1 \Rightarrow m_3 = \tan^{-1}(-1) = -45^\circ = 135^\circ$
 (ii) $\theta = \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right) = 36.9^\circ$ (1 d.p.)

3.2 The equation of a straight line (Page 19)

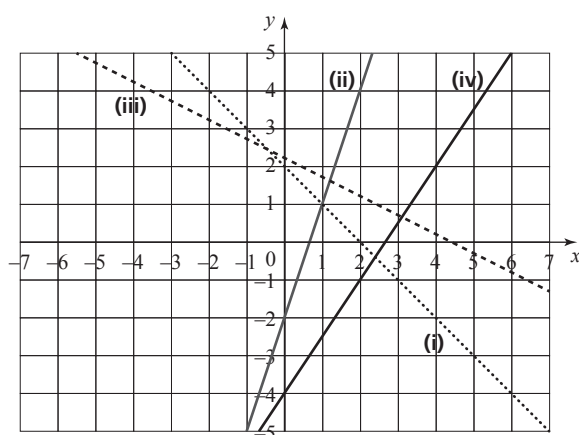
1

	Gradient	y-intercept
(i)	3	-1
(ii)	-2	3
(iii)	$\frac{3}{2}$	1
(iv)	$-\frac{1}{2}$	-2



2

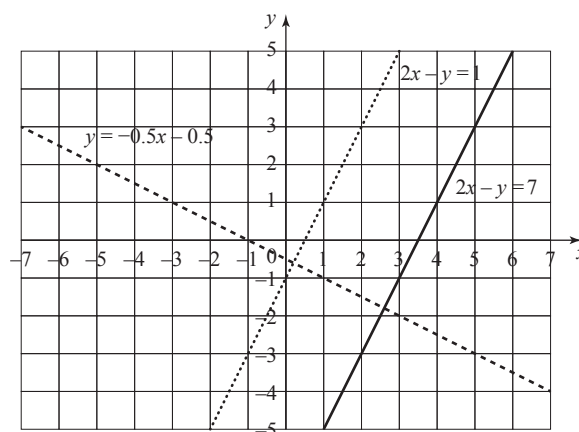
	Equation	Gradient	y-intercept
(i)	$y = -x + 2$	-1	2
(ii)	$y = 3x - 2$	3	-2
(iii)	$y = -\frac{1}{2}x + \frac{9}{4}$	$-\frac{1}{2}$	$\frac{9}{4}$
(iv)	$y = \frac{3}{2}x - 4$	$\frac{3}{2}$	-4



- 3 (i) $3y = -x - 6$
 $x + 3y + 6 = 0$
 (ii) $15y = 12x + 5$
 $12x - 15y + 5 = 0$
- 4 A: $y = -x + 1$ or $x + y = 1$
 B: $y = 2x - 3$
 C: $y = \frac{1}{3}x$
 D: $y = -3$
 E: $x = 6$
 F: $y = -\frac{2}{3}x - 2$

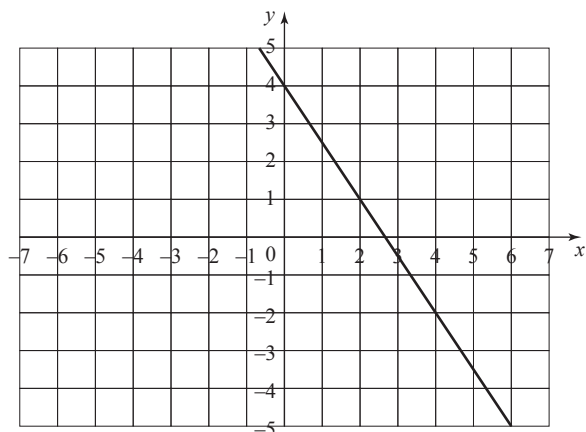
Gradient of line	Gradient of perpendicular
1	-1
$\frac{1}{4}$	-4
$-2\frac{1}{3} = -\frac{7}{3}$	$\frac{3}{7}$
$-\frac{3}{2}$	$\frac{2}{3}$
$0.3 = \frac{3}{10}$	$-\frac{10}{3}$

- 6 (i) $2x - y = 1 \Rightarrow y = 2x - 1$
 The gradient is 2 so the equation is
 $y = 2x + c$
 $1 = 2 \times 4 + c \Rightarrow c = -7$
 $y = 2x - 7$
- (ii) The gradient of the perpendicular is $-\frac{1}{2}$
 The equation is $y = -\frac{1}{2}x + c$
 $1 = -\frac{1}{2} \times -3 + c \Rightarrow c = -\frac{1}{2}$
 $y = -\frac{1}{2}x - \frac{1}{2}$



3.3 The intersection of two lines (Page 22)

1



The x -intercept is when $y = 0$, so

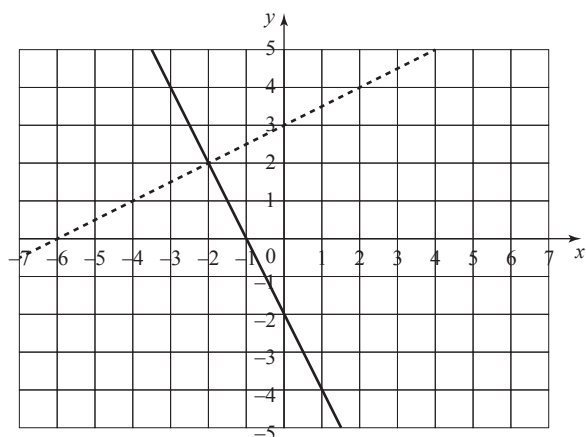
$$3x = 8 \Rightarrow x = \frac{8}{3}$$

The y -intercept is when $x = 0$, so

$$2y = 8 \Rightarrow y = 4$$

$$\text{Area} = \frac{1}{2} \times \frac{8}{3} \times 4 = \frac{16}{3} = 5\frac{1}{3}$$

2



The midpoint of AB is $\left(\frac{2+(-6)}{2}, \frac{4+0}{2}\right) = (-2, 2)$

The gradient of AB is $\frac{1}{2}$ so the gradient of the perpendicular line is -2 .

The equation is

$$y = -2x + c \Rightarrow 2 = -2 \times -2 + c \Rightarrow c = -2$$

$$y = -2x - 2$$

3 Midpoint of BC is $\left(\frac{2+4}{2}, \frac{3+(-5)}{2}\right) = (3, -1)$

Call the midpoint M.

The gradient of AM is $-\frac{3}{8}$.

The equation of AM is

$$y = -\frac{3}{8}x + c$$

$$2 = -\frac{3}{8} \times -5 + c \Rightarrow c = \frac{1}{8}$$

$$y = -\frac{3}{8}x + \frac{1}{8}$$

$$4 \quad ax - 3y + 1 = 0 \Rightarrow 3y = ax + 1 \Rightarrow y = \frac{a}{3}x + \frac{1}{3}$$

$$2x + by - 6 = 0 \Rightarrow by = -2x + 6 \Rightarrow y = -\frac{2}{b}x + \frac{6}{b}$$

If the lines are perpendicular,

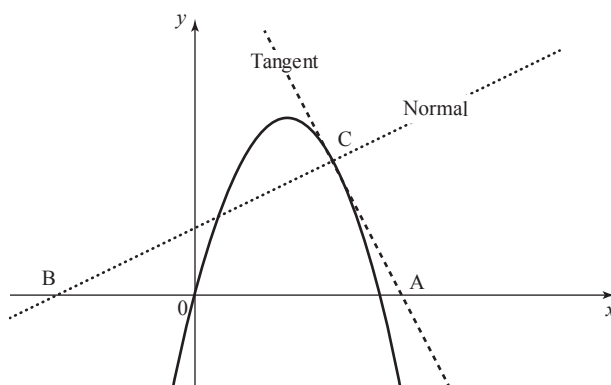
$$m_1 \times m_2 = -1$$

$$\frac{a}{3} \times -\frac{2}{b} = -1$$

$$\frac{a}{b} = \frac{3}{2}$$

$$a : b = 3 : 2$$

5



When $x = 3$, $y = 3 \times (4 - 3) = 3$

so the point C is (3, 3)

The x -intercept of the line $y = 9 - 2x$

is when $0 = 9 - 2x$, so $x = \frac{9}{2}$

The normal at (3, 3) has gradient $\frac{1}{2}$

The equation of the normal is

$$y = \frac{1}{2}x + c \Rightarrow 3 = \frac{1}{2} \times 3 + c \Rightarrow c = \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{3}{2}$$

The x -intercept of this line is when

$$0 = \frac{1}{2}x + \frac{3}{2} \Rightarrow x = -3$$

$$\text{Length of AB} = 7\frac{1}{2}$$

$$\text{Area of triangle ABC} = \frac{1}{2} \times 7\frac{1}{2} \times 3 = 11\frac{1}{4}$$

3.4 The circle (Page 24)

1 (i) Centre (1, -3), radius 4

(ii) Centre (-3, 1), radius 3

2 (i) $(x-4)^2 + (y+1)^2 = 36$ or

$$x^2 + y^2 - 8x + 2y - 19 = 0$$

(ii) $(x+2)^2 + (y-7)^2 = 8$ or

$$x^2 + y^2 + 4x - 14y + 45 = 0$$

(iii) Radius = $\sqrt{(2-(-1))^2 + (4-0)^2} = \sqrt{25} = 5$

$$(x-2)^2 + (y-4)^2 = 25 \text{ or}$$

$$x^2 + y^2 - 4x - 8y - 5 = 0$$

(iv) $(x-3)^2 + (y+3)^2 = 9$ or

$$x^2 + y^2 - 6x + 6y + 9 = 0$$

3 A: $(x+1)^2 + y^2 = 9$ or $x^2 + y^2 + 2x - 8 = 0$

B: $(x-2)^2 + (y-4)^2 = 16$ or

$$x^2 + y^2 - 4x - 8y + 4 = 0$$

4 (i) $x^2 + y^2 - 2x + 4y - 20 = 0$

$$(x-1)^2 + (y+2)^2 = 25$$

Centre C is (1, -2)

(ii) Gradient of DC is $\frac{4.5}{1.5} = 3$

Gradient of AB is $-\frac{1}{3}$

Equation of AB is

$$2.5 = -\frac{1}{3} \times 2.5 + c \Rightarrow c = \frac{10}{3}$$

$$\text{So } y = -\frac{1}{3}x + \frac{10}{3}$$

$$x + 3y - 10 = 0$$

(iii) $(10-3y)^2 + y^2 - 2(10-3y) + 4y - 20 = 0$

$$100 - 60y + 9y^2 + y^2 - 20 + 6y + 4y - 20 = 0$$

$$10y^2 - 50y + 60 = 0$$

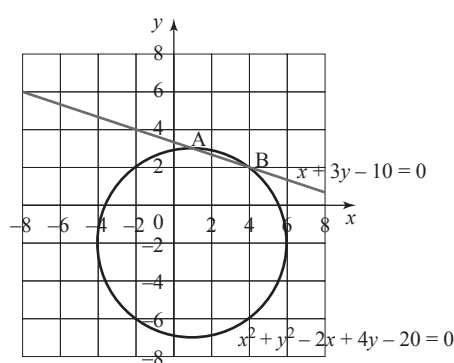
$$y^2 - 5y + 6 = 0$$

$$(y-2)(y-3) = 0$$

$$y = 2 \text{ or } 3$$

$$x = 4 \text{ or } 1$$

A(1, 3), B(4, 2)



3.5 The intersection of a line and a curve (Page 26)

1 (i) $x^2 + 4x = x - 2$

$$x^2 + 3x + 2 = 0$$

$$(x+2)(x+1) = 0$$

$$x = -1 \text{ or } -2$$

$$y = -3 \text{ or } -4$$

$$(-1, -3) \text{ or } (-2, -4)$$

(ii) $x^2 + 4x - 2 = -2x - 2$

$$x^2 + 6x = 0$$

$$x(x+6) = 0$$

$$x = 0 \text{ or } -6$$

$$y = -2 \text{ or } 10$$

$$(0, -2) \text{ or } (-6, 10)$$

(iii) $x(7-2x) = 3$

$$7x - 2x^2 = 3$$

$$0 = 2x^2 - 7x + 3$$

$$0 = (2x-1)(x-3)$$

$$x = \frac{1}{2} \text{ or } 3$$

$$y = 6 \text{ or } 1$$

$$\left(\frac{1}{2}, 6\right) \text{ or } (3, 1)$$

(iv) $(x-2)^2 + (9-2x)^2 = 5$

$$x^2 - 4x + 4 + (81 - 36x + 4x^2) = 5$$

$$5x^2 - 40x + 80 = 0$$

$$x^2 - 8x + 16 = 0$$

$$(x-4)^2 = 0$$

$$x = 4, y = 1$$

$$(4, 1)$$

2 $x^2 + 3x + 9 = x + k$

$$x^2 + 2x + 9 - k = 0$$

For one solution, discriminant = 0

$$2^2 - 4 \times 1 \times (9 - k) = 0$$

$$4 - 36 + 4k = 0$$

$$-32 + 4k = 0$$

$$4k = 32$$

$$k = 8$$

3 $2 - x^2 = 3 - kx$

$$0 = x^2 - kx + 1$$

For two points of intersection, discriminant > 0

$$(-k)^2 - 4 \times 1 \times 1 > 0$$

$$k^2 - 4 > 0$$

$$k^2 > 4$$

$$k > 2 \text{ or } k < -2$$

4 (i) Centre of circle is (-1, 0)

$$\text{Gradient of radius} = -\frac{4}{3}$$

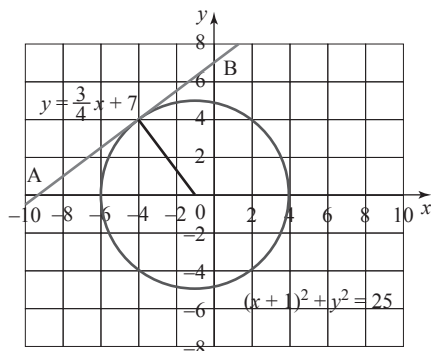
$$\text{Gradient of tangent} = \frac{3}{4}$$

Equation of tangent is

$$4 = \frac{3}{4} \times -4 + c \Rightarrow c = 7$$

$$\text{So } y = \frac{3}{4}x + 7$$

$$m = \frac{3}{4}, c = 7$$



(ii) A is $(-\frac{28}{3}, 0)$, B is $(0, 7)$

$$\text{Area AOB} = \frac{1}{2} \times \frac{28}{3} \times 7 = \frac{98}{3}$$

5 $y + kx = 8 \Rightarrow y = 8 - kx$

$$2x^2 + 3x + 10 = 8 - kx$$

$$2x^2 + 3x + kx + 2 = 0$$

$$2x^2 + (3+k)x + 2 = 0$$

For a tangent, $b^2 - 4ac = 0$

$$(3+k)^2 - 4 \times 2 \times 2 = 0$$

$$9 + 6k + k^2 - 16 = 0$$

$$k^2 + 6k - 7 = 0$$

$$(k-1)(k+7) = 0$$

$$k = -7 \text{ or } 1$$

6 (i) $2 + kx^2 = kx + 1$

$$kx^2 - kx + 1 = 0$$

For no common points, $b^2 - 4ac < 0$

$$(-k)^2 - 4 \times k \times 1 < 0$$

$$k^2 - 4k < 0$$

$$k(k-4) < 0$$

$$0 < k < 4$$

(ii) For a tangent, $b^2 - 4ac = 0$

$$k^2 - 4k = 0$$

$$k(k-4) = 0$$

$$k = 0 \text{ or } 4$$

Since $k \neq 0$, $k = 4$

When $k = 4$,

$$2 + 4x^2 = 4x + 1$$

$$4x^2 - 4x + 1 = 0$$

$$(2x-1)(2x-1) = 0$$

$$x = \frac{1}{2}$$

$$y = 4 \times \frac{1}{2} + 1 = 3$$

Point is $(\frac{1}{2}, 3)$

Further practice (Page 28)

1 (i) Length = $EF = \sqrt{8^2 + 4^2} = \sqrt{80} \approx 8.94$

Midpoint = $(-1, 1)$

Gradient = $\frac{-4}{8} = -\frac{1}{2}$

(ii) Length = $GH = \sqrt{6^2 + 12^2} = \sqrt{180} \approx 13.4$

Midpoint = $(-7, -3)$

Gradient = $\frac{12}{6} = 2$

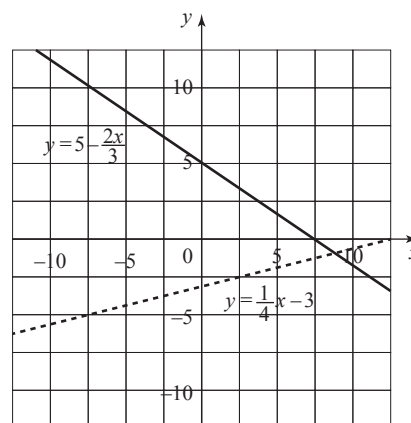
2 $m(\text{line A}) = \tan^{-1} 2 = 63.4^\circ$

$m(\text{line B}) = \tan^{-1}(-\frac{1}{3}) = -18.4^\circ$

Angle between lines = $63.43^\circ + 18.43^\circ = 81.0^\circ$ (1 d.p.)

3 (i) Gradient = $\frac{1}{4}$, y -intercept = -3

(ii) Gradient = $-\frac{2}{3}$, y -intercept = 5



4 (i) $y = \frac{1}{2}x + \frac{1}{2}$

(ii) $y = -\frac{1}{2}x - \frac{2}{5}$

5 (i) $3x - 2y + 8 = 0$

(ii) $8x + 5y + 5 = 0$

- 6 (i) Line is of the form $4x + 3y = k$

Substituting $(-2, 0)$ gives

$$4 \times -2 + 3 \times 0 = k$$

$$k = -8$$

$$4x + 3y = -8$$

$$4x + 3y + 8 = 0$$

- (ii) Rearranging the equation gives

$$4x + 3y = 1$$

$$3y = -4x + 1$$

$$y = -\frac{4}{3}x + \frac{1}{3}$$

The gradient of the line is $-\frac{4}{3}$ so the gradient of the perpendicular line is $\frac{3}{4}$.

The equation is $y = \frac{3}{4}x + c$

Substituting the point $(3, -1)$ gives

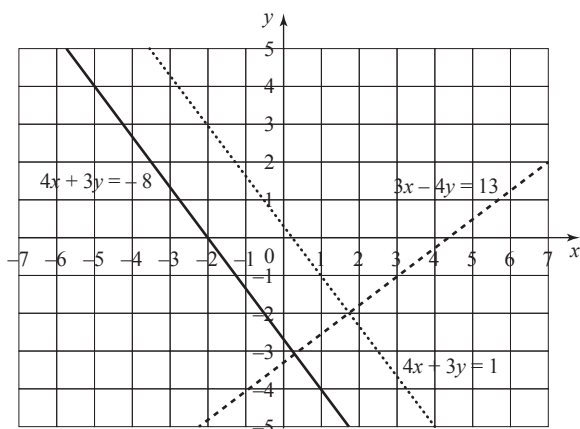
$$-1 = \frac{3}{4} \times 3 + c$$

$$c = -\frac{13}{4}$$

$$y = \frac{3}{4}x - \frac{13}{4}$$

$$4y = 3x - 13$$

$$3x - 4y - 13 = 0$$

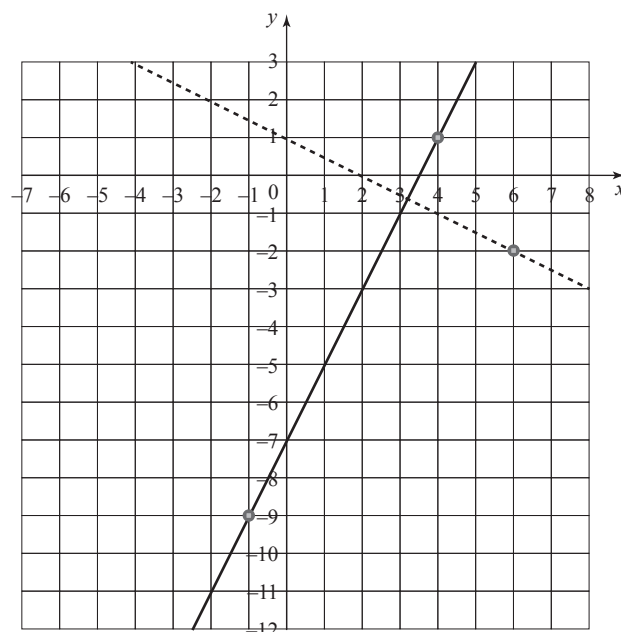


- 7 (i) Midpoint is $\left(\frac{4+1}{2}, \frac{1+9}{2}\right) = \left(\frac{5}{2}, 5\right)$

- (ii) $m(AC) = 2$ so $m(\perp) = -\frac{1}{2}$

$$y = mx + c \Rightarrow -2 = -\frac{1}{2} \times 6 + c \Rightarrow c = 1$$

Equation is $y = -\frac{1}{2}x + 1$



- 8 (i) $x - 3y = 6 \Rightarrow 3y = x - 6 \Rightarrow y = \frac{1}{3}x - 2$

The gradient is $\frac{1}{3}$ so the gradient of the perpendicular line is -3 .

Equation of the perpendicular line is

$$y = -3x + c \Rightarrow 2 = -3 \times -3 + c \Rightarrow c = -7$$

$$y = -3x - 7$$

- (ii) Solving simultaneously,

$$y = -3x - 7$$

$$x - 3y = 6$$

Substitution gives

$$x - 3(-3x - 7) = 6$$

$$x + 9x + 21 = 6$$

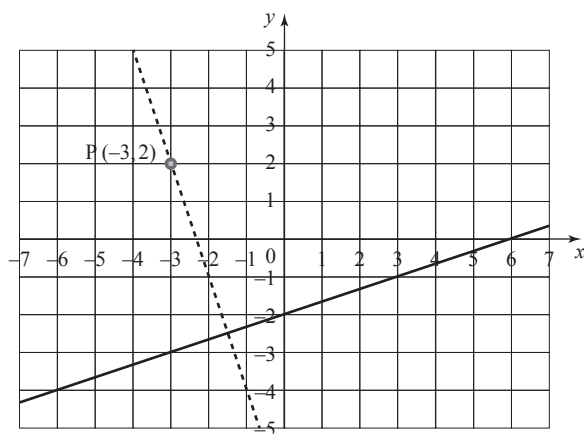
$$10x = -15$$

$$x = -\frac{3}{2}$$

$$y = -3 \times -\frac{3}{2} - 7 = -\frac{5}{2}$$

The distance between $\left(-\frac{3}{2}, -\frac{5}{2}\right)$ and $(-3, 2)$ is

$$d = \sqrt{\left(-3 - -\frac{3}{2}\right)^2 + \left(2 - -\frac{5}{2}\right)^2} = 4.74 \text{ (3 s.f.)}$$



9 (i) Centre $(0, -5)$, radius 8

(ii) Centre $(-2, 1)$, radius 3

10 (i) $(x-5)^2 + (y+1)^2 = 49$

(ii) $(x-3)^2 + (y+3)^2 = 9$

11 (i) $x^2 + 9 = 4x + 5$
 $x^2 - 4x + 4 = 0$
 $(x-2)(x-2) = 0$
 $x = 2, y = 13$
 $(2, 13)$

(ii) $x^2 + 2(1-x)^2 = 9$
 $x^2 + 2(1-2x+x^2) = 9$
 $x^2 + 2 - 4x + 2x^2 = 9$
 $3x^2 - 4x - 7 = 0$
 $(3x-7)(x+1) = 0$
 $x = \frac{7}{3}$ or -1
 $y = -\frac{4}{3}$ or 2
 $(\frac{7}{3}, -\frac{4}{3})$ or $(-1, 2)$

12 $2 - x^2 = 3 - kx$

$0 = x^2 - kx + 1$

For two points of intersection, discriminant > 0

$(-k)^2 - 4 \times 1 \times 1 > 0$

$k^2 - 4 > 0$

$k^2 > 4$

$k > 2$ or $k < -2$

13 $2x + \frac{k}{x} = 3$

$2x^2 + k = 3x$

$2x^2 - 3x + k = 0$

No points of intersection $\Rightarrow$ discriminant < 0

$(-3)^2 - 4 \times 2 \times k < 0$

$9 - 8k < 0$

$9 < 8k$

$\frac{9}{8} < k$

$k > \frac{9}{8}$

14 $x^2 - x(kx - \sqrt{8}) = k$

$x^2 - kx^2 + \sqrt{8}x - k = 0$

$(1-k)x^2 + \sqrt{8}x - k = 0$

$(\sqrt{8})^2 - 4 \times (1-k) \times -k = 0$

$8 + 4k - 4k^2 = 0$

$2 + k - k^2 = 0$

$(2-k)(1+k) = 0$

$k = 2$ or -1

Past exam questions (Page 29)

1 (i) Gradient of perpendicular bisector $= \frac{3}{2}$

Gradient of AB $= -\frac{2}{3}$

Equation of AB:

$6 = -\frac{2}{3} \times -2 + c \Rightarrow c = \frac{14}{3}$

So $y = -\frac{2}{3}x + \frac{14}{3}$ or $2x + 3y - 14 = 0$

(ii) Solving simultaneously,

$2y = 3x + 5$ or $3x - 2y = -5$

$2x + 3y = 14$

Multiply ① by 3, ② by 2:

$9x - 6y = -15$

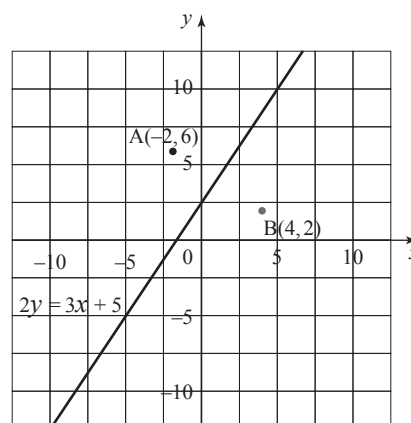
$4x + 6y = 28$

$13x = 13$

$x = 1, y = 4$

Midpoint is $(1, 4)$

By symmetry, B is $(4, 2)$



2 (i) Midpoint is $\left(\frac{5+9}{2}, \frac{7+(-1)}{2}\right) = (7, 3)$

Gradient of AB is $\frac{(-1)-7}{9-5} = -2$

Gradient of perpendicular line is $\frac{1}{2}$

Equation is

$$3 = \frac{1}{2} \times 7 + c \Rightarrow c = -\frac{1}{2}$$

$$\text{So } y = \frac{1}{2}x - \frac{1}{2}$$

(ii) Equation of CX is

$$2 = -2 \times 1 + c \Rightarrow c = 4$$

$$\text{So } y = -2x + 4$$

Solving simultaneously,

$$-2x + 4 = \frac{1}{2}x - \frac{1}{2}$$

$$-4x + 8 = x - 1$$

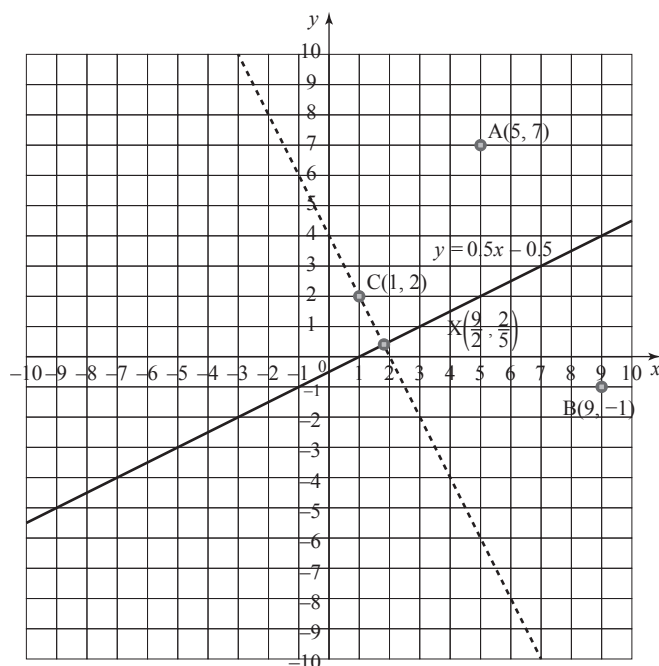
$$9 = 5x$$

$$x = \frac{9}{5}, y = \frac{2}{5}$$

$$\text{So X is } \left(\frac{9}{5}, \frac{2}{5}\right) \text{ or } (1.8, 0.4)$$

$$BX^2 = (9-1.8)^2 + (-1-0.4)^2$$

$$BX = 7.33$$



3 (i) $(9-p)^2 + (3p)^2 = 13^2$

$$(81 - 18p + p^2) + 9p^2 = 169$$

$$10p^2 - 18p - 88 = 0$$

$$5p^2 - 9p - 44 = 0$$

$$(5p+11)(p-4) = 0$$

$$p = -\frac{11}{5} \text{ or } 4$$

(ii) Gradient of $2x + 3y = 9$ is $-\frac{2}{3}$

$$\text{Gradient of AB is } \frac{3}{2}$$

$$\frac{3p}{9-p} = \frac{3}{2}$$

$$6p = 27 - 3p$$

$$9p = 27$$

$$p = 3$$

4 $x^2 - 4x + c = 2x - 7$

$$x^2 - 6x + (c+7) = 0$$

$$b^2 - 4ac = 0$$

$$(-6)^2 - 4 \times 1 \times (c+7) = 0$$

$$36 - 4c - 28 = 0$$

$$4c = 8$$

$$c = 2$$

5 Solving simultaneously,

$$y = 3x$$

$$4y = x + 11 \Rightarrow y = \frac{1}{4}x + \frac{11}{4}$$

$$3x = \frac{1}{4}x + \frac{11}{4}$$

$$12x = x + 11$$

$$11x = 11$$

$$x = 1, y = 3$$

A is (1, 3)

By symmetry, C is (12, 14).

Equation of BC is

$$4y = x + c$$

$$4 \times 14 = 12 + c \Rightarrow c = 44$$

$$4y = x + 44$$

Solving simultaneously to find B,

$$4(3x) = x + 44$$

$$11x = 44$$

$$x = 4, y = 12$$

B is (4, 12)

By symmetry, D is (9, 5).

6 $3x^2 - 4x + 7 = mx + 4$

$$3x^2 + (-4-m)x + 3 = 0$$

For two distinct points of intersection, $b^2 - 4ac > 0$

$$(-4-m)^2 - 4 \times 3 \times 3 > 0$$

$$16 + 8m + m^2 - 36 > 0$$

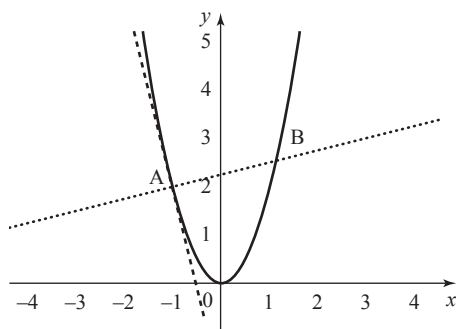
$$m^2 + 8m - 20 > 0$$

$$(m+10)(m-2) > 0$$

$$m < -10 \text{ or } m > 2$$

Stretch and challenge (Page 30)

- 1 The diagram shows the relationship between the tangent and normal at the point A.



You need to find the coordinates of both A and B. To find where the line meets the curve, solve simultaneously:

$$4x + y + 2 = 0 \Rightarrow y = -4x - 2$$

$$2x^2 = -4x - 2$$

$$2x^2 + 4x + 2 = 0$$

$$x^2 + 2x + 1 = 0$$

$$(x+1)(x+1) = 0$$

$$x = -1$$

$$y = 2(-1)^2 = 2$$

So A is $(-1, 2)$

The gradient of the tangent line is -4 so the gradient of the normal is $\frac{1}{4}$

The equation of the normal:

$$y = mx + c \Rightarrow 2 = \frac{1}{4} \times -1 + c \Rightarrow c = \frac{9}{4}$$

So the equation is $y = \frac{1}{4}x + \frac{9}{4}$

To find B, solve simultaneously:

$$2x^2 = \frac{1}{4}x + \frac{9}{4}$$

$$8x^2 = x + 9 \Rightarrow 8x^2 - x - 9 = 0$$

$$(8x-9)(x+1) = 0$$

$$x = \frac{9}{8} \text{ or } -1$$

Substitute $x = \frac{9}{8}$ into the equation of the curve to find

the y -coordinate:

$$y = 2\left(\frac{9}{8}\right)^2 = 2 \times \frac{81}{64} = \frac{81}{32}$$

So B is $\left(\frac{9}{8}, \frac{81}{32}\right)$

Finally, use Pythagoras' theorem to find the length of AB:

$$AB = \sqrt{\left(-1 - \frac{9}{8}\right)^2 + \left(2 - \frac{81}{32}\right)^2} = 2.19 \text{ (3 s.f.)}$$

2 $\frac{x}{a} - \frac{y}{b} = 4$

Multiplying every term by ab :

$$bx - ay = 4ab$$

$$bx - 4ab = ay$$

$$y = \frac{bx - 4ab}{a}$$

$$y = \frac{b}{a}x - 4b$$

From this form the gradient of the line is $\frac{b}{a}$,

$$\text{so } \frac{b}{a} = \frac{1}{2} \Rightarrow 2b = a$$

The x -intercept is when $y = 0$ so

$$\frac{x}{a} - \frac{0}{b} = 4 \Rightarrow \frac{x}{a} = 4 \Rightarrow x = 4a, \text{ so M is } (4a, 0)$$

The y -intercept is when $x = 0$ so

$$\frac{0}{a} - \frac{y}{b} = 4 \Rightarrow -\frac{y}{b} = 4 \Rightarrow y = -4b, \text{ so N is } (0, -4b)$$

By Pythagoras' theorem,

$$MN = \sqrt{(4a)^2 + (4b)^2} = \sqrt{16a^2 + 16b^2}$$

$$\sqrt{16a^2 + 16b^2} = \sqrt{720}$$

$$16a^2 + 16b^2 = 720$$

$$16(2b)^2 + 16b^2 = 720$$

$$16 \times 4b^2 + 16b^2 = 720$$

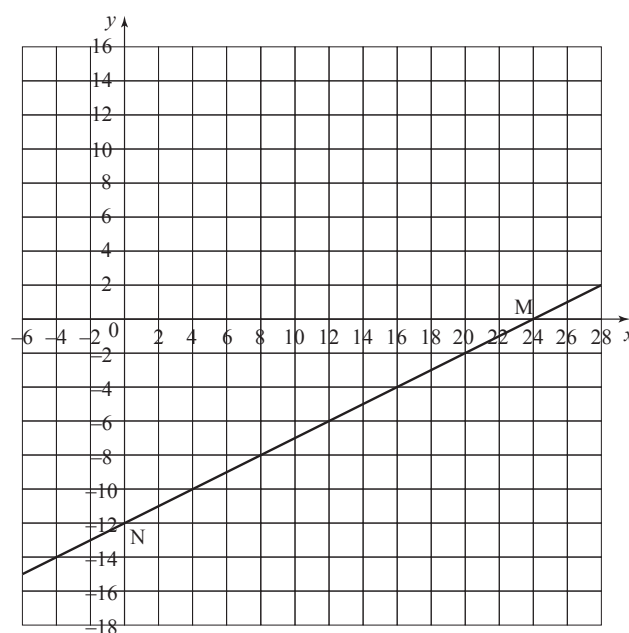
$$64b^2 + 16b^2 = 720$$

$$80b^2 = 720$$

$$b^2 = 9$$

$$b = 3$$

$$a = 2b = 6$$



- 3 Midpoint of the line joining A(-6, 1) and B(k, -3) is

$$\left(\frac{-6+k}{2}, \frac{1+(-3)}{2}\right) = \left(\frac{k-6}{2}, -1\right)$$

$$\text{Gradient of AB is } \frac{1-(-3)}{-6-k} = \frac{4}{-6-k}$$

$$\text{Gradient of perpendicular is } -\frac{-6-k}{4} = \frac{6+k}{4}$$

Equation of perpendicular bisector is

$$y = mx + c \Rightarrow -1 = \frac{6+k}{4} \times \frac{k-6}{2} + c$$

$$-1 = \frac{k^2 - 36}{8} + c$$

$$c = -\frac{k^2 - 36}{8} - 1$$

Since the y-intercept is -9,

$$-\frac{k^2 - 36}{8} - 1 = -9$$

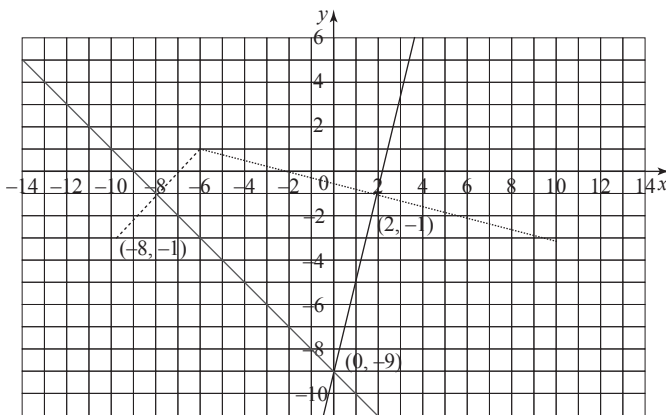
$$-\frac{k^2 - 36}{8} = -8$$

$$\frac{k^2 - 36}{8} = 8$$

$$k^2 - 36 = 64$$

$$k^2 = 100$$

$$k = \pm 10$$



- 4 (i) $x^2 - 4x + 3 = (x-1)(x-3)$ so the x-intercepts are $x = 1$ and 3 .

So the line must pass through $(1, 0)$ or $(3, 0)$.

$$y = kx + 1 \Rightarrow 0 = k \times 1 + 1 \Rightarrow k = -1$$

$$\text{or } y = kx + 1 \Rightarrow 0 = k \times 3 + 1 \Rightarrow k = -\frac{1}{3}$$

- (ii) The x-intercepts are $x = a$ and b .

So the line must pass through $(a, 0)$ or $(b, 0)$.

$$y = kx + 1 \Rightarrow 0 = k \times a + 1 \Rightarrow k = -\frac{1}{a}$$

$$y = kx + 1 \Rightarrow 0 = k \times b + 1 \Rightarrow k = -\frac{1}{b}$$

- (iii) The x-intercepts are $x = a$ and b .

So the line must pass through $(a, 0)$ or $(b, 0)$.

$$y = mx + c \Rightarrow 0 = k \times a + c \Rightarrow k = -\frac{c}{a}$$

$$y = mx + c \Rightarrow 0 = k \times b + c \Rightarrow k = -\frac{c}{b}$$

$$5 \quad 2y = kx + 1 \Rightarrow y = \frac{k}{2}x + \frac{1}{2}$$

$$2x^2 + x + 1 = \frac{k}{2}x + \frac{1}{2}$$

$$4x^2 + 2x + 2 = kx + 1$$

$$4x^2 + (2-k)x + 1 = 0$$

For two points of intersection, $b^2 - 4ac > 0$

$$(2-k)^2 - 4 \times 4 \times 1 > 0$$

$$4 - 4k + k^2 - 16 > 0$$

$$k^2 - 4k - 12 > 0$$

$$(k+2)(k-6) > 0$$

$$k < -2 \text{ or } k > 6$$

4 Sequences and series

4.1 Arithmetic progressions (Page 31)

- 1 (i) 5, 11, 17, 23, ...

$$a = 5, d = 6, n = 12$$

$$u_{12} = 5 + (12-1) \times 6 = 71$$

- (ii) -3, -8, -13, ...

$$a = -3, d = -5, n = 15$$

$$u_{15} = -3 + (15-1) \times -5 = -73$$

- (iii) $\frac{2}{3}, \frac{6}{5}, \frac{26}{15}, \dots$

$$a = \frac{2}{3}, d = \frac{8}{15}, n = 7$$

$$u_7 = \frac{2}{3} + (7-1) \times \frac{8}{15} = \frac{58}{15}$$

- (iv) 0.85, 0.55, 0.25, ...

$$a = 0.85, d = -0.3, n = 11$$

$$u_{11} = 0.85 + (11-1) \times -0.3 = -2.15$$

- 2 (i) 21, 15, 9, ...

$$a = 21, d = -6, n = 9$$

$$S_9 = \frac{9}{2} [2 \times 21 + (9-1) \times -6] = -27$$

- (ii) -8, 0, 8, ...

$$a = -8, d = 8, n = 18$$

$$S_{18} = \frac{18}{2} [2 \times -8 + (18-1) \times 8] = 1080$$

(iii) 7, 9, 11, ..., 79

$$a = 7, d = 2, n = ?, u_n = l = 79$$

$$u_n = 7 + (n-1) \times 2 = 79$$

$$2(n-1) = 72$$

$$n-1 = 36$$

$$n = 37$$

$$S_{37} = \frac{37}{2}[7 + 79] = 1591$$

(iv) 0.05, 0.15, 0.25, ..., 4.05

$$a = 0.05, d = 0.1, n = ?, u_n = l = 4.05$$

$$u_n = 0.05 + (n-1) \times 0.1 = 4.05$$

$$0.1(n-1) = 4$$

$$n-1 = 40$$

$$n = 41$$

$$S_{41} = \frac{41}{2}[0.05 + 4.05] = 84.05$$

3 $34 + 5d = 43$

$$5d = 9$$

$$d = \frac{9}{5} = 1.8$$

$$a = 34 - 4 \times 1.8 = 26.8$$

$$S_{20} = \frac{20}{2}[2 \times 26.8 + (20-1) \times 1.8] = 878$$

4 $d = (2m+n) - m = m+n$

$$u_{10} = m + (10-1) \times (m+n)$$

$$= m + 9(m+n)$$

$$= 10m + 9n$$

5 $a = 40, S_{30} = 3375, n = 30, d = ?$

$$S_{30} = 3375 \Rightarrow \frac{30}{2}[2 \times 40 + (30-1)d] = 3375$$

$$15(80 + 29d) = 3375$$

$$1200 + 435d = 3375$$

$$435d = 2175$$

$$d = 5$$

$$u_5 = 40 + (5-1) \times 5 = \$60$$

6 (i) $u_7 = 32 \Rightarrow a + 6d = 32$

$$S_5 = 130 \Rightarrow \frac{5}{2}[2a + 4d] = 130$$

$$5a + 10d = 130$$

$$a + 2d = 26$$

Solving simultaneously, $d = 1.5, a = 23$ (ii) $u_n = 56 \Rightarrow 23 + (n-1) \times 1.5 = 56$

$$23 + 1.5n - 1.5 = 56$$

$$1.5n + 21.5 = 56$$

$$1.5n = 34.5$$

$$n = 23$$

$$7 \quad a = 4u_4 = 4(a + 3d) \Rightarrow a = 4a + 12d$$

$$3a + 12d = 0$$

$$a + 4d = 0$$

$$u_6 = u_4 - 4 \Rightarrow a + 5d = a + 3d - 4$$

$$2d = -4$$

$$d = -2$$

$$a + 4 \times -2 = 0 \Rightarrow a = 8$$

$$u_8 = 8 + 7 \times -2 = -6$$

4.2 Geometric progressions (Page 33)1 (i) $\frac{1}{2}, 1, 2, 4, \dots$

$$a = \frac{1}{2}, r = 2, n = 10$$

$$u_{10} = \frac{1}{2} \times 2^{10-1} = 256$$

(ii) $-16, 8, -4, 2, \dots$

$$a = -16, r = -\frac{1}{2}, n = 8$$

$$u_8 = -16 \times \left(-\frac{1}{2}\right)^{8-1} = 0.125 = \frac{1}{8}$$

(iii) $0.1, 0.01, 0.001, \dots$

$$a = 0.1, r = 0.1, n = 9$$

$$u_9 = 0.1 \times (0.1)^{9-1} = 0.000\,000\,001 = 1 \times 10^{-9}$$

(iv) $6, 9, 13.5, \dots$

$$a = 6, r = 1.5, n = 12$$

$$u_{12} = 6 \times (1.5)^{12-1} = 519 \text{ (3 s.f.)}$$

2 (i) $3, 6, 12, 24, \dots$

$$a = 3, r = 2, n = 8$$

$$S_8 = \frac{3(1-2^8)}{1-2} = 765$$

(ii) $81, 27, 9, 3, \dots$

$$a = 81, r = \frac{1}{3}, n = 20$$

$$S_{20} = \frac{81\left(1 - \left(\frac{1}{3}\right)^{20}\right)}{1 - \frac{1}{3}} = 121.5$$

(iii) $2, -5, 12.5, -31.25, \dots$

$$a = 2, r = -2.5, n = 14$$

$$S_{14} = \frac{2(1 - (-2.5)^{14})}{1 - (-2.5)} = -212\,874.3028\dots$$

$$= -213\,000 \text{ (3 s.f.)}$$

(iv) $-240, 48, -9.6, 1.92, \dots$

$$a = -240, r = -0.2, n = 10$$

$$S_{10} = \frac{-240(1 - (-0.2)^{10})}{1 - (-0.2)} = -200$$

- 3 The sequences in parts (ii) and (iv) have a sum to infinity:

$$(ii) S_{\infty} = \frac{81}{1 - \frac{1}{3}} = 121.5$$

$$(iv) S_{\infty} = \frac{-240}{1 - (-0.2)} = -200$$

$$4 \quad \frac{x}{32} = \frac{2}{x}$$

$$x^2 = 64$$

$$x = \pm 8$$

$$5 \quad a = 8, S_{\infty} = 12 \Rightarrow \frac{8}{1-r} = 12$$

$$8 = 12(1-r)$$

$$\frac{8}{12} = 1-r$$

$$r = 1 - \frac{8}{12} = \frac{1}{3}$$

$$u_5 = 8 \times \left(\frac{1}{3}\right)^{5-1} = \frac{8}{81}$$

$$6 \quad 3, 6x, 12x^2, \dots$$

$$r = \frac{6x}{3} = \frac{12x^2}{6x} = 2x$$

For the sequence to have a sum to infinity,

$$-1 < r < 1 \text{ so } -1 < 2x < 1$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

$$7 \quad (i) \quad \frac{5x+1}{2} \times r^2 = x+2 \Rightarrow r^2 = \frac{x+2}{\frac{5x+1}{2}} = \frac{2x+4}{5x+1}$$

$$(x+2) \times r^2 = \frac{x}{2} \Rightarrow r^2 = \frac{\frac{x}{2}}{x+2} = \frac{x}{2x+4}$$

$$\text{so } \frac{2x+4}{5x+1} = \frac{x}{2x+4}$$

$$(2x+4)^2 = x(5x+1)$$

$$4x^2 + 16x + 16 = 5x^2 + x$$

$$0 = x^2 - 15x - 16$$

$$0 = (x-16)(x+1)$$

$$x = -1 \text{ or } 16$$

$$(ii) \quad r^2 = \frac{x}{2x+4} = \frac{16}{36} = \frac{4}{9} \Rightarrow r = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

$$a = \frac{5x+1}{2} = \frac{5 \times 16 + 1}{2} = 40.5$$

$$S_{\infty} = \frac{40.5}{1 - \frac{2}{3}} = 121.5$$

$$8 \quad (i) \quad \text{Plan A: G.P. with } a = 2, r = 1.3, n = 20$$

$$S_{20} = \frac{2(1 - (1.3)^{20})}{1 - 1.3} = 1260 \text{ minutes (3 s.f.)}$$

$$(ii) \quad \text{Plan B: } a = 1, d = t, n = 20$$

$$S_{20} = \frac{20}{2} [2 \times 1 + (20-1) \times t]$$

$$= 10(2 + 19t)$$

$$= 20 + 190t$$

So to be the same,

$$20 + 190t = 1260$$

$$t = 6.53 \text{ minutes (3 s.f.)}$$

$$9 \quad (i) \quad \text{For the A.P., } a = 18$$

$$u_4 = a + (4-1)d = 18 + 3d$$

$$u_6 = a + (6-1)d = 18 + 5d$$

First three terms of G.P. are 18, $18 + 3d$, $18 + 5d$

$$\text{so } r = \frac{18+3d}{18} = \frac{18+5d}{18+3d}$$

$$(18+3d)^2 = 18(18+5d)$$

$$324 + 108d + 9d^2 = 324 + 90d$$

$$9d^2 + 18d = 0$$

$$9d(d+2) = 0$$

$$d = 0 \text{ or } -2$$

Since $d \neq 0$, $d = -2$

When $d = -2$, $r = \frac{18+3d}{18} = \frac{18+3 \times -2}{18} = \frac{12}{18} = \frac{2}{3}$

$$(ii) \quad S_{\infty} = \frac{18}{1 - \frac{2}{3}} = 54$$

$$(iii) \quad \text{You want } S_n = 0 \text{ with } a = 18, d = -2$$

$$\frac{n}{2} [2 \times 18 + (n-1) \times -2] = 0$$

$$\frac{n}{2} [36 - 2n + 2] = 0$$

$$\frac{n}{2} [38 - 2n] = 0$$

$$n = 0 \text{ or } 19$$

$$\text{So } n = 19$$

4.3 Binomial expansions (Page 36)

$$1 \quad (i) \quad (x+2)^4$$

$$= x^4 + \binom{4}{1} x^3 \times 2 + \binom{4}{2} x^2 \times 2^2 + \binom{4}{3} x^1 \times 2^3 + 2^4$$

$$= x^4 + 8x^3 + 24x^2 + 32x + 16$$

$$(ii) \quad (1-3x)^3$$

$$= 1^3 + \binom{3}{1} 1^2 \times (-3x) + \binom{3}{2} 1 \times (-3x)^2 + (-3x)^3$$

$$= 1 - 9x + 27x^2 - 27x^3$$

$$2 \quad (i) \quad (2x-3)^8$$

$$= (2x)^8 + \binom{8}{1} (2x)^7 \times (-3) + \binom{8}{2} (2x)^6 \times (-3)^2 + \dots$$

$$= 256x^8 - 3072x^7 + 16128x^6 - \dots$$

$$\begin{aligned} \text{(ii)} \quad (3a-b)^7 &= (3a)^7 + \binom{7}{1}(3a)^6 \times (-b)^1 + \binom{7}{2}(3a)^5 \times (-b)^2 + \dots \\ &= 2187a^7 - 5103a^6b + 5103a^5b^2 - \dots \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \left(\frac{1}{x^2} + x^3\right)^7 &= \left(\frac{1}{x^2}\right)^7 + \binom{7}{1}\left(\frac{1}{x^2}\right)^6 \times (x^3)^1 + \binom{7}{2}\left(\frac{1}{x^2}\right)^5 \times (x^3)^2 + \dots \\ &= \frac{1}{x^{14}} + \frac{7}{x^9} + \frac{21}{x^4} + \dots \end{aligned}$$

3 General term is $\binom{10}{r}(2x)^{10-r}\left(\frac{1}{x}\right)^r$
 x terms are $\frac{(x)^{10-r}}{x^r} = x^{10-r-r} = x^{10-2r}$
 You want $10 - 2r = 2$
 $r = 4$

So the x^2 term is $\binom{10}{4}(2x)^6\left(\frac{1}{x}\right)^4 = 13\,440x^2$
 So the coefficient is 13 440

4 General term is $\binom{15}{r}(x^2)^{15-r}\left(\frac{5}{x^3}\right)^r$
 x terms are $\frac{(x^2)^{15-r}}{(x^3)^r} = x^{30-2r-3r} = x^{30-5r}$
 You want $30 - 5r = 0$
 $r = 6$

So the term independent of x is

$$\binom{15}{6}(x^2)^9\left(\frac{5}{x^3}\right)^6 = 78\,203\,125$$

5 (i) General term is $\binom{6}{r}2^{6-r}(-3x)^r$

You need $r = 2$,

$$\binom{6}{2}2^4(-3x)^2 = 2160x^2$$

So the coefficient is 2160.

(ii) $(3+2x)(\dots x + 2160x^2 + \dots)$

You need the x term in the expansion of $(2-3x)^6$.

Here $r = 1$ so

$$\binom{6}{1}2^5(-3x)^1 = -576x$$

The x^2 term is

$$2x \times -576x + 3 \times 2160x^2 = 5328x^2$$

Hence the coefficient is 5328

6 The x^3 term in the expansion of $(1+bx)(x+2)^5$ will be
 $1 \times x^3$ term + $bx \times x^2$ term
 $(x+2)^5 = \dots + \binom{5}{2}x^3 \times 2^2 + \binom{5}{3}x^2 \times 2^3 + \dots$
 $= \dots + 40x^3 + 80x^2 + \dots$

The x^3 term is then

$$1 \times 40x^3 + bx \times 80x^2 = (40 + 80b)x^3$$

In order that there is no term in x^3 ,

$$40 + 80b = 0 \Rightarrow b = -\frac{1}{2}$$

7 (i) $(3+u)^6$
 $= 3^6 + \binom{6}{1}3^5 \times u^1 + \binom{6}{2}3^4 \times u^2 + \dots$
 $= 729 + 1458u + 1215u^2 + \dots$

(ii) $u = x - x^2$
 $= 729 + 1458(x - x^2) + 1215(x - x^2)^2 + \dots$
 $= 729 + 1458x - 1458x^2 + 1215(x^2 - 2x^3 + x^4) + \dots$

The x^2 term is $-1458x^2 + 1215x^2 = -243x^2$

The coefficient is -243

8 The general term is $\binom{7}{r}4^{7-r}(-ax)^r$

For the x^3 term you need $r = 3$ so

$$\begin{aligned} \binom{7}{3}4^4(-ax)^3 &= -8960a^3x^3 \\ -8960a^3 &= -1120 \Rightarrow a^3 = \frac{1}{8} \Rightarrow a = \frac{1}{2} \end{aligned}$$

The x^2 term is when $r = 2$

$$\binom{7}{2}4^5\left(-\frac{1}{2}x\right)^2 = 5376x^2$$

The coefficient of the x^2 term is 5376

9 (i) First three terms of $(3-2x^2)^7$ are
 $3^7 + \binom{7}{1}3^6(-2x^2)^1 + \binom{7}{2}3^5(-2x^2)^2$
 $= 2187 - 10\,206x^2 + 20\,412x^4$

(ii) $(1+x^2)(3-2x^2)^7$
 $= (1+x^2)(2187 - 10\,206x^2 + 20\,412x^4 \dots)$
 The x^4 term is
 $1 \times 20\,412x^4 + x^2 \times -10\,206x^2$
 $= 20\,412x^4 - 10\,206x^4$
 $= 10\,206x^4$

The coefficient of the x^4 term is 10 206

- 10 The general term is $\binom{6}{r} k^{6-r} \left(-\frac{1}{2}x\right)^r$

The x^3 term is when $r = 3$

$$\begin{aligned}\binom{6}{3} k^3 \left(-\frac{1}{2}x\right)^3 &= 20 \times k^3 \times -\frac{1}{8}x^3 \\ &= -\frac{5}{2}k^3x^3 \\ -\frac{5}{2}k^3 &= 160 \\ k^3 &= -64 \\ k &= -4\end{aligned}$$

Further practice (Page 39)

- 1 $u_3 = 12 \Rightarrow a + 2d = 12$ ①

$$\begin{aligned}S_8 = 168 &\Rightarrow \frac{8}{2}[2a + (8-1)d] = 168 \\ &\Rightarrow 8a + 28d = 168 \quad \text{②}\end{aligned}$$

Solving simultaneously,

$$\begin{aligned}8a + 16d &= 96 \quad \text{②} \times 8 = \text{②} \\ 12d &= 72 \quad \text{①} - \text{②} \\ d &= 6 \\ a &= 0\end{aligned}$$

$$u_{14} = 0 + (14-1) \times 6 = 78$$

- 2 $a = 25, d = ?, u_9 = 5, u_n = l = -45$

$$\begin{aligned}25 + 8d &= 5 \\ 8d &= -20 \\ d &= -2.5\end{aligned}$$

$$\begin{aligned}u_n = l = -45 &\Rightarrow -45 = 25 + (n-1) \times -2.5 \\ -45 &= 25 - 2.5n + 2.5 \\ 2.5n &= 72.5 \\ n &= 29 \\ S_{29} &= \frac{29}{2}[25 + -45] = -290\end{aligned}$$

- 3 $128r^4 = 40.5$

$$\begin{aligned}r^4 &= \frac{40.5}{128} \\ r &= \sqrt[4]{\frac{40.5}{128}} = 0.75 \\ u_5 &= 128 \times 0.75^2 = 72 \\ a &= \frac{128}{0.75^2} = 227.5 \\ S_{\infty} &= \frac{227.5}{1-0.75} = 910.2\end{aligned}$$

- 4 $\dots, \dots, 12, \dots, \dots, 6, \dots$

From the above, you can write $12 + 4d = 6 \Rightarrow d = -1.5$

Subtracting d from 12 twice gives the first term, $a = 15$

Now using the term formula,

$$u_{21} = 15 + (21-1) \times -1.5 = -15$$

Alternatively, you can form two equations from the information and solve them simultaneously:

$$3\text{rd term} = 12 \Rightarrow 12 = a + 2d$$

$$7\text{th term} = 6 \Rightarrow 6 = a + 6d$$

$$\begin{aligned}\text{Subtracting the equations gives } 4d &= -6 \\ &\Rightarrow d = -1.5\end{aligned}$$

The rest of the answer follows from there.

$$5 \quad r = \frac{2}{\sqrt{2}} = \frac{\sqrt{8}}{2} = \sqrt{2}$$

$$u_6 = \sqrt{2} \times (\sqrt{2})^{6-1} = 8$$

$$6 \quad \dots, \dots, \dots, \dots, \underbrace{-40, \dots, \dots, \dots, \dots, -20}_{+d, +d, +d, +d, +d}, \dots$$

$$-40 + 5d = -20$$

$$5d = 20$$

$$d = 4$$

Subtracting 4 from -40 four times gives

$$a = -56$$

You want $S_n = 0$.

$$\frac{n}{2}[2 \times -56 + (n-1) \times 4] = 0$$

$$\frac{n}{2}[-112 + 4n - 4] = 0$$

$$\frac{n}{2}[4n - 116] = 0$$

$$2n^2 - 58n = 0$$

$$2n(n-29) = 0$$

$$n = 0 \text{ or } 29$$

Since $n = 0$ is not a possible answer, the answer is $n = 29$

$$7 \quad \dots, 9, \dots, \dots, \underbrace{1\frac{1}{8}, \dots, \dots, \dots}_{\times r, \times r, \times r, \times r}, \dots$$

$$\text{From this, write } 9r^3 = 1\frac{1}{8} \Rightarrow r^3 = \frac{1}{8} \Rightarrow r = \frac{1}{2}$$

So the first term, $a = 18$

$$\text{Alternatively, write } ar = 9 \text{ and } ar^4 = 1\frac{1}{8}$$

Dividing the equations gives

$$\frac{ar^4}{ar} = \frac{1\frac{1}{8}}{9} \text{ so } r^3 = \frac{1}{8}$$

and the answer is the same.

$$S_{\infty} = \frac{a}{1-r} = \frac{18}{1-0.5} = 36$$

- 8 $r = 0.95$

$$\begin{aligned}u_6 &= 45\,000 \times (0.95)^8 = \$29\,853.92 \\ &= \$29\,900 \quad (3 \text{ s.f.})\end{aligned}$$

$$\begin{aligned}
 9 \quad & \left(\frac{1}{x} + 2x\right)^6 \\
 &= \left(\frac{1}{x}\right)^6 + \binom{6}{1}\left(\frac{1}{x}\right)^5 \times (2x)^1 + \binom{6}{2}\left(\frac{1}{x}\right)^4 \times (2x)^2 \\
 &= \frac{1}{x^6} + \frac{12}{x^4} + \frac{60}{x^2} + \dots
 \end{aligned}$$

$$\begin{aligned}
 10 \quad & \text{General term is } \binom{8}{r}\left(\frac{x}{2}\right)^{8-r}\left(-\frac{3}{x}\right)^r \\
 & x \text{ terms are } \frac{(x)^{8-r}}{x^r} = x^{8-r-r} = x^{8-2r} \\
 & 8 - 2r = 4 \\
 & r = 2
 \end{aligned}$$

$$\text{So the } x^4 \text{ term is } \binom{8}{2}\left(\frac{x}{2}\right)^6\left(-\frac{3}{x}\right)^2 = \frac{63}{16}x^4$$

$$11 \quad \text{General term is } \binom{12}{r}\left(\frac{2}{x}\right)^{12-r}(x^2)^r$$

To find the value of r that gives us the term independent of x , it is possible to guess and check until one that works is found OR use an algebraic approach.

$$\frac{(x^2)^r}{x^{12-r}} = \frac{x^{2r}}{x^{12-r}} = x^{2r-(12-r)} = x^{3r-12}$$

To get the term independent of x , the power of x must be 0, so $3r - 12 = 0 \Rightarrow r = 4$

Substituting into the general term,

$$\binom{12}{4}\left(\frac{2}{x}\right)^{12-4}(x^2)^4 = 495 \times \frac{2^8}{x^8} \times x^8 = 126\,720$$

$$12 \quad \text{General term is } \binom{5}{r}(kx)^{5-r}\left(\frac{2}{x}\right)^r$$

$$x \text{ terms are } \frac{x^{5-r}}{x^r} = x^{5-2r}$$

To get the x^{-1} term:

$$5 - 2r = -1$$

$$r = 3$$

$$\binom{5}{3}(kx)^2\left(\frac{2}{x}\right)^3 = 80k^2x^{-1}$$

$$80k^2 = 720$$

$$k^2 = 9$$

$$k = \pm 3$$

$$13 \quad \text{The general term for } (3-x)^4 \text{ is } \binom{4}{r}3^{4-r}(-x)^r$$

$$\text{The general term for } (2+ax)^5 \text{ is } \binom{5}{r}2^{5-r}(ax)^r$$

In both cases, $r = 2$ is the value needed to get the x^2 term.

$$\binom{4}{2}3^2(-x)^2 + \binom{5}{2}2^3(ax)^2$$

$$= 54x^2 + 80a^2x^2$$

$$= (54 + 80a^2)x^2$$

$$54 + 80a^2 = 554$$

$$80a^2 = 500$$

$$a^2 = 6.25$$

$$a = 2.5$$

Past exam questions (Page 39)

$$1 \quad \text{(i)} \quad \left(2x - \frac{1}{x}\right)^5$$

$$\text{General term is } \binom{5}{r}(2x)^{5-r}\left(-\frac{1}{x}\right)^r$$

$$x \text{ terms are } \frac{(x)^{5-r}}{x^r} = x^{5-r-r} = x^{5-2r}$$

$$5 - 2r = 1$$

$$r = 2$$

$$\text{So the } x \text{ term is } \binom{5}{2}(2x)^3\left(-\frac{1}{x}\right)^2 = 80x$$

Coefficient of x is 80

$$\text{(ii)} \quad (1+3x^2)\left(2x - \frac{1}{x}\right)^5$$

$$x^{-1} \text{ term in } \left(2x - \frac{1}{x}\right)^5 :$$

$$5 - 2r = -1$$

$$r = 3$$

$$\text{So the } x^{-1} \text{ term is } \binom{5}{3}(2x)^2\left(-\frac{1}{x}\right)^3 = -40x^{-1}$$

$$x \text{ terms in } (1+3x^2)\left(2x - \frac{1}{x}\right)^5 \text{ are}$$

$$1 \times x \text{ term in } \left(2x - \frac{1}{x}\right)^5 + 3x^2 \times x^{-1} \text{ term in } \left(2x - \frac{1}{x}\right)^5$$

$$= 1 \times 80x + 3x^2 \times -40x^{-1}$$

$$= 80x - 120x$$

$$= -40x$$

Coefficient of x is -40

$$2 \quad a + ar = 50 \Rightarrow a(1+r) = 50$$

$$ar + ar^2 = 30 \Rightarrow ar(1+r) = 30$$

Dividing the equations,

$$\frac{ar(1+r)}{a(1+r)} = \frac{30}{50}$$

$$r = \frac{3}{5} = 0.6$$

$$a = 31.25$$

$$S_{\infty} = \frac{31.25}{1-0.6} = 78.125$$

- 3 (i) (a) A.P. with $a = 10$, $d = 2$

$$u_{30} = 10 + (30 - 1) \times 2 = 68 \text{ litres}$$

$$(b) S_n = \frac{n}{2} [2 \times 10 + (n - 1) \times 2] = 2000$$

$$\frac{n}{2} [20 + 2n - 2] = 2000$$

$$18n + 2n^2 = 4000$$

$$2n^2 + 18n - 4000 = 0$$

$$n^2 + 9n - 2000 = 0$$

Solving gives

$$n = 40.4 \text{ or } -49.4$$

$$n = 41$$

- (ii) G.P. with $a = 10$, $r = 1.1$

$$S_{30} = \frac{10(1 - (1.1)^{30})}{1 - (1.1)} = 1644.94L$$

$$\text{Percentage left} = \frac{2000 - 1644.94}{2000} \times 100\% = 17.8\%$$

- 4 (i) $(a - x)^5$

$$= a^5 + \binom{5}{1} a^4 (-x)^1 + \binom{5}{2} a^3 (-x)^2 + \binom{5}{3} a^2 (-x)^3 + \dots$$

$$= a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + \dots$$

- (ii) $(1 - ax)(a - x)^5$

$$= (1 - ax)(a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + \dots)$$

Terms in x^3 :

$$-10a^2x^3 + (-ax)(10a^3x^2)$$

$$= -10a^2x^3 - 10a^4x^3$$

$$= (-10a^2 - 10a^4)x^3$$

$$-10a^2 - 10a^4 = -200$$

$$10a^4 + 10a^2 - 200 = 0$$

$$a^4 + a^2 - 20 = 0$$

$$(a^2 - 4)(a^2 + 5) = 0$$

$$a^2 = -5 \text{ (ignore)}$$

$$a^2 = 4 \Rightarrow a = \pm 2$$

- 5 (i) (a) $a = -15$, $n = 25$

$$S_{25} = 525$$

$$\Rightarrow \frac{25}{2} [2 \times -15 + (25 - 1) \times d] = 525$$

$$\frac{25}{2} [-30 + 24d] = 525$$

$$-375 + 300d = 525$$

$$d = 3$$

$$(b) u_{25} = -15 + (25 - 1) \times 3 = 57$$

- (c) The first six terms are

$$-15, -12, -9, -6, -3, 0$$

So the sum of the 7th to 25th terms is needed.

$$S_7 = -15 + -12 + -9 + -6 + -3 + 0 = -45$$

$$S_+ = S_{25} - S_7 = 525 - (-45) = 570$$

- (ii) (a) 2022 is 10 years later, or the 11th term:

$$u_{11} = 4000 \times 1.05^{10} = \$6515.58 \text{ or } \$6516$$

$$(b) S_{11} = \frac{4000(1.05^{11} - 1)}{1.05 - 1} = \$56\,827 \text{ or } \$56\,800$$

Stretch and challenge (Page 40)

$$1 \quad u_7 = 400 \Rightarrow a + 6d = 400$$

$$S_{30} = 1800 \Rightarrow \frac{30}{2} [2a + (30 - 1) \times d] = 1800$$

$$15[2a + 29d] = 1800 \Rightarrow 30a + 435d = 1800$$

$$2a + 29d = 120$$

Solving simultaneously,

$$d = -40 \text{ and } a = 640$$

- 2 General term for the first expansion is

$$\binom{6}{r} (kx^3)^{6-r} \left(-\frac{7}{x^3}\right)^r$$

Term independent of x is when $r = 3$

$$\binom{6}{3} (kx^3)^{6-3} \left(-\frac{7}{x^3}\right)^3 = 20k^3x^9 \times \frac{-343}{x^9} = -6860k^3$$

General term for the second expansion is

$$\binom{8}{r} (kx^4)^{8-r} \left(\frac{m}{x^4}\right)^r$$

Term independent of x is when $r = 4$

$$\binom{8}{4} (kx^4)^{8-4} \left(\frac{m}{x^4}\right)^4 = 70k^4x^{16} \times \frac{m^4}{x^{16}} = 70k^4m^4$$

$$\text{So } 70k^4m^4 = -6860k^3$$

$$k = -\frac{98}{m^4}$$

- 3 $a + ar = -3 \Rightarrow a(1 + r) = -3$

$$ar^5 + ar^6 = 729 \Rightarrow ar^5(1 + r) = 729$$

$$\frac{ar^5(1 + r)}{a(1 + r)} = \frac{729}{-3}$$

$$r^5 = -243$$

$$r = \sqrt[5]{-243} = -3$$

$$a(1 + (-3)) = -3 \Rightarrow -2a = -3 \Rightarrow a = 1.5$$

$$4 \quad u_{12} = 3 \times u_6 \Rightarrow a + 11d = 3(a + 5d)$$

$$a + 11d = 3a + 15d$$

$$2a + 4d = 0$$

$$a + 2d = 0$$

$$S_{30} = 450 \Rightarrow \frac{30}{2}[2a + 29d] = 450$$

$$30a + 435d = 450$$

$$6a + 87d = 90$$

Solving simultaneously,

$$a = -2.4, d = 1.2$$

5 Stamp 1:

55 000, 52 600, ...

A.P. with $a = 55\,000$, $d = -2400$

Stamp 2:

G.P. with $r = 0.96$

For Stamp 1, $u_{10} = 55\,000 + (10 - 1) \times -2400 = 33\,400$

$$a \times 0.96^{10} = 33\,400$$

$$a = \frac{33\,400}{0.96^{10}} = \$50\,238$$

6 (i) The general term is $\binom{10}{r}(x)^{10-r}(a)^r$

The expansion is

$$x^{10} + 10ax^9 + 45a^2x^8 + 120a^3x^7 + 210a^4x^6 + 252a^5x^5$$

$$+ 210a^6x^4 + 120a^7x^3 + 45a^8x^2 + 10a^9x + a^{10}$$

Comparing the coefficients of the terms either side of the x^3 term:

$$120a^7 > 210a^6 \Rightarrow a > \frac{7}{4}$$

$$120a^7 > 45a^8 \Rightarrow a < \frac{8}{3}$$

$$\text{So } \frac{7}{4} < a < \frac{8}{3}$$

(ii) The terms that are of interest are

$$\dots \binom{100}{56}(x)^{44}(a)^{56} + \binom{100}{57}(x)^{43}(a)^{57}$$

$$+ \binom{100}{58}(x)^{42}(a)^{58} \dots$$

$$\binom{100}{57}a^{57} > \binom{100}{56}a^{56} \Rightarrow a > \frac{57}{44}$$

$$\text{and } \binom{100}{57}a^{57} > \binom{100}{58}a^{58} \Rightarrow a < \frac{58}{43}$$

$$\text{So } \frac{57}{44} < a < \frac{58}{43}$$

5 Functions and transformations

5.1 The language of functions (Page 41)

1 (i) $f(x) = x + 3$

(ii) $f(x) = x^2 + 1$

(iii) $f(x) = -2x + 1 = 1 - 2x$

(iv) $f(x) = 10x + 2$

2 All are functions. (ii) is *not* one—one; the rest are one—one.

3

	Function	One—one
(i)	✓	✗
(ii)	✗	✗
(iii)	✓	✓
(iv)	✓	✗

4 (i) $f(-2) = (-2)^2 - 2(-2) + 1 = 9$

(ii) $f(x+1) = (x+1)^2 - 2(x+1) + 1$
 $= x^2 + 2x + 1 - 2x - 2 + 1$
 $= x^2$

5 (i) Domain: $x \in \mathbb{R}$

Range: $f(x) \geq -2$

(ii) Domain: $x \in \mathbb{R}$

Range: $f(x) \leq 3$

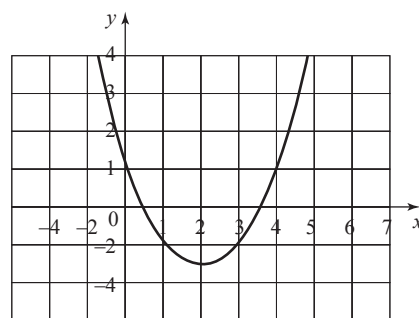
(iii) Domain: $x \geq 1$

Range: $f(x) \geq 2$

(iv) Domain: $x \in \mathbb{R}$

Range: $f(x) \in \mathbb{R}$

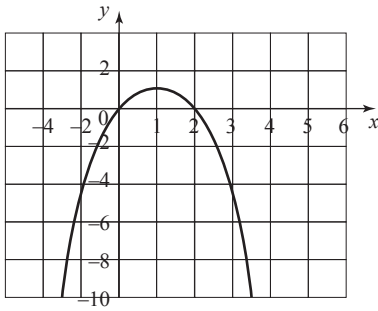
6 Range is $f(x) \geq -2$



7 Domain: $0 \leq x \leq 4$

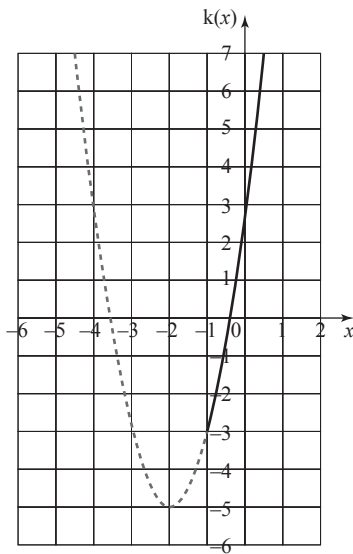
Range: $0 \leq f(x) \leq \sqrt{2}$

8


 Domain: $x \in \mathbb{R}$

 Range: $f(x) \leq 1$

- 9 Given that $x \geq -1$, the range (from the diagram) is $f(x) \geq -3$.



5.2 Composite functions (Page 43)

- 1 (i) $f(-2) = 1 - (-2) = 3$
 (ii) $g(-2) = 1 - (-2)^2 = -3$
 (iii) $fg(-2) = f(-3) = 1 - (-3) = 4$
 (iv) $gf(-2) = g(3) = 1 - (3)^2 = -8$
 (v) $fg(x) = f(1 - x^2) = 1 - (1 - x^2) = x^2$
 (vi) $gf(x) = g(1 - x)$

$$= 1 - (1 - x)^2$$
$$= 1 - (1 - 2x + x^2)$$
$$= 2x - x^2$$

 (vii) $ff(x) = f(1 - x) = 1 - (1 - x) = x$
 (viii) $gg(x) = g(1 - x^2)$

$$= 1 - (1 - x^2)^2$$
$$= 1 - (1 - 2x^2 + x^4)$$
$$= 2x^2 - x^4$$

- 2 $ff(x) = f(ax + b)$

$$= a(ax + b) + b$$
$$= a^2x + ab + b$$

 so $a^2x + ab + b = 9x - 4$
 $\therefore a^2 = 9 \Rightarrow a = \pm 3$
 When $a = 3$ $3b + b = -4$

$$4b = -4$$
$$b = -1$$

 When $a = -3$ $-3b + b = -4$

$$-2b = -4$$
$$b = 2$$

 $a = 3, b = -1$ or $a = -3, b = 2$

- 3 (i) $fg(x) = f\left(4 + \frac{2}{x}\right)$

$$= 3\left(4 + \frac{2}{x}\right) + 2$$
$$= 12 + \frac{6}{x} + 2$$
$$= 14 + \frac{6}{x}$$

 (ii) $gf(x) = g(3x + 2)$

$$= 4 + \frac{2}{3x + 2}$$
$$= \frac{4(3x + 2) + 2}{3x + 2}$$
$$= \frac{12x + 8 + 2}{3x + 2}$$
$$= \frac{12x + 10}{3x + 2}$$

- (iii) $ff(x) = f(3x + 2)$

$$= 3(3x + 2) + 2$$
$$= 9x + 6 + 2$$
$$= 9x + 8$$

 $9x + 8 = -3$
 $9x = -11$
 $x = -\frac{11}{9}$

- (iv) $4 + \frac{2}{x} = kx$
 $4x + 2 = kx^2$
 $0 = kx^2 - 4x - 2$
 For two solutions, $b^2 - 4ac > 0$

$$(-4)^2 - 4 \times k \times -2 > 0$$
$$16 + 8k > 0$$
$$8k > -16$$
$$k > -2$$

$$4 \quad g(1) = -1 \Rightarrow b - a = -1$$

$$gg(1) = 5 \Rightarrow g(-1) = 5 \Rightarrow b + a = 5$$

Solving simultaneously gives

$$a = 3, b = 2$$

$$5 \quad (i) \quad fg(x) = f(x^2 + 4x)$$

$$= 3((x^2 + 4x) + 2)$$

$$= 3x^2 + 12x + 6$$

$$3x^2 + 12x + 6 \geq x^2 + 4x$$

$$2x^2 + 8x + 6 \geq 0$$

$$x^2 + 4x + 3 \geq 0$$

$$(x+3)(x+1) \geq 0$$

$$x \leq -3 \text{ or } x \geq -1$$

$$(ii) \quad gf(x) = g(3(x+2))$$

$$= g(3x+6)$$

$$= (3x+6)^2 + 4(3x+6)$$

$$= 9x^2 + 36x + 36 + 12x + 24$$

$$= 9x^2 + 48x + 60$$

$$gf(x) \leq 45$$

$$9x^2 + 48x + 60 \leq 45$$

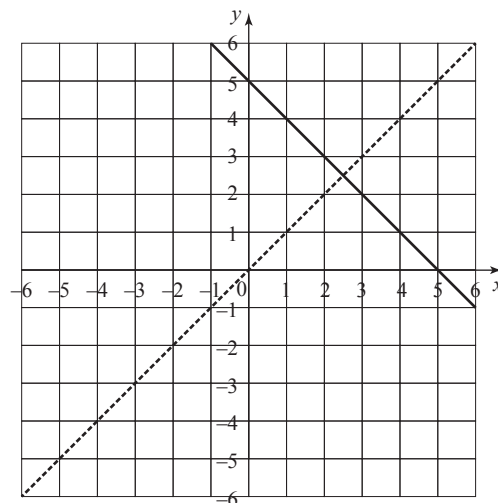
$$9x^2 + 48x + 15 \leq 0$$

$$3x^2 + 16x + 5 \leq 0$$

$$(3x+1)(x+5) \leq 0$$

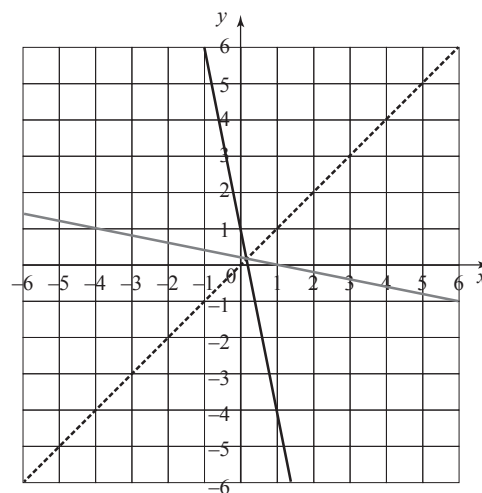
$$-5 \leq x \leq -\frac{1}{3}$$

$$(ii) \quad f^{-1}(x) = 5 - x$$

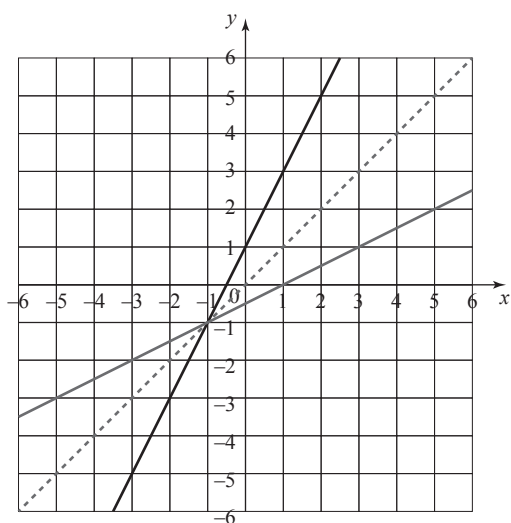
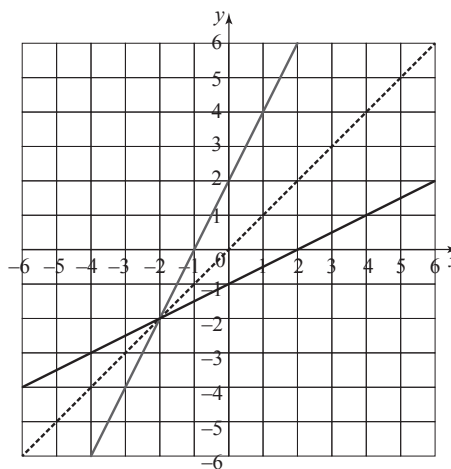


$$(iii) \quad f(x) = 1 - 5x = -5x + 1$$

$$f^{-1}(x) = \frac{x-1}{-5} = \frac{1-x}{5}$$

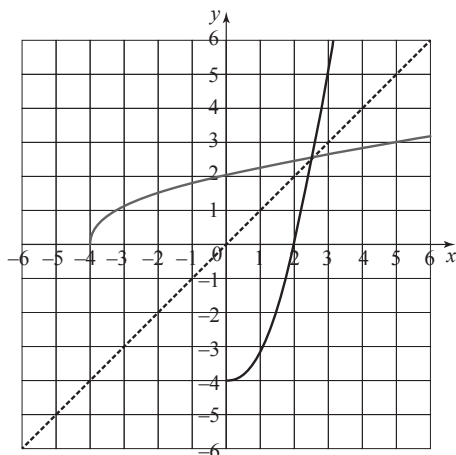


$$(iv) \quad f^{-1}(x) = 2(x+1) = 2x+2$$



5.3 Inverse functions (Page 45)

$$1 \quad (i) \quad f^{-1}(x) = \frac{x-1}{2}$$

2 (i) $k=0$

 Inverse is $x = y^2 - 4$

$$x + 4 = y^2$$

$$\sqrt{x+4} = y$$

$$f^{-1}(x) = \sqrt{x+4}$$

 Domain $f(x)$: $x \geq 0$

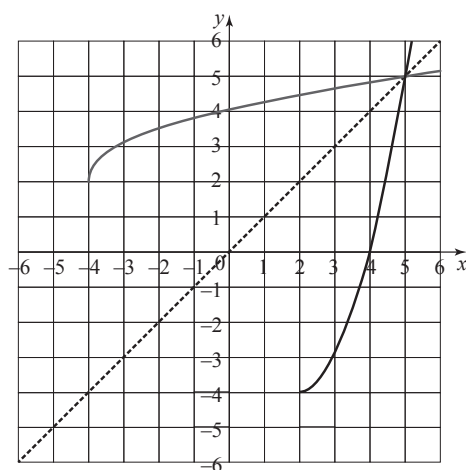
 Range $f(x)$: $f(x) \geq -4$

 Domain $f^{-1}(x)$: $x \geq -4$

 Range $f^{-1}(x)$: $f^{-1}(x) \geq 0$

 (ii) $k=2$

$$y = x^2 - 4x = (x-2)^2 - 4$$


 Inverse is $x = (y-2)^2 - 4$

$$x + 4 = (y-2)^2$$

$$\sqrt{x+4} = y-2$$

$$\sqrt{x+4} + 2 = y$$

$$f^{-1}(x) = \sqrt{x+4} + 2$$

 Domain $f(x)$: $x \geq 2$

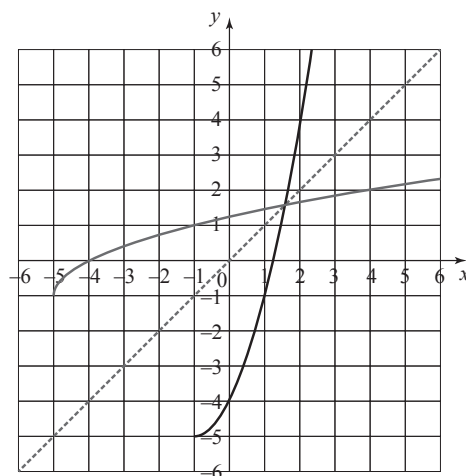
 Range $f(x)$: $f(x) \geq -4$

 Domain $f^{-1}(x)$: $x \geq -4$

 Range $f^{-1}(x)$: $f^{-1}(x) \geq 2$

 (iii) $k=-1$

$$y = x^2 + 2x - 4 = (x+1)^2 - 5$$


 Inverse is $x = (y+1)^2 - 5$

$$x + 5 = (y+1)^2$$

$$\sqrt{x+5} = y+1$$

$$\sqrt{x+5} - 1 = y$$

$$f^{-1}(x) = \sqrt{x+5} - 1$$

 Domain $f(x)$: $x \geq -1$

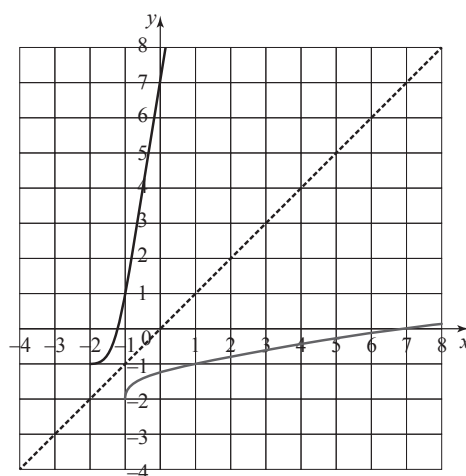
 Range $f(x)$: $f(x) \geq -5$

 Domain $f^{-1}(x)$: $x \geq -5$

 Range $f^{-1}(x)$: $f^{-1}(x) \geq -1$

 (iv) $k=-2$

$$y = 2x^2 + 8x + 7 = 2(x+2)^2 - 1$$



Inverse is $x = 2(y+2)^2 - 1$

$$x+1 = 2(y+2)^2$$

$$\frac{x+1}{2} = (y+2)^2$$

$$\sqrt{\frac{x+1}{2}} = y+2$$

$$\sqrt{\frac{x+1}{2}} - 2 = y$$

$$f^{-1}(x) = \sqrt{\frac{x+1}{2}} - 2$$

Domain $f(x)$: $x \geq -2$

Range $f(x)$: $f(x) \geq -1$

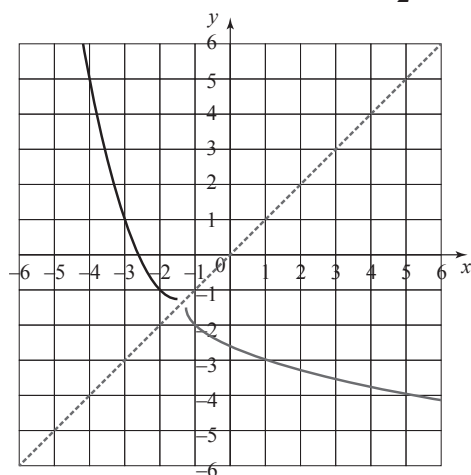
Domain $f^{-1}(x)$: $x \geq -1$

Range $f^{-1}(x)$: $f^{-1}(x) \geq -2$

3 $g(x) = x^2 + 3x + 1 = \left(x + \frac{3}{2}\right)^2 - \frac{5}{4}$

so $x \leq -\frac{3}{2}$ means g will be one-to-one

The largest possible value of k is $-\frac{3}{2}$



4 Inverse is $x = 2(y+1)^3 - 1$

$$x+1 = 2(y+1)^3$$

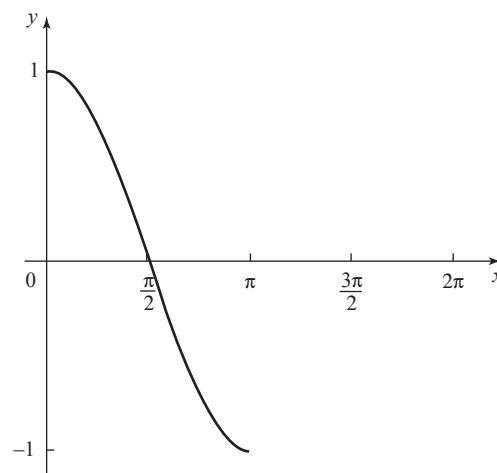
$$\frac{x+1}{2} = (y+1)^3$$

$$\sqrt[3]{\frac{x+1}{2}} = y+1$$

$$\sqrt[3]{\frac{x+1}{2}} - 1 = y$$

$$f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}} - 1$$

5 $k = \pi$

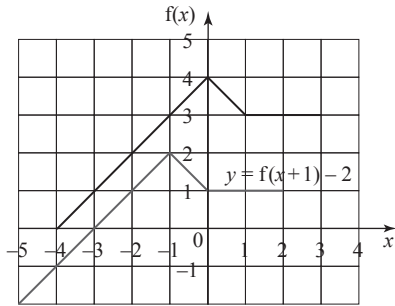


5.4 Transformations (Page 48)

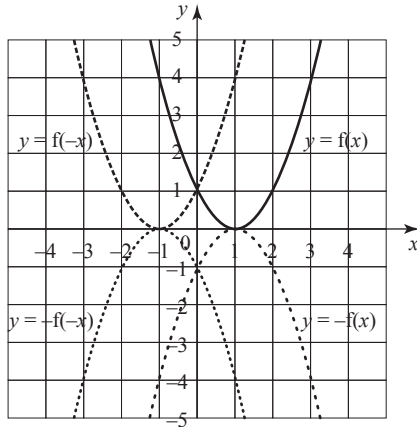
1

Transformation	Detail
For $y = f(x) + b$, the effect of b is to translate the graph of $y = f(x)$ vertically through b units.	- If $b > 0$ it moves upwards. - If $b < 0$ it moves downwards.
For $y = f(x - a)$, the effect of a is to translate the graph of $y = f(x)$ horizontally through a units.	- If $a > 0$ it moves to the right. - If $a < 0$ it moves to the left.
For $y = f(x - a) + b$, the graph of $y = f(x)$ has been translated horizontally by a units and translated vertically by b units.	You say it has been translated by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$
For $y = pf(x)$, $p > 0$, the effect of p is to vertically stretch the graph of $y = f(x)$ by a factor of p .	- If $p > 1$ it moves points on $y = f(x)$ further away from the x-axis. - If $0 < p < 1$ it moves points on $y = f(x)$ closer to the x-axis.
For $y = f(kx)$, $k > 0$, the effect of k is to horizontally stretch the graph of $y = f(x)$ by a factor of $\frac{1}{k}$.	- If $k > 1$ it moves points on $y = f(x)$ closer to the y-axis. - If $0 < k < 1$ it moves points on $y = f(x)$ further away from the y-axis.
For $y = -f(x)$, the effect on the graph of $y = f(x)$ is to reflect it in the x -axis.	
For $y = f(-x)$, the effect on the graph of $y = f(x)$ is to reflect it in the y -axis.	

2



3 (i)



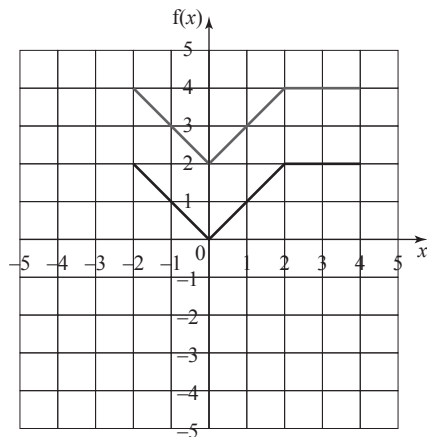
(ii) Translating by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is the same as $y = f(x) + 2$

$$\text{Equation is } y = (x-1)^2 + 2$$

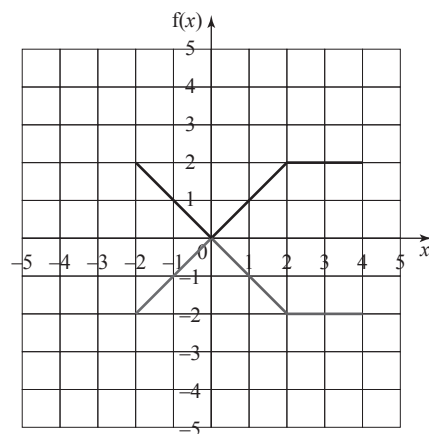
$$y = x^2 - 2x + 1 + 2$$

$$y = x^2 - 2x + 3$$

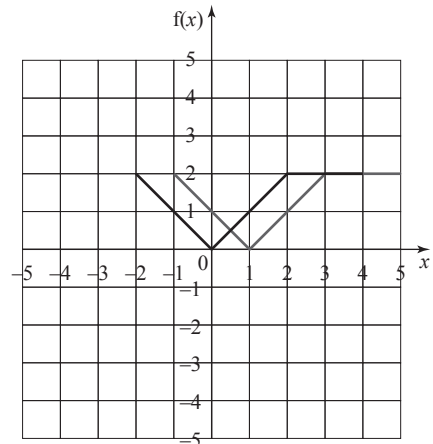
4 (i)



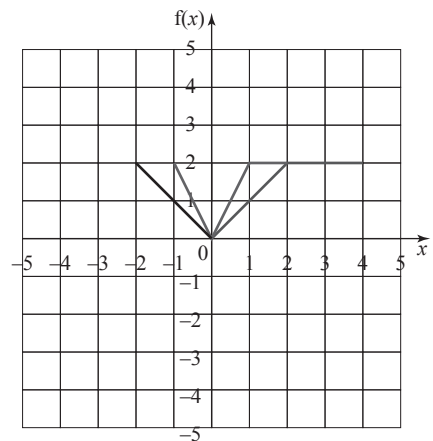
(ii)



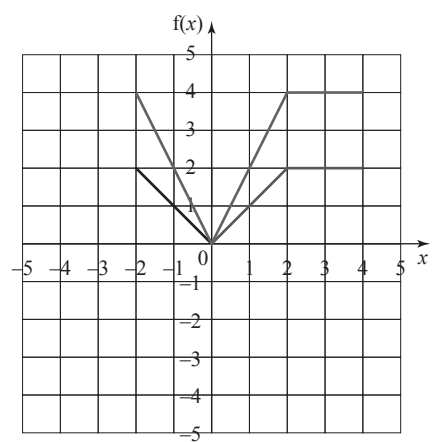
(iii)



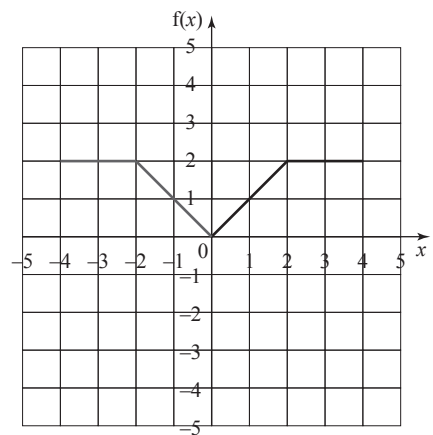
(iv)



(v)

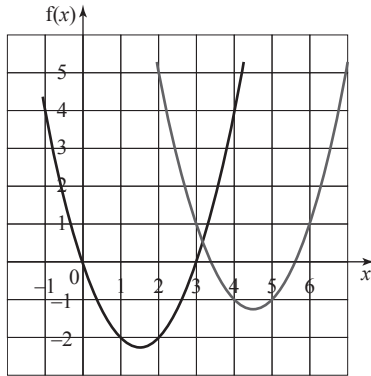


(vi)



- 5 Translation by $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is the same as $f(x-3)+1$

$$\begin{aligned} f(x-3)+1 &= (x-3)^2 - 3(x-3)+1 \\ &= x^2 - 6x + 9 - 3x + 9 + 1 \\ &= x^2 - 9x + 19 \end{aligned}$$



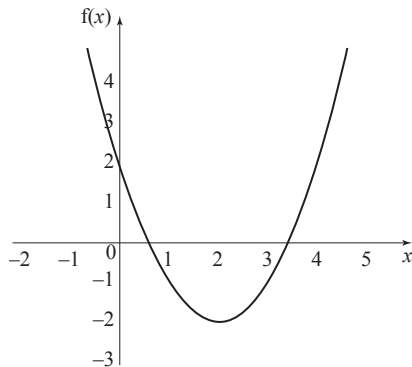
- 6 • Reflection in the y -axis
• Vertical stretch, scale factor 2

Further practice (Page 50)

- 1 Completing the square:

$$f(x) = x^2 - 4x + 2 = (x-2)^2 - 2$$

So the vertex is at $(2, -2)$, as shown on the graph.



The domain of $f(x)$ is $x \in \mathbb{R}$.

The range of $f(x)$ is $y \geq -2$.

The domain of $f^{-1}(x)$ is $x \geq -2$.

The range of $f^{-1}(x)$ is $y \in \mathbb{R}$.

$$\begin{aligned} 2 \text{ (ii)} \quad ff(x) &= f\left(\frac{x+2}{4x-1}\right) = \frac{\left(\frac{x+2}{4x-1}\right)+2}{4\left(\frac{x+2}{4x-1}\right)-1} \\ &= \frac{x+2+2(4x-1)}{4(x+2)-(4x-1)} \\ &= \frac{x+2+8x-2}{4x+8-4x+1} \\ &= \frac{9x}{9} \\ &= x \end{aligned}$$

- (ii) The definition of an inverse is $ff^{-1}(x) = x$ or $f^{-1}f(x) = x$. Since $ff(x) = x$ here, the function f must be its own inverse so $f^{-1}(x) = \frac{x+2}{4x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{4}$.

OR

The original function is $y = \frac{x+2}{4x-1}$ so the inverse is

$$x = \frac{y+2}{4y-1}$$

$$x(4y-1) = y+2$$

$$4xy - x = y+2$$

$$4xy - y = x+2$$

$$y(4x-1) = x+2$$

$$y = \frac{x+2}{4x-1}$$

$$\text{So } f^{-1}(x) = \frac{x+2}{4x-1}, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{4}$$

- 3 (i) $f(-1)=1 \Rightarrow a(-1)+b=1$ so $-a+b=1$

$$f(2)=7 \Rightarrow a(2)+b=7 \text{ so } 2a+b=7$$

$$3a=6 \Rightarrow a=2$$

$$-2+b=1 \Rightarrow b=3$$

- (ii) Given that $f(x) = 2x+3$

$$ff(x) = f(2x+3) = 2(2x+3)+3 = 4x+9$$

$$4x+9=1 \Rightarrow x=-2$$

- 4 Inverse is $x = \frac{2}{3-y}$

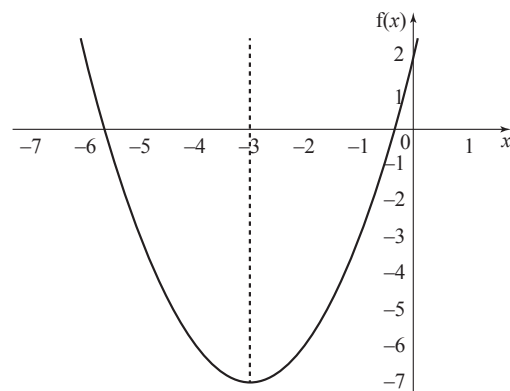
$$x(3-y) = 2$$

$$3-y = \frac{2}{x}$$

$$3 - \frac{2}{x} = y$$

$$j^{-1}(x) = 3 - \frac{2}{x} = \frac{3x-2}{x}$$

- 5 (i) When you are asked this kind of question, what is meant is what is the value of k so that $x \geq k$ means $g(x)$ has an inverse function, or said another way, that the inverse of $g(x)$ is a function.



Completing the square:

$$y = (x+3)^2 - 7$$

This form shows us the vertex is at $(-3, -7)$ as shown.

The value of x that divides the parabola down the middle is the value of x needed. This is -3 .

So $k = -3$.

$$(ii) \quad y = (x+3)^2 - 7$$

$$x = (y+3)^2 - 7$$

$$x+7 = (y+3)^2$$

$$\sqrt{x+7} = y+3$$

$$\sqrt{x+7} - 3 = y$$

$$g^{-1}(x) = \sqrt{x+7} - 3$$

$$6 \quad (ii) \quad gf(x) = g(2-3x) = \frac{(2-3x)+2}{2(2-3x)+7} = \frac{4-3x}{11-6x}$$

$$gf(x) = x \Rightarrow \frac{4-3x}{11-6x} = x$$

$$4-3x = x(11-6x)$$

$$4-3x = 11x-6x^2$$

$$6x^2 - 14x + 4 = 0$$

$$3x^2 - 7x + 2 = 0$$

$$(3x-1)(x-2) = 0$$

$$x = \frac{1}{3} \text{ or } x = 2$$

$$(iii) \quad f^{-1}(x) \text{ is } x = 2-3y \Rightarrow 3y = 2-x \Rightarrow y = \frac{2-x}{3}$$

$$\text{so } f^{-1}(x) = \frac{2-x}{3}$$

$$g^{-1}(x) \text{ is } x = \frac{y+2}{2y+7}$$

$$x(2y+7) = y+2$$

$$2xy+7x = y+2$$

$$2xy - y = 2-7x$$

$$y(2x-1) = 2-7x$$

$$y = \frac{2-7x}{2x-1}$$

$$g^{-1}(x) = \frac{2-7x}{2x-1}$$

$$(iii) \quad g^{-1}(x) = x-4 \Rightarrow \frac{2-7x}{2x-1} = x-4$$

$$2-7x = (x-4)(2x-1)$$

$$2-7x = 2x^2-9x+4$$

$$0 = 2x^2-2x+2$$

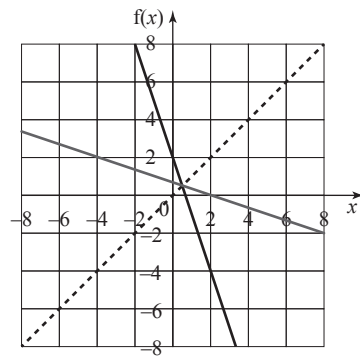
$$0 = x^2-x+1$$

Since $b^2 - 4ac = (-1)^2 - 4 \times 1 \times 1 = -3$, the equation has no real solutions.

$$(iv) \quad f^{-1}g(x) = f^{-1}\left(\frac{x+2}{2x+7}\right) \\ = \frac{2 - \left(\frac{x+2}{2x+7}\right)}{3} \\ = \frac{2(2x+7) - (x+2)}{3(2x+7)} \\ = \frac{4x+14-x-2}{3(2x+7)} \\ = \frac{3x+12}{3(2x+7)} \\ = \frac{3(x+4)}{3(2x+7)} \\ = \frac{x+4}{2x+7}$$

(v)

Point on $y = f(x)$	Point on $y = f^{-1}(x)$
$(0, 2)$	$(2, 0)$
$(1, -1)$	$(-1, 1)$
$(-1, 5)$	$(5, -1)$



7 Translation by $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is the equivalent of

$$f(x+1)+2 = ((x+1)^3+1)+2 \\ = (x^3+3x^2+3x+1+1)+2 \\ = x^3+3x^2+3x+4$$

- 8 • Reflection in the x -axis
• Horizontal stretch, scale factor $\frac{1}{2}$

Past exam questions (Page 51)

$$\begin{aligned}
 1 \quad (i) \quad fg(x) &= f\left(\frac{4}{5x+2}\right) \\
 &= \frac{\frac{4}{5x+2}}{\frac{4}{5x+2}} - 2 \\
 &= \frac{4(5x+2)}{4} - 2 \\
 &= 5x
 \end{aligned}$$

Range of fg is $y \geq 0$

$$\begin{aligned}
 (ii) \quad g^{-1}(x) \text{ is given by } x &= \frac{4}{5y+2} \\
 x(5y+2) &= 4 \\
 5y+2 &= \frac{4}{x} \\
 5y &= \frac{4}{x} - 2 \\
 y &= \frac{4}{5x} - \frac{2}{5} \\
 y &= \frac{4-2x}{5x}
 \end{aligned}$$

Range of g is $0 < y \leq 2$ sodomain of g^{-1} is $0 < x \leq 2$

$$\begin{aligned}
 2 \quad (i) \quad f(x) \text{ is given by } x &= \frac{1-5y}{2y} \\
 2xy &= 1-5y \\
 2xy+5y &= 1 \\
 y(2x+5) &= 1 \\
 y &= \frac{1}{2x+5} \\
 f(x) &= \frac{1}{2x+5}
 \end{aligned}$$

Range of f^{-1} is $y \geq -\frac{9}{4}$ Domain of f is $x \geq -\frac{9}{4}$

$$\begin{aligned}
 (ii) \quad f^{-1}g(x) &= f^{-1}\left(\frac{1}{x}\right) \\
 &= \frac{1-5\left(\frac{1}{x}\right)}{2\left(\frac{1}{x}\right)} \\
 &= \frac{1-\frac{5}{x}}{\frac{2}{x}} \\
 &= \left(1-\frac{5}{x}\right)\left(\frac{x}{2}\right) \\
 &= \frac{x}{2} - \frac{5}{2} \\
 a &= \frac{1}{2}, b = -\frac{5}{2}
 \end{aligned}$$

3 (i) Since g is defined for $x \leq -1$, the domain of f must be $x \leq -1$.

$$3x+1 \leq -1$$

$$3x \leq -2$$

$$x \leq -\frac{2}{3}$$

Largest value of a is $-\frac{2}{3}$

$$(ii) \quad fg(x)+14=0$$

$$f(-1-x^2)+14=0$$

$$3(-1-x^2)+1+14=0$$

$$-3-3x^2+15=0$$

$$3x^2=12$$

$$x^2=4$$

$$x=\pm 2$$

But since $x \leq -1$, $x = -2$

$$(iii) \quad gf(x) \leq -50$$

$$g(3x+1) \leq -50$$

$$-1-(3x+1)^2 \leq -50$$

$$49 \leq (3x+1)^2$$

$$(3x+1)^2 \geq 49$$

$$3x+1 \geq 7 \text{ or } 3x+1 \leq -7$$

$$x \geq 2 \text{ or } x \leq -\frac{8}{3}$$

But since $x \leq -1$, $x \leq -\frac{8}{3}$

$$4 \quad (i) \quad gf(x)=g(2x^2+3)$$

$$=3(2x^2+3)+2$$

$$=6x^2+11$$

$$fg(x)=f(3x+2)$$

$$=2(3x+2)^2+3$$

$$=2(9x^2+12x+4)+3$$

$$=18x^2+24x+11$$

$$(ii) \quad (fg)^{-1}(x) \text{ is}$$

$$x=2(3y+2)^2+3$$

$$x-3=2(3y+2)^2$$

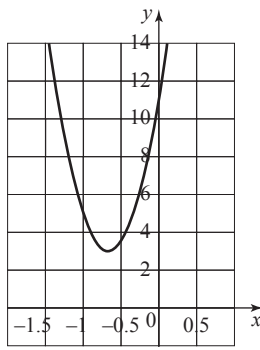
$$\pm\sqrt{\frac{x-3}{2}}=3y+2$$

Since $x \geq 0$, only use positive root:

$$(fg)^{-1}(x)=\frac{1}{3}\sqrt{\frac{x-3}{2}}-\frac{2}{3}$$

Domain of $(fg)^{-1}(x)$ is range of $fg(x)$

Since $x \geq 0$, range is $y \geq 11$



(iii) $gf(2x) = fg(x)$

$$6(2x)^2 + 11 = 18x^2 + 24x + 11$$

$$24x^2 + 11 = 18x^2 + 24x + 11$$

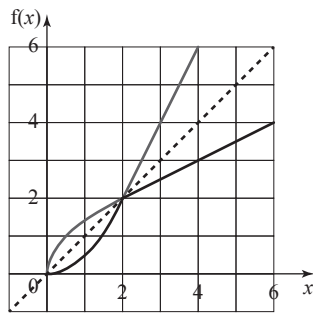
$$6x^2 - 24x = 0$$

$$6x(x - 4) = 0$$

$$x = 0 \text{ or } x = 4$$

5 (i) Range of f is $0 \leq f \leq 4$

(ii)



(iii) For the first function, the inverse is

$$x = \frac{1}{2}y^2$$

$$2x = y^2$$

$$y = \sqrt{2x} \text{ for } 0 \leq x \leq 2$$

For the second function, the inverse is

$$x = \frac{1}{2}y + 1$$

$$2x = y + 2$$

$$y = 2x - 2 \text{ for } 2 < x \leq 4$$

$$f^{-1}(x) = \begin{cases} \sqrt{2x} & \text{for } 0 \leq x \leq 2 \\ 2x - 2 & \text{for } 2 < x \leq 4 \end{cases}$$

6 (i) $f(2) = 6 + a$

$$ff(2) = f(6 + a) = 3(6 + a) + a = 18 + 4a$$

$$18 + 4a = 10 \Rightarrow a = -2$$

$$g^{-1}(2) = 3 \Rightarrow g(3) = 2$$

$$b - 2 \times 3 = 2 \Rightarrow b = 8$$

(ii) $f(x) = 3x - 2$ and $g(x) = 8 - 2x$

$$fg(x) = f(8 - 2x) = 3(8 - 2x) - 2$$

$$= 24 - 6x - 2$$

$$= 22 - 6x$$

Stretch and challenge (Page 52)

1 $g(1) = f(0) = 1$

$$g(2) = fg(1) = f(1) = 3$$

$$g(3) = fg(2) = f(3) = 7$$

$$g(4) = fg(3) = f(7) = 15$$

A pattern can be seen here: every number is 1 less than a power of 2 so $g(n) = 2^n - 1$.

$$h(0) = g(2) = 3$$

$$h(1) = gh(0) = g(3) = 7$$

$$h(2) = gh(1) = g(7) = 2^7 - 1 = 127$$

2 $f(x+1) + f(x-1) = af(x)$

$$3^{x+1} + 3^{x-1} = a3^x$$

$$a = \frac{3^{x+1} + 3^{x-1}}{3^x} = \frac{3^x(3 + 3^{-1})}{3^x} = 3 + 3^{-1} = \frac{10}{3}$$

3 $f(2n-1) = a(2n-1) + b = 2an - a + b$

$$f(2n) - 1 = a(2n) + b - 1 = 2an + b - 1$$

$$2f(n) - 1 = 2(an + b) - 1 = 2an + 2b - 1$$

$2an + b$ is common to all three expressions so the three expressions can be simplified to $-a, -1$ and $b - 1$

Since -1 is one of the terms, the possible consecutive terms are (i) $-3, -2, -1$ (ii) $-2, -1, 0$ or (iii) $-1, 0, 1$

(i) $a = 3, b = -1 \Rightarrow f(n) = 3n - 1$

$$a = 2, b = -2 \Rightarrow f(n) = 2n - 2$$

(ii) $a = 2, b = 1 \Rightarrow f(n) = 2n + 1$

(iii) $a = -1, b = 1 \Rightarrow f(n) = -n + 1$

6 Differentiation

6.1 Basic differentiation (Page 53)

- 1 (i) $y = 10x^2 - 3x + 1 \Rightarrow \frac{dy}{dx} = 20x - 3$
- (ii) $y = 4 + x - 5x^4 \Rightarrow \frac{dy}{dx} = 1 - 20x^3$
- (iii) $y = \frac{5}{x^2} = 5x^{-2} \Rightarrow \frac{dy}{dx} = -10x^{-3} = -\frac{10}{x^3}$
- (iv) $y = \frac{2x^3}{3} \Rightarrow \frac{dy}{dx} = \frac{6x^2}{3} = 2x^2$
- (v) $y = \frac{4}{x} - \frac{x}{4} = 4x^{-1} - \frac{1}{4}x$
 $\Rightarrow \frac{dy}{dx} = -4x^{-2} - \frac{1}{4} = -\frac{4}{x^2} - \frac{1}{4}$
- (vi) $y = \frac{1}{4x^3} = \frac{x^{-3}}{4} \Rightarrow \frac{dy}{dx} = \frac{-3x^{-4}}{4} = -\frac{3}{4x^4}$
- (vii) $y = 2\sqrt{x} - 3x = 2x^{\frac{1}{2}} - 3x$
 $\Rightarrow \frac{dy}{dx} = x^{-\frac{1}{2}} - 3 = \frac{1}{\sqrt{x}} - 3$
- (viii) $y = \sqrt[3]{x} + \frac{2}{3x} = x^{\frac{1}{3}} + \frac{2x^{-1}}{3}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} + \frac{-2x^{-2}}{3} = \frac{1}{3\sqrt[3]{x^2}} - \frac{2}{3x^2}$

2 (i) $f(x) = \frac{2x+1}{x^2} = \frac{2x}{x^2} + \frac{1}{x^2} = 2x^{-1} + x^{-2}$
 $f'(x) = -2x^{-2} - 2x^{-3} = -\frac{2}{x^2} - \frac{2}{x^3}$

(ii) $f(x) = 6\sqrt[3]{x^5} = 6x^{\frac{5}{3}}$
 $f'(x) = 6 \times \frac{5}{3}x^{\frac{2}{3}} = 10\sqrt[3]{x^2}$

3 $f(x) = \frac{6}{\sqrt{x}} + 5x = 6x^{-\frac{1}{2}} + 5x$
 $f'(x) = -3x^{-\frac{3}{2}} + 5 = -\frac{3}{\sqrt{x}} + 5$
 $f'(9) = -\frac{3}{\sqrt{9}} + 5 = 4$

4 $y = 3 - 8x - x^2 \Rightarrow \frac{dy}{dx} = -8 - 2x$
 $-8 - 2x = 2$
 $-2x = 10$
 $x = -5$
 $y = 3 - 8 \times -5 - (-5)^2 = 18$
 Point is $(-5, 18)$

5 $f(x) = x^3 + x^2 + 2x - 1 \Rightarrow f'(x) = 3x^2 + 2x + 2$
 $3x^2 + 2x + 2 = 3$

$$3x^2 + 2x - 1 = 0$$

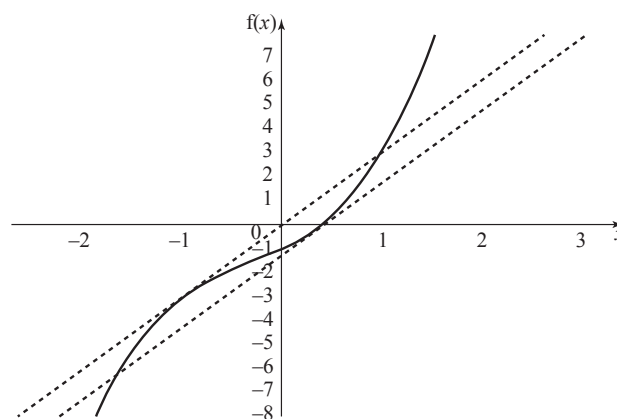
$$(3x - 1)(x + 1) = 0$$

$$x = \frac{1}{3} \text{ or } -1$$

$$y = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 + 2 \times \left(\frac{1}{3}\right) - 1 = -\frac{5}{27}$$

$$y = (-1)^3 + (-1)^2 + 2 \times (-1) - 1 = -3$$

Points are $\left(\frac{1}{3}, -\frac{5}{27}\right)$ and $(-1, -3)$



6 (i)	Chord	AE	BE	CE	DE
	Gradient of chord	6	7	7.4	7.9

(ii) $f'(2) = 8$

6.2 Tangents and normals (Page 55)

1 $\frac{dy}{dx} = 8x^3 + 2x$

At $x = -1$, $\frac{dy}{dx} = 8 \times (-1)^3 + 2 \times (-1) = -10$

Equation of the tangent:

$$y = mx + c \Rightarrow 2 = -10 \times -1 + c \Rightarrow c = -8$$

$$y = -10x - 8$$

2 $g(x) = \frac{4}{\sqrt{x}} - 3 = 4x^{-\frac{1}{2}} - 3$

$$g(4) = \frac{4}{\sqrt{4}} - 3 = -1$$

$$g'(x) = -2x^{-\frac{3}{2}} = -\frac{2}{\sqrt{x^3}}$$

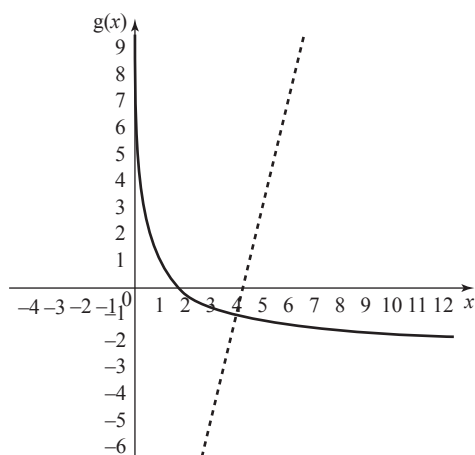
$$g'(4) = -\frac{2}{\sqrt{4^3}} = -\frac{1}{4}$$

So the gradient of the normal is 4.

Equation is

$$y = mx + c \Rightarrow -1 = 4 \times 4 + c \Rightarrow c = -17$$

$$y = 4x - 17$$



- 3 (i) $h(x) = 6x - x^2$
 $h'(x) = 6 - 2x$
 $h'(2) = 6 - 2 \times 2 = 2$
 $y = mx + c \Rightarrow 8 = 2 \times 2 + c \Rightarrow c = 4$
 $y = 2x + 4 \Rightarrow 2x - y + 4 = 0$

(ii) Gradient of normal is $-\frac{1}{2}$

Equation of normal is

$$y = mx + c \Rightarrow 8 = -\frac{1}{2} \times 2 + c \Rightarrow c = 9$$

$$y = -\frac{1}{2}x + 9$$

$$-\frac{1}{2}x + 9 = 6x - x^2$$

$$-x + 18 = 12x - 2x^2$$

$$2x^2 - 13x + 18 = 0$$

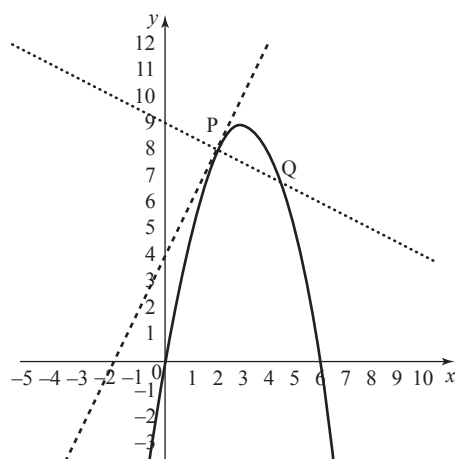
$$(2x - 9)(x - 2) = 0$$

$$x = \frac{9}{2} \text{ or } 2$$

So the y -coordinate of Q is

$$-\frac{1}{2} \times \frac{9}{2} + 9 = 6\frac{3}{4}$$

$$Q \text{ is } \left(4\frac{1}{2}, 6\frac{3}{4}\right)$$



$$4 \quad y = x + \frac{k}{x} = x + kx^{-1}$$

$$\frac{dy}{dx} = 1 - kx^{-2} = 1 - \frac{k}{x^2}$$

Gradient of the tangent at $x = -1$ is

$$1 - \frac{k}{(-1)^2} = 1 - k$$

Gradient of the normal is $-\frac{1}{1-k}$

$$-\frac{1}{1-k} = \frac{1}{4} \Rightarrow -4 = 1 - k \Rightarrow k = 5$$

6.3 Maximum and minimum points and increasing and decreasing functions

(Page 56)

$$1 \quad (i) \quad y = 2x^3 - 3x^2$$

$$\frac{dy}{dx} = 6x^2 - 6x$$

Stationary points when

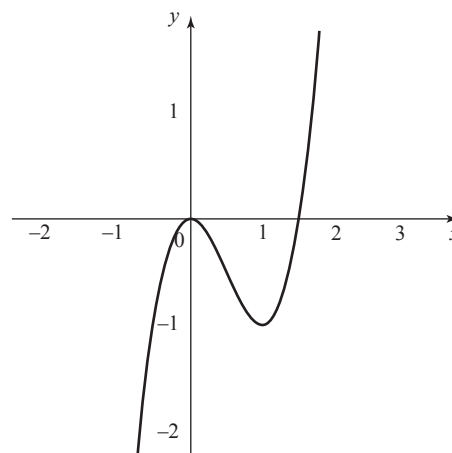
$$6x^2 - 6x = 0$$

$$6x(x - 1) = 0$$

$$x = 0 \text{ or } 1$$

$$y = 0 \text{ or } -1$$

$$(0, 0) \text{ or } (1, -1)$$



$$\frac{d^2y}{dx^2} = 12x - 6$$

When $x = 0$, $\frac{d^2y}{dx^2} < 0$ so $(0, 0)$ is a local maximum

When $x = 1$, $\frac{d^2y}{dx^2} > 0$ so $(1, -1)$ is a local minimum

$$(ii) \quad f(x) = 4x + \frac{1}{x} = 4x + x^{-1}$$

$$f'(x) = 4 - x^{-2} = 4 - \frac{1}{x^2}$$

For stationary points,

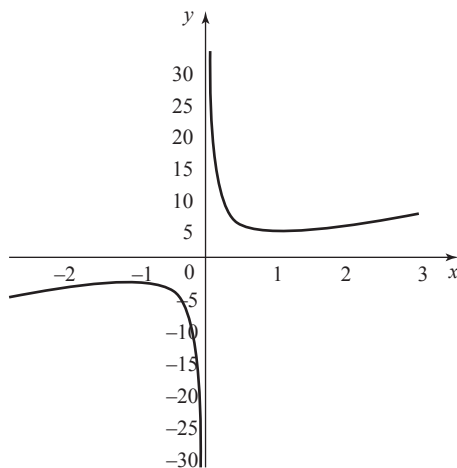
$$4 - \frac{1}{x^2} = 0 \Rightarrow 4 = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{4}$$

$$x = \frac{1}{2} \text{ or } -\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = 4 \times \frac{1}{2} + \frac{1}{\frac{1}{2}} = 4$$

$$f\left(-\frac{1}{2}\right) = 4 \times -\frac{1}{2} + \frac{1}{-\frac{1}{2}} = -4$$

Points are $\left(\frac{1}{2}, 4\right)$ and $\left(-\frac{1}{2}, -4\right)$



$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$

$$f''\left(\frac{1}{2}\right) > 0 \Rightarrow \left(\frac{1}{2}, 4\right) \text{ is a local minimum}$$

$$f''\left(-\frac{1}{2}\right) < 0 \Rightarrow \left(-\frac{1}{2}, -4\right) \text{ is a local maximum}$$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ (note that $x \neq 0$)

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}} = 0 \Rightarrow \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x^3}}$$

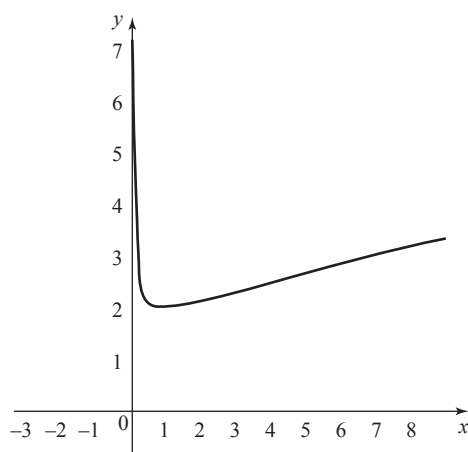
$$\Rightarrow 2\sqrt{x^3} = 2\sqrt{x}$$

$$\Rightarrow \sqrt{x^3} - \sqrt{x} = 0$$

$$x = 1 \quad (x \neq 0)$$

$$y = \sqrt{1} + \frac{1}{\sqrt{1}} = 2$$

Point is $(1, 2)$



$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}} = -\frac{1}{4\sqrt{x^3}} + \frac{3}{4\sqrt{x^5}}$$

When $x = 1$, $\frac{d^2y}{dx^2} = \frac{1}{2} > 0$ so $x = 1$ is a local min.

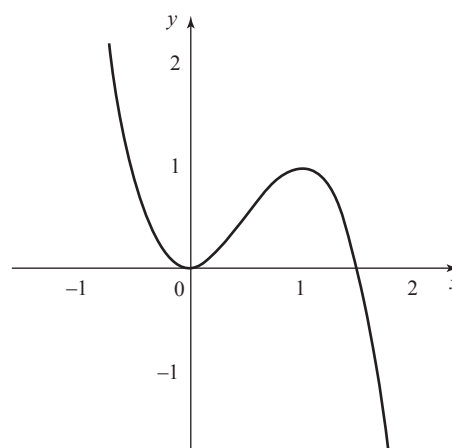
2 $g(x) = 3x^2 - 2x^3$

$$g'(x) = 6x - 6x^2 = 6x(1-x)$$

Increasing when $6x(1-x) > 0$

Solving the quadratic inequality,

$$0 < x < 1$$



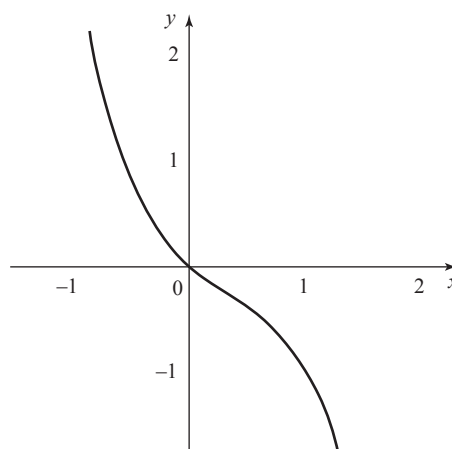
3 $y = -x + x^2 - x^3 \Rightarrow \frac{dy}{dx} = -1 + 2x - 3x^2$

Decreasing when $-1 + 2x - 3x^2 < 0$

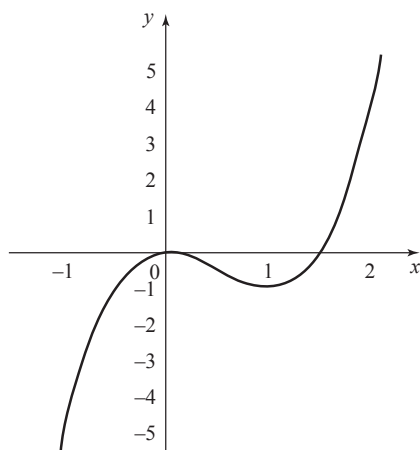
For this quadratic,

$$b^2 - 4ac = 2^2 - 4 \times -3 \times -1 = -8$$

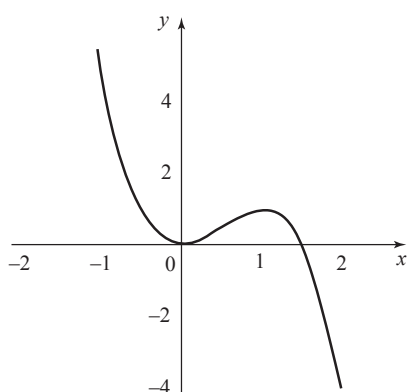
Since the discriminant is < 0 , there are no solutions when $-1 + 2x - 3x^2 = 0$ so $-1 + 2x - 3x^2$ is always < 0 , as shown in the diagram.



4 (i) $y = 2x^3 - 3x^2$



(ii) $g(x) = 3x^2 - 2x^3$



5 (i) Stationary point at $x = 1$ means that $f'(1) = 0$

$$3 \times 1^2 + k \times 1 - 8 = 0$$

$$3 + k - 8 = 0$$

$$k - 5 = 0$$

$$k = 5$$

(ii) Stationary points when $3x^2 + 5x - 8 = 0$

$$(3x + 8)(x - 1) = 0$$

$$x = -\frac{8}{3} \text{ or } 1$$

The other stationary point is when $x = -\frac{8}{3} = -2\frac{2}{3}$

(iii) $f'(x) = 3x^2 + 5x - 8 \Rightarrow f''(x) = 6x + 5$

$$f''(1) = 6 \times 1 + 5 = 11 > 0$$

$\Rightarrow x = 1$ is a minimum

$$f''\left(-\frac{8}{3}\right) = 6 \times \left(-\frac{8}{3}\right) + 5 = -11 < 0$$

$\Rightarrow x = -\frac{8}{3}$ is a maximum

6.4 The chain rule (Page 58)

1 (i) $y = (x-4)^7 \Rightarrow \frac{dy}{dx} = 7(x-4)^6$

(ii) $y = (5-2x)^9 \Rightarrow \frac{dy}{dx} = 9(5-2x)^8 \times -2$
 $= -18(5-2x)^8$

(iii) $y = \sqrt{3+4x} = (3+4x)^{\frac{1}{2}}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{2}(3+4x)^{-\frac{1}{2}} \times 4$
 $= \frac{2}{\sqrt{3+4x}}$

(iv) $y = \frac{2}{x+3} = 2(x+3)^{-1}$
 $\Rightarrow \frac{dy}{dx} = -2(x+3)^{-2} = -\frac{2}{(x+3)^2}$

(v) $y = \frac{4}{(1-2x)^2} = 4(1-2x)^{-2}$
 $\Rightarrow \frac{dy}{dx} = -8(1-2x)^{-3} \times -2 = \frac{16}{(1-2x)^3}$

2 $f(x) = 3\left(1 - \frac{x}{6}\right)^4$
 $f(12) = 3\left(1 - \frac{12}{6}\right)^4 = 3$
 $f'(x) = 12\left(1 - \frac{x}{6}\right)^3 \times -\frac{1}{6} = -2\left(1 - \frac{x}{6}\right)^3$
 $f'(12) = -2\left(1 - \frac{12}{6}\right)^3 = 2$

Equation of the tangent is

$$y = mx + c \Rightarrow 3 = 2 \times 12 + c \Rightarrow c = -21$$

$$y = 2x - 21$$

3 $y = \frac{8}{x^2 - 4x} = 8(x^2 - 4x)^{-1}$

$$\frac{dy}{dx} = -8(x^2 - 4x)^{-2} \times (2x - 4)$$

$$= \frac{-8(2x - 4)}{(x^2 - 4x)^2}$$

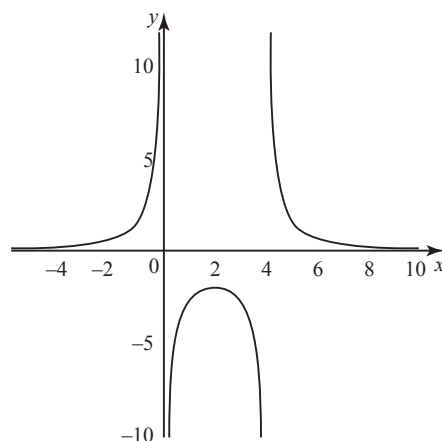
Stationary point when $\frac{-8(2x - 4)}{(x^2 - 4x)^2} = 0$

$$-8(2x - 4) = 0$$

$$x = 2$$

$$y = \frac{8}{(2)^2 - 4 \times 2} = -2$$

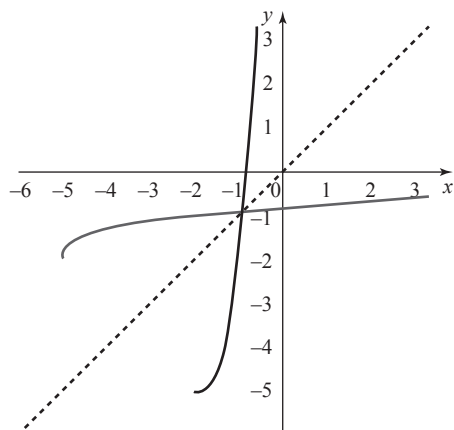
Point is $(2, -2)$



4 $g'(x) = 9(x+2)^2$

Since $g'(x) > 0$ for all values of x , the curve is always increasing so there are no stationary points. Hence the function is one-to-one so it has an inverse.

The graphs of $y = g(x)$ for $x > -2$ and its inverse are shown below.



5 (i) $\frac{dy}{dx} = -2(x-1)^{-2} + 2 = -\frac{2}{(x-1)^2} + 2$
 $\frac{d^2y}{dx^2} = 4(x-1)^{-3} = \frac{4}{(x-1)^3}$

(ii) Stationary points when $\frac{dy}{dx} = 0$

$$-\frac{2}{(x-1)^2} + 2 = 0 \Rightarrow 2 = \frac{2}{(x-1)^2}$$

$$2(x-1)^2 = 2$$

$$(x-1)^2 = 1$$

$$x-1 = \pm 1$$

$$x = 0 \text{ or } 2$$

When $x = 0$, $y = \frac{2}{0-1} + 2 \times 0 = -2$

When $x = 2$, $y = \frac{2}{2-1} + 2 \times 2 = 6$

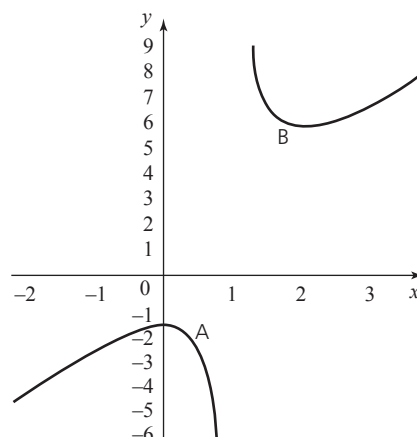
When $x = 0$,

$$\frac{d^2y}{dx^2} = \frac{4}{(0-1)^3} < 0 \Rightarrow x = 0 \text{ is a maximum}$$

When $x = 2$,

$$\frac{d^2y}{dx^2} = \frac{4}{(2-1)^3} > 0 \Rightarrow x = 2 \text{ is a minimum}$$

So maximum point A is $(0, -2)$, minimum point B is $(2, 6)$.



6.5 Applications (Page 59)

1 Let the two numbers be x and y .

$$x + y = 18 \Rightarrow y = 18 - x$$

$$P = xy = x(18 - x) = 18x - x^2$$

$$\frac{dP}{dx} = 18 - 2x$$

Maximum when $\frac{dP}{dx} = 0$

$$18 - 2x = 0$$

$$x = 9$$

$$y = 18 - 9 = 9$$

Maximum value of P is $9 \times 9 = 81$

2 $6x + 4y = 36 \Rightarrow y = \frac{36 - 6x}{4} = 9 - 1.5x$

$$A = xy = x(9 - 1.5x) = 9x - 1.5x^2$$

$$\frac{dA}{dx} = 9 - 3x$$

Maximum when $\frac{dA}{dx} = 0$

$$9 - 3x = 0 \Rightarrow x = 3 \text{ m}$$

$$y = 9 - 1.5 \times 3 = 4.5 \text{ m}$$

Dimensions are 3 m by 4.5 m

3 (i) $V = \pi r^2 h = 5000 \Rightarrow h = \frac{5000}{\pi r^2}$

$$S = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi r \left(\frac{5000}{\pi r^2} \right)$$

$$= 2\pi r^2 + \frac{10000}{r}$$

$$(ii) S = 2\pi r^2 + \frac{10\,000}{r} = 2\pi r^2 + 10\,000r^{-1}$$

$$\frac{dS}{dr} = 4\pi r - 10\,000r^{-2} = 4\pi r - \frac{10\,000}{r^2}$$

Stationary value when $\frac{dS}{dr} = 0$

$$4\pi r - \frac{10\,000}{r^2} = 0$$

$$4\pi r = \frac{10\,000}{r^2}$$

$$4\pi r^3 = 10\,000$$

$$r = \sqrt[3]{\frac{10\,000}{4\pi}} = 9.27 \text{ cm}$$

$$(iii) \frac{d^2S}{dr^2} = 4\pi + 20\,000r^{-3} = 4\pi + \frac{20\,000}{r^3}$$

$$\text{When } r = 9.27, \frac{d^2S}{dr^2} = 4\pi + \frac{20\,000}{9.27^3} > 0$$

So $r = 9.27$ is a minimum value

- 4 Let the length of the square end be x .

Let the length be y .

$$4x + y = 120 \text{ so } y = 120 - 4x$$

$$V = x^2y$$

$$= x^2(120 - 4x)$$

$$= 120x^2 - 4x^3$$

$$\frac{dV}{dx} = 240x - 12x^2$$

Maximum volume when $\frac{dV}{dx} = 0$

$$240x - 12x^2 = 0$$

$$12x(20 - x) = 0$$

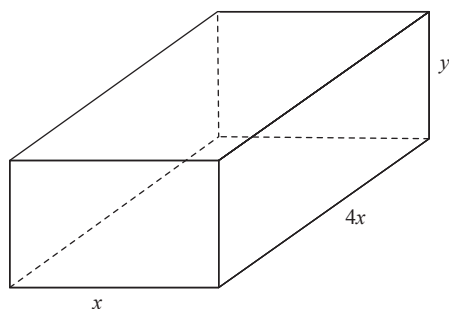
$$x = 0 \text{ cm or } 20 \text{ cm}$$

Since $x \neq 0$, $x = 20$ cm

$$y = 120 - 4 \times 20 = 40 \text{ cm}$$

Volume is $16\,000 \text{ cm}^3$

5



$$V = 4x^2y$$

$$4x^2y = 50 \Rightarrow y = \frac{50}{4x^2}$$

The cost of the open box is

$$C = 12 \times 4x^2 + 5(xy + xy + 4xy + 4xy)$$

$$= 48x^2 + 5(10xy)$$

$$= 48x^2 + 50xy$$

$$= 48x^2 + 50x \left(\frac{50}{4x^2} \right)$$

$$= 48x^2 + \frac{625}{x}$$

$$= 48x^2 + 625x^{-1}$$

$$\frac{dC}{dx} = 96x - 625x^{-2} = 96x - \frac{625}{x^2}$$

Minimum when $\frac{dC}{dx} = 0$

$$96x - \frac{625}{x^2} = 0$$

$$96x^3 - 625 = 0$$

$$96x^3 = 625$$

$$x^3 = \frac{625}{96}$$

$$x = \sqrt[3]{\frac{625}{96}} = 1.87 \text{ m (3 s.f.)}$$

$$\text{When } x = 1.87, \frac{d^2C}{dx^2} = 96 + \frac{1250}{x^3} > 0$$

So $x = 1.87$ is a minimum.

Dimensions are

length = 1.87 m, width = 7.47 m, height = 3.56 m

6.6 Rates of change (Page 61)

- 1 Let the length of the side of the square be x .

Let the area of the square be A .

$$A = x^2 \Rightarrow \frac{dA}{dx} = 2x \Rightarrow \frac{dx}{dA} = \frac{1}{2x}$$

$$\frac{dx}{dt} = \frac{dA}{dt} \times \frac{dx}{dA}$$

$$= 8 \times \frac{1}{2x}$$

$$= 8 \times \frac{1}{2 \times 20}$$

$$= 0.2 \text{ cm/s}$$

- 2 $V = \pi r^2 h = \pi \times 1.2^2 \times h = 1.44\pi h$

$$\frac{dV}{dh} = 1.44\pi \Rightarrow \frac{dh}{dV} = \frac{1}{1.44\pi}$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= 0.5 \times \frac{1}{1.44\pi}$$

$$= 0.111 \text{ m/min (3 s.f.)}$$

$$= 11.1 \text{ cm/min (3 s.f.)}$$

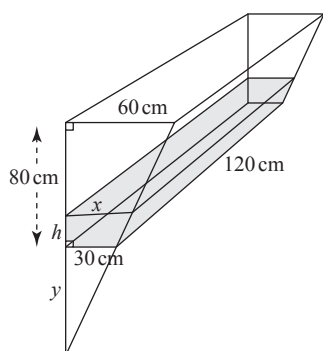
$$3 \quad \frac{dy}{dx} = 10(0.2x - 1)^9 \times 0.2 = 2(0.2x - 1)^9$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{dx}{dt} \times \frac{dy}{dx} \\ &= 0.08 \times 2(0.2x - 1)^9 \\ &= 0.08 \times 2(0.2 \times 10 - 1)^9 \\ &= 0.16 \text{ units/s} \end{aligned}$$

$$4 \quad x = 200\theta \Rightarrow \frac{dx}{d\theta} = 200 \Rightarrow \frac{d\theta}{dx} = \frac{1}{200}$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{dx}{dt} \times \frac{d\theta}{dx} \\ &= 10 \times \frac{1}{200} \\ &= 0.05 \text{ rad/s} \end{aligned}$$

- 5 (i) Using similar triangles:



$$\frac{y+80}{60} = \frac{y}{30} \Rightarrow y = 80 \text{ cm}$$

$$\frac{h+80}{x} = \frac{80}{30} \Rightarrow 3h+240=8x \Rightarrow x = \frac{3}{8}h+30$$

Area of trapezium cross section

$$= \frac{\left(\frac{3}{8}h+30\right)+30}{2} \times h = \frac{\frac{3}{8}h^2+60h}{2} = \frac{3}{16}h^2+30h$$

$$\begin{aligned} \text{Volume} &= 120 \times \left(\frac{3}{16}h^2+30h\right) \\ &= \frac{45}{2}h^2+3600h \text{ cm}^3 \end{aligned}$$

$$(ii) \quad \frac{45}{2}h^2+3600h = 40\,000$$

$$45h^2+7200h-80\,000=0$$

Using the quadratic formula,

$$h = 10.4 \text{ cm or } -170.4 \text{ cm}$$

Since h cannot be negative, $h = 10.4$ cm.

$$(iii) \quad 4 \text{ litres/min} = 4000 \text{ cm}^3/\text{min}$$

$$V = \frac{45}{2}h^2+3600h \Rightarrow \frac{dV}{dh} = 45h+3600$$

$$\Rightarrow \frac{dh}{dV} = \frac{1}{45h+3600}$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= 4000 \times \frac{1}{45h+3600}$$

$$\begin{aligned} &= 4000 \times \frac{1}{45 \times 10.4 + 3600} \\ &= 0.983 \text{ cm/min} \end{aligned}$$

Further practice (Page 62)

- 1 You want to find the points where $\frac{dy}{dx} = -1$.

$$\frac{dy}{dx} = x^2 + x - 3$$

$$\text{so } x^2 + x - 3 = -1$$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$x = 1 \text{ or } x = -2$$

Substitute into the original equation to find the y values:

$$y = \frac{1}{3} \times 1^3 + \frac{1}{2} \times 1^2 - 3 \times 1 = -\frac{13}{6}$$

$$y = \frac{1}{3} \times (-2)^3 + \frac{1}{2} \times (-2)^2 - 3 \times (-2) = \frac{16}{3}$$

The points are $\left(1, -\frac{13}{6}\right)$ and $\left(-2, \frac{16}{3}\right)$

- 2 (i) $y = \frac{1}{x^2} + x = x^{-2} + x$

$$\frac{dy}{dx} = -2x^{-3} + 1 = -\frac{2}{x^3} + 1$$

$$\text{When } x = 1, \frac{dy}{dx} = -\frac{2}{1^3} + 1 = -1$$

Equation of tangent is

$$y = mx + c \Rightarrow 2 = -1 \times 1 + c \Rightarrow c = 3$$

$$y = -x + 3$$

$$x\text{-intercept is } 0 = -x + 3 \Rightarrow x = 3$$

So Q is (3, 0)

$$y\text{-intercept is } y = 0 + 3 \Rightarrow y = 3$$

So R is (0, 3)

$$RQ = \sqrt{3^2 + 3^2} = \sqrt{18} = 4.24 \text{ (3 s.f.)}$$

- (ii) $m(\text{tangent}) = -1$ so $m(\text{normal}) = 1$

Equation of normal:

$$2 = 1 \times 1 + c \Rightarrow c = 1 \text{ so equation is } y = x + 1$$

$$\frac{1}{x^2} + x = x + 1$$

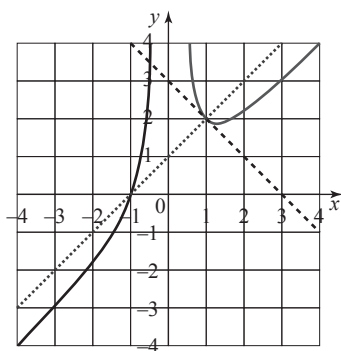
$$\frac{1}{x^2} = 1$$

$$x^2 = 1$$

$$x = \pm 1$$

$$x = 1 \Rightarrow y = 2, x = -1 \Rightarrow y = 0$$

So S is (-1, 0)



3 $f(x) = x + 4x^{-1}$ so $f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$

$f'(1) = 1 - \frac{4}{1^2} = -3$ so the gradient of the tangent is -3

$f(1) = 1 + \frac{4}{1} = 5$ so the point is $(1, 5)$

$y = mx + c \Rightarrow 5 = -3 \times 1 + c \Rightarrow c = 8$

The equation of the tangent is

$y = -3x + 8$ or $3x + y - 8 = 0$

The gradient of the normal is $\frac{1}{3}$

$y = mx + c \Rightarrow 5 = \frac{1}{3} \times 1 + c \Rightarrow c = \frac{14}{3}$

The equation of the normal is

$y = \frac{1}{3}x + \frac{14}{3}$ or $x - 3y + 14 = 0$

4 The stationary points are when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 4x^3 - 16x$ so solve $4x^3 - 16x = 0$

$4x(x^2 - 4) = 0 \Rightarrow 4x = 0$ or $x^2 - 4 = 0$

$x = 0$ or 2 or -2

Finding the y -values,

$y = 0^4 - 8 \times 0^2 + 2 = 2$

$y = 2^4 - 8 \times 2^2 + 2 = -14$

$y = (-2)^4 - 8 \times (-2)^2 + 2 = -14$

The stationary points are $(0, 2)$, $(2, -14)$ and $(-2, -14)$

$\frac{d^2y}{dx^2} = 12x^2 - 16$

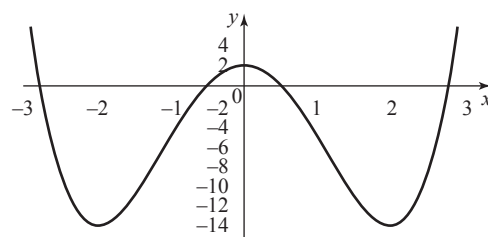
At $x = 0$, $\frac{d^2y}{dx^2} = -16$ so $(0, 2)$ is a maximum

At $x = 2$, $\frac{d^2y}{dx^2} = 12 \times (2)^2 - 16 = 32$

so $(2, -14)$ is a minimum

At $x = -2$, $\frac{d^2y}{dx^2} = 12 \times (-2)^2 - 16 = 32$

so $(-2, -14)$ is a minimum

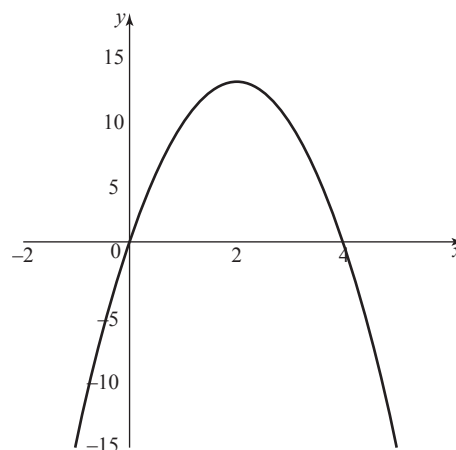


5 $f'(x) = 12x - 3x^2$ so the function is increasing when

$12x - 3x^2 > 0$

$3x(4 - x) > 0$

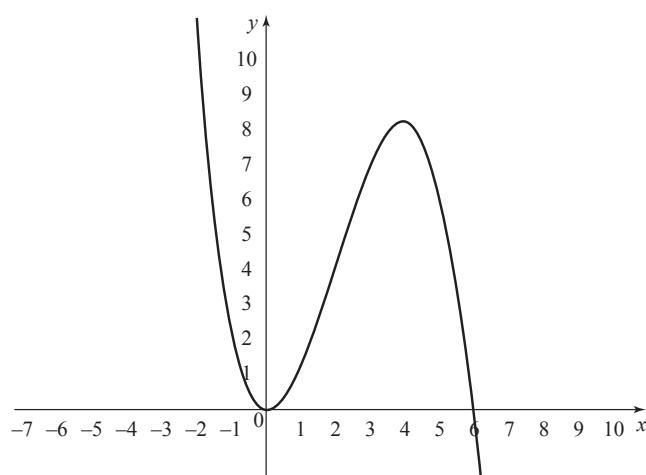
From the graph, the solutions to $3x(4 - x) > 0$ are the values of x where the graph is above the x -axis.



So the function is increasing for $0 < x < 4$

The graph above is the graph of the derivative.

The graph of $f(x)$ is



6 (i) $y = (3x + 2)^8 \Rightarrow \frac{dy}{dx} = 8(3x + 2)^7 \times 3$
 $= 24(3x + 2)^7$

(ii) $y = \sqrt[3]{1 - 9x} = (1 - 9x)^{\frac{1}{3}}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{3}(1 - 9x)^{-\frac{2}{3}} \times -9$
 $= -\frac{3}{\sqrt[3]{(1 - 9x)^2}}$

$$\text{(iii)} \quad y = \frac{5}{\sqrt{5x+3}} = 5(5x+3)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{5}{2}(5x+3)^{-\frac{3}{2}} \times 5 = -\frac{25}{2\sqrt{(5x+3)^3}}$$

$$7 \quad \text{(i)} \quad y = \frac{k+1}{2x+3} + x = (k+1)(2x+3)^{-1} + x$$

$$\frac{dy}{dx} = -(k+1)(2x+3)^{-2} \times 2 + 1$$

$$= \frac{-2(k+1)}{(2x+3)^2} + 1$$

$$\text{When } x = -1, \frac{dy}{dx} = 2$$

$$\frac{-2(k+1)}{(2 \times -1 + 3)^2} + 1 = 2$$

$$-2(k+1) + 1 = 2$$

$$-2k - 2 = 1$$

$$-2k = 3$$

$$k = -\frac{3}{2}$$

(ii) When $k = 1$,

$$\frac{dy}{dx} = \frac{-2(1+1)}{(2x+3)^2} + 1 = \frac{-4}{(2x+3)^2} + 1$$

$$\text{Increasing when } \frac{-4}{(2x+3)^2} + 1 > 0$$

$$\frac{-4}{(2x+3)^2} > -1$$

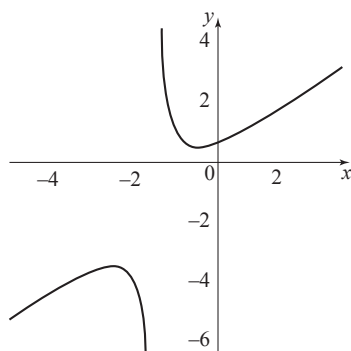
$$-4 > -(2x+3)^2$$

$$4 < (2x+3)^2$$

$$x^2 > 4 \Rightarrow x < -2 \text{ or } x > 2$$

$$\text{so } 2x+3 < -2 \text{ or } 2x+3 > 2$$

$$x < -\frac{5}{2} \text{ or } x > -\frac{1}{2}$$



$$8 \quad \text{(i)} \quad 2b + 2\pi r = 400$$

$$2b = 400 - 2\pi r$$

$$b = 200 - \pi r$$

$$\text{(ii)} \quad A = 2rb = 2r(200 - \pi r) = 400r - 2\pi r^2$$

$$\frac{dA}{dr} = 400 - 4\pi r$$

$$\text{Maximum area is when } \frac{dA}{dr} = 0$$

$$400 - 4\pi r = 0$$

$$400 = 4\pi r$$

$$r = \frac{400}{4\pi} = 31.8 \text{ m (3 s.f.)}$$

$$b = 200 - \pi \times 31.8 \dots = 100 \text{ m}$$

9 The y -intercepts of both lines are 1, so the equation of the line that P lies on is $y = -x + 1$

The coordinates of P are $(x, 1-x)$

The area of the rectangle, A, is

$$A = 2x(1-x) = 2x - 2x^2$$

$$\frac{dA}{dx} = 2 - 4x$$

$$2 - 4x = 0 \Rightarrow x = 0.5$$

$$y = 1 - 0.5 = 0.5$$

$$A = 2 \times 0.5 \times (1 - 0.5) = 0.5 \text{ units}^2$$

The largest area is 0.5 units^2

10 Let S be the surface area of the cube, and L the length of its side.

The equation that relates S and L is $S = 6L^2$ so

$$\frac{dS}{dL} = 12L$$

$$\frac{dS}{dt} = \frac{dL}{dt} \times \frac{dS}{dL}$$

$$= 5 \times 12L$$

$$= 5 \times 12 \times 20$$

$$= 1200 \text{ mm}^2/\text{min}$$

$$11 \quad y = \sqrt{1+6x} = (1+6x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(1+6x)^{-\frac{1}{2}} \times 6 = \frac{3}{\sqrt{1+6x}}$$

$$\text{The rate you are given is } \frac{dx}{dt} = 0.02 \text{ units/s}$$

$$\text{The rate you want is } \frac{dy}{dt}.$$

$$\text{Using the chain rule, } \frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$$

$$\frac{dy}{dt} = 0.02 \times \frac{3}{\sqrt{1+6x}}$$

$$= 0.02 \times \frac{3}{\sqrt{1+6 \times 4}}$$

$$= 0.02 \times \frac{3}{\sqrt{25}}$$

$$= 0.012 \text{ units/s}$$

$$12 \text{ (i) Volume} = \pi r^2 h + \frac{1}{2} \left(\frac{4}{3} \pi r^3 \right) = \pi r^2 h + \frac{2}{3} \pi r^3$$

$$\pi r^2 h + \frac{2}{3} \pi r^3 = 100$$

$$\pi r^2 h = 100 - \frac{2}{3} \pi r^3$$

$$h = \frac{100 - \frac{2}{3} \pi r^3}{\pi r^2} = \frac{100}{\pi r^2} - \frac{\frac{2}{3} \pi r^3}{\pi r^2}$$

$$= \frac{100}{\pi r^2} - \frac{2r}{3}$$

$$(ii) S = 2\pi r h + \pi r^2 + \frac{1}{2}(4\pi r^2)$$

$$S = 2\pi r \left(\frac{100}{\pi r^2} - \frac{2r}{3} \right) + \pi r^2 + 2\pi r^2$$

$$= \frac{200}{r} - \frac{4\pi r^2}{3} + 3\pi r^2$$

$$= \frac{200}{r} + \left(3\pi - \frac{4\pi}{3} \right) r^2$$

$$= \frac{200}{r} + \frac{5\pi}{3} r^2$$

$$(iii) S = 200r^{-1} + \frac{5\pi}{3} r^2$$

$$\frac{dS}{dr} = -200r^{-2} + \frac{10\pi}{3} r$$

$$= -\frac{200}{r^2} + \frac{10\pi}{3} r$$

Stationary point is when

$$-\frac{200}{r^2} + \frac{10\pi}{3} r = 0$$

$$-200 + \frac{10\pi}{3} r^3 = 0$$

$$r^3 = \frac{200}{\frac{10\pi}{3}} = 19.098\dots$$

$$r = 2.67 \text{ cm (3 s.f.)}$$

$$(iv) \frac{d^2S}{dr^2} = 400r^{-3} + \frac{10\pi}{3} = \frac{400}{r^3} + \frac{10\pi}{3}$$

$$\text{When } r = 2.67 \text{ cm, } \frac{d^2S}{dr^2} = \frac{400}{2.67^3} + \frac{10\pi}{3} > 0$$

So $r = 2.67 \text{ cm}$ is a minimum value.

$$\frac{dy}{dx} = -12(2-x)^{-2} \times -1$$

$$= \frac{12}{(2-x)^2}$$

$$\text{When } x = 6, \frac{dy}{dx} = \frac{12}{(2-6)^2} = \frac{3}{4}$$

Equation of the tangent is

$$y = mx + c$$

$$0 = \frac{3}{4} \times 6 + c \Rightarrow c = -\frac{9}{2}$$

$$y = \frac{3}{4}x - \frac{9}{2} \text{ or } 3x - 4y - 18 = 0$$

$$(ii) \frac{dx}{dt} = 0.04$$

$$\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$$

$$= 0.04 \times \frac{12}{(2-4)^2} = \frac{3}{25} \text{ units/s (0.12 units/s)}$$

$$2 \text{ (i) } y = \frac{1}{x-1} - \frac{9}{x-5}$$

$$y = (x-1)^{-1} - 9(x-5)^{-1}$$

$$\frac{dy}{dx} = -(x-1)^{-2} + 9(x-5)^{-2}$$

$$= -\frac{1}{(x-1)^2} + \frac{9}{(x-5)^2}$$

$$\text{When } x = 3, \frac{dy}{dx} = -\frac{1}{(3-1)^2} + \frac{9}{(3-5)^2} = 2$$

$$\text{Gradient of normal} = -\frac{1}{2}$$

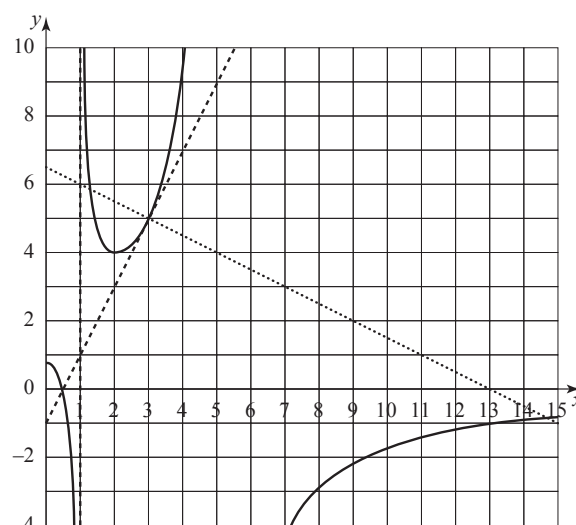
Equation of normal:

$$5 = -\frac{1}{2} \times 3 + c \Rightarrow c = \frac{13}{2}$$

$$y = -\frac{1}{2}x + \frac{13}{2}$$

x -intercept is when $y = 0$

$$0 = -\frac{1}{2}x + \frac{13}{2} \Rightarrow x = 13$$



Past exam questions (Page 64)

$$1 \text{ (i) } y = 3 + \frac{12}{2-x}$$

Curve crosses x -axis when

$$3 + \frac{12}{2-x} = 0$$

$$3(2-x) + 12 = 0$$

$$6 - 3x + 12 = 0$$

$$x = 6$$

$$y = 3 + 12(2-x)^{-1}$$

$$(ii) \frac{dy}{dx} = -\frac{1}{(x-1)^2} + \frac{9}{(x-5)^2} = 0$$

$$\frac{9}{(x-5)^2} = \frac{1}{(x-1)^2}$$

$$9(x-1)^2 = (x-5)^2$$

$$9x^2 - 18x + 9 = x^2 - 10x + 25$$

$$8x^2 - 8x - 16 = 0$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1 \text{ or } 2$$

$$\frac{d^2y}{dx^2} = -2(x-1)^{-3} - 18(x-5)^{-3}$$

$$= \frac{2}{(x-1)^3} - \frac{18}{(x-5)^3}$$

$$\text{When } x = -1, \frac{d^2y}{dx^2} = -\frac{1}{6} < 0 \Rightarrow \text{maximum}$$

$$\text{When } x = 2, \frac{d^2y}{dx^2} = \frac{8}{3} > 0 \Rightarrow \text{minimum}$$

$$3 \quad f(x) = 2x + (x+1)^{-2}$$

$$f'(x) = 2 - 2(x+1)^{-3} = 2 - \frac{2}{(x+1)^3}$$

$$f''(x) = 6(x+1)^{-4} = \frac{6}{(x+1)^4}$$

$$f'(0) = 2 - 2(0+1)^{-3} = 2 - \frac{2}{(0+1)^3} = 0$$

$$\Rightarrow x = 0 \text{ is a stationary point}$$

$$f''(0) = 6(0+1)^{-4} = \frac{6}{(0+1)^4} > 0$$

$$\Rightarrow x = 0 \text{ is a minimum}$$

$$4 \quad (ii) \text{ Let the height of the cuboid be } h \text{ cm.}$$

$$V = 3x^2h = 288$$

$$\Rightarrow h = \frac{288}{3x^2}$$

$$S = 2(3x^2) + 2xh + 2(3xh)$$

$$= 6x^2 + 8xh$$

$$= 6x^2 + 8x \left(\frac{288}{3x^2} \right)$$

$$= 6x^2 + \frac{768}{x}$$

$$(ii) \frac{dS}{dx} = 12x - \frac{768}{x^2} = 0$$

$$12x^3 - 768 = 0$$

$$x^3 = \frac{768}{12} = 64$$

$$x = 4 \text{ cm}$$

$$\frac{d^2S}{dx^2} = 12 + \frac{1536}{x^3}$$

$$\text{When } x = 4, \frac{d^2S}{dx^2} = 12 + \frac{1536}{4^3} > 0$$

$$\text{So } x = 4 \text{ is a minimum}$$

$$5 \quad (i) \quad y = 12(3-2x)^{-1}$$

$$\frac{dy}{dx} = -12(3-2x)^{-2} \times -2$$

$$= \frac{24}{(3-2x)^2}$$

$$(ii) \text{ At A, } \frac{dx}{dt} = 0.15, \quad \frac{dy}{dt} = 0.4$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{24}{(3-2x)^2} = 0.4 \times \frac{1}{0.15}$$

$$\frac{24}{(3-2x)^2} = \frac{8}{3}$$

$$72 = 8(3-2x)^2$$

$$9 = (3-2x)^2$$

$$\pm 3 = 3 - 2x$$

$$x = 0 \text{ or } 3$$

$$6 \quad y + 3x = 9 \Rightarrow y = 9 - 3x$$

$$u = x^2y = x^2(9-3x) = 9x^2 - 3x^3$$

$$\frac{du}{dx} = 18x - 9x^2$$

$$18x - 9x^2 = 0$$

$$9x(2-x) = 0$$

$$x = 0 \text{ or } 2$$

$$\text{Since } x \neq 0, \quad x = 2$$

$$u = 9 \times 2^2 - 3 \times 2^3 = 12$$

$$\frac{d^2u}{dx^2} = 18 - 18x$$

$$\text{When } x = 2, \frac{d^2u}{dx^2} = 18 - 18 \times 2 < 0$$

$$\text{So } u = 12 \text{ is a maximum}$$

$$7 \quad y = k^2(x+2)^{-1} + x$$

$$\frac{dy}{dx} = -\frac{k^2}{(x+2)^2} + 1$$

$$-\frac{k^2}{(x+2)^2} + 1 = 0$$

$$-k^2 + (x+2)^2 = 0$$

$$(x+2)^2 = k^2$$

$$x+2 = \pm k$$

$$x = -2 \pm k$$

$$\frac{d^2y}{dx^2} = \frac{2k^2}{(x+2)^3}$$

$$\text{When } x = -2 + k, \frac{d^2y}{dx^2} = \frac{2k^2}{k^3} = \frac{2}{k} > 0 \Rightarrow \text{minimum}$$

$$\text{When } x = -2 - k, \frac{d^2y}{dx^2} = \frac{2k^2}{-k^3} = -\frac{2}{k} < 0 \Rightarrow \text{maximum}$$

Stretch and challenge (page 65)

- 1 (i) $g'(x) = k - 2x$ so if the gradient of the normal is $\frac{1}{2}$, the gradient of the tangent is -2

$$k - 2x = -2$$

$$k - 2 \times 2 = -2$$

$$k - 4 = -2$$

$$k = 2$$

- (ii) The y -value when $x = 2$ is $g(2) = 2 \times 2 - 2^2 = 0$

Equation of the normal is

$$y = mx + c \Rightarrow 0 = \frac{1}{2} \times 2 + c \Rightarrow c = -1 \text{ so } y = \frac{1}{2}x - 1$$

To find the other point, solve simultaneously.

$$\frac{1}{2}x - 1 = 2x - x^2$$

$$x - 2 = 4x - 2x^2$$

$$2x^2 - 3x - 2 = 0$$

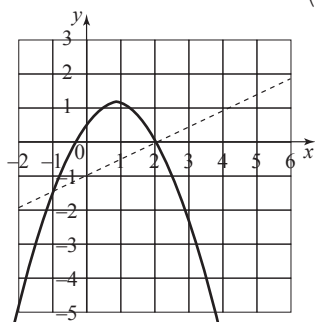
$$(2x+1)(x-2) = 0$$

$$x = -\frac{1}{2} \text{ or } 2$$

So the other point is $x = -\frac{1}{2}$

$$g\left(-\frac{1}{2}\right) = 2 \times -\frac{1}{2} - \left(-\frac{1}{2}\right)^2 = -1\frac{1}{4}$$

The coordinates of the point P are $\left(-\frac{1}{2}, -1\frac{1}{4}\right)$



$$2 \quad \text{SP} = \sqrt{x^2 + 10^2} \text{ so } \cos \alpha = \frac{x}{\sqrt{x^2 + 10^2}}$$

$$\text{PW} = \sqrt{(30-x)^2 + 5^2}$$

$$= \sqrt{(900 - 60x + x^2) + 25}$$

$$= \sqrt{925 - 60x + x^2}$$

$$\cos \beta = \frac{30-x}{\sqrt{925 - 60x + x^2}}$$

$$\begin{aligned} \text{Time} &= \frac{\sqrt{x^2 + 10^2}}{10} + \frac{\sqrt{925 - 60x + x^2}}{5} \\ &= \frac{\sqrt{x^2 + 10^2} + 2\sqrt{925 - 60x + x^2}}{10} \end{aligned}$$

$$T' = \frac{1}{10} \left[\frac{1}{2}(x^2 + 100)^{-\frac{1}{2}} \times 2x + 2 \times \frac{1}{2}(925 - 60x + x^2)^{-\frac{1}{2}} \times (2x - 60) \right]$$

$$= \frac{1}{10} \left[\frac{x}{\sqrt{x^2 + 100}} + \frac{2(x - 30)}{\sqrt{925 - 60x + x^2}} \right]$$

$$= \frac{1}{10} \left[\frac{x}{\sqrt{x^2 + 100}} - 2 \frac{30 - x}{\sqrt{925 - 60x + x^2}} \right]$$

$$= \frac{1}{10} [\cos \alpha - 2 \cos \beta]$$

$$T' = 0 \text{ when } \frac{1}{10} [\cos \alpha - 2 \cos \beta] = 0$$

$$\cos \alpha = 2 \cos \beta$$

- 3 (i) $V = \pi r^2 h$

$$R^2 = r^2 + \left(\frac{h}{2}\right)^2 \Rightarrow r^2 = R^2 - \frac{h^2}{4}$$

$$V = \pi \left(R^2 - \frac{h^2}{4} \right) h$$

$$= \pi R^2 h - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4}$$

$$\frac{dV}{dh} = 0 \text{ when } \pi R^2 - \frac{3\pi h^2}{4} = 0$$

$$\pi R^2 = \frac{3\pi h^2}{4}$$

$$\frac{4\pi R^2}{3\pi} = h^2$$

$$h = \frac{2R}{\sqrt{3}}$$

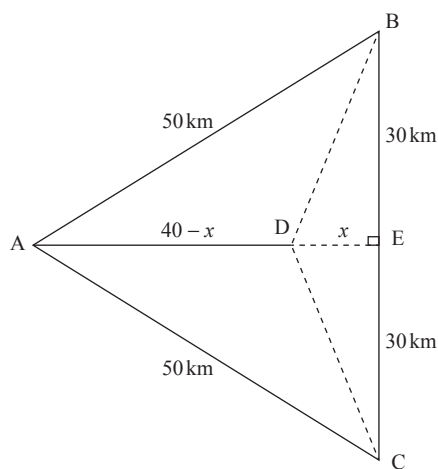
$$r^2 = R^2 - \frac{h^2}{4} = R^2 - \frac{\left(\frac{2R}{\sqrt{3}}\right)^2}{4} = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}$$

$$r = \sqrt{\frac{2}{3}} R$$

$$\begin{aligned}
 \text{(ii)} \quad \frac{V_{\text{cylinder}}}{V_{\text{sphere}}} &= \frac{\pi r^2 h}{\frac{4}{3}\pi R^3} = \frac{3r^2 h}{4R^3} = \frac{3\left(\sqrt{\frac{2}{3}}R\right)^2 \left(\frac{2R}{\sqrt{3}}\right)}{4R^3} \\
 &= \frac{3\left(\frac{2R^2}{3}\right)\left(\frac{2R}{\sqrt{3}}\right)}{4R^3} \\
 &= \frac{4R^3}{4R^3} \\
 &= \frac{\sqrt{3}}{\sqrt{3}} \\
 &= \frac{1}{\sqrt{3}}
 \end{aligned}$$

The ratio is $1:\sqrt{3}$ or $\sqrt{3}:3$

$$4 \quad AE^2 + 30^2 = 50^2 \Rightarrow AE = 40 \text{ km}$$



$$\text{In } \triangle BDE, BD^2 = x^2 + 30^2 \Rightarrow BD = \sqrt{x^2 + 900}$$

Total length of roads, L , is

$$\begin{aligned}
 L &= AD + BD + CD \\
 &= (40 - x) + \sqrt{x^2 + 900} + \sqrt{x^2 + 900} \\
 &= (40 - x) + 2\sqrt{x^2 + 900} \\
 &= (40 - x) + 2(x^2 + 900)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dL}{dx} &= -1 + 2 \times \frac{1}{2}(x^2 + 900)^{-\frac{1}{2}} \times 2x \\
 &= -1 + \frac{2x}{\sqrt{x^2 + 900}}
 \end{aligned}$$

Minimum length is when

$$\begin{aligned}
 -1 + \frac{2x}{\sqrt{x^2 + 900}} &= 0 \\
 \frac{2x}{\sqrt{x^2 + 900}} &= 1 \\
 2x &= \sqrt{x^2 + 900} \\
 4x^2 &= x^2 + 900 \\
 3x^2 &= 900 \\
 x^2 &= 300 \\
 x &= \sqrt{300} \approx 17.3 \text{ km}
 \end{aligned}$$

$$\begin{aligned}
 L &= (40 - x) + 2\sqrt{x^2 + 900} \\
 &= (40 - 17.3...) + 2\sqrt{(17.3...) ^2 + 900} \\
 &= 91.9615... \text{ km} \\
 &= 92 \text{ km (3 s.f.)}
 \end{aligned}$$

7 Integration

7.1 Reversing differentiation (Page 66)

- 1 (i) $\int (2x^3 + x) dx = \frac{2x^4}{4} + \frac{x^2}{2} + c = \frac{x^4}{2} + \frac{x^2}{2} + c$
- (ii) $\int (1 + 3x) dx = x + \frac{3x^2}{2} + c$
- (iii) $\int \frac{4}{x^3} dx = 4 \int x^{-3} dx = 4 \left(\frac{x^{-2}}{-2} \right) + c = -\frac{2}{x^2} + c$
- (iv) $\int 6\sqrt[3]{x} dx = 6 \int x^{\frac{1}{3}} dx = 6 \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{9\sqrt[3]{x^4}}{2} + c$
- 2 (i) $\int \left(\frac{2}{\sqrt{x}} \right) dx = 2 \int x^{-\frac{1}{2}} dx = 2 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 4\sqrt{x} + c$
- (ii) $\int 10\sqrt{x^3} dx = 10 \int x^{\frac{3}{2}} dx$
 $= 10 \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$
 $= 4\sqrt{x^5} + c$
- 3 $\int \frac{5x^2 - 1}{x^4} dx = \int \left(\frac{5x^2}{x^4} - \frac{1}{x^4} \right) dx$
 $= \int (5x^{-2} - x^{-4}) dx$
 $= \frac{5x^{-1}}{-1} - \frac{x^{-3}}{-3} + c$
 $= -\frac{5}{x} + \frac{1}{3x^3} + c$
- 4 $\frac{dy}{dx} = \frac{3}{x^4} + 1 \Rightarrow y = \int \left(\frac{3}{x^4} + 1 \right) dx$
 $= \int (3x^{-4} + 1) dx$
 $= \frac{3x^{-3}}{-3} + x + c$
 $= -\frac{1}{x^3} + x + c$

Substitute $(-1, 2)$ into the equation:

$$2 = -\frac{1}{(-1)^3} + (-1) + c \Rightarrow c = 2$$

$$\text{So } y = -\frac{1}{x^3} + x + 2$$

$$5 \quad f'(x) = 3\sqrt{x} - 2x = 3x^{\frac{1}{2}} - 2x$$

$$f(x) = \int (3x^{\frac{1}{2}} - 2x) dx$$

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^2}{2} + c$$

$$= 2\sqrt{x^3} - x^2 + c$$

$$f(4) = 30 \Rightarrow 30 = 2\sqrt{4^3} - 4^2 + c \Rightarrow c = 30$$

$$\text{So } f(x) = 2\sqrt{x^3} - x^2 + 30$$

$$6 \quad \frac{dy}{dx} = 1 - \frac{k}{x^2} \Rightarrow y = \int \left(1 - \frac{k}{x^2}\right) dx$$

$$= \int (1 - kx^{-2}) dx$$

$$= x - \frac{kx^{-1}}{-1} + c$$

$$= x + \frac{k}{x} + c$$

Substitute the two points into the equation:

$$2 = 1 + \frac{k}{1} + c \Rightarrow k + c = 1$$

$$-6 = -3 + \frac{k}{-3} + c \Rightarrow -\frac{k}{3} + c = -3$$

Subtract the equations:

$$\frac{4k}{3} = 4 \Rightarrow k = 3$$

$$3 + c = 1 \Rightarrow c = -2$$

$$\text{The equation is } y = x + \frac{3}{x} - 2$$

7.2 Finding areas (Page 67)

1 Finding the x-intercepts first,

$$x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0$$

$$\Rightarrow x = 1 \text{ or } 3$$

$$A = \int_1^3 (x^2 - 4x + 3) dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3$$

$$= \left(\frac{3^3}{3} - 2 \times 3^2 + 3 \times 3 \right) - \left(\frac{1^3}{3} - 2 \times 1^2 + 3 \times 1 \right)$$

$$= 0 - 1\frac{1}{3}$$

$$= -1\frac{1}{3}$$

So the area is $1\frac{1}{3}$

2 Area between the curve and the x-axis:

$$A = \int_0^4 2\sqrt{x} dx = 2 \int_0^4 x^{\frac{1}{2}} dx$$

$$= 2 \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^4 = 2 \left[\frac{2\sqrt{x^3}}{3} \right]_0^4$$

$$= 2 \left[\left(\frac{2\sqrt{4^3}}{3} \right) - \left(\frac{2\sqrt{0^3}}{3} \right) \right]$$

$$= 2 \times \frac{16}{3} = 10\frac{2}{3}$$

Area between the curve and the y-axis

$$= 4 \times 4 - 10\frac{2}{3} = 5\frac{1}{3}$$

$$3 \quad A = \int_{-1}^2 (3x^2 + 1) dx$$

$$= \left[x^3 + x \right]_{-1}^2$$

$$= (2^3 + 2) - ((-1)^3 + (-1))$$

$$= 10 - (-2) = 12$$

4 Points of intersection:

$$x^2 - 2x - 1 = x - 1$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0 \text{ or } 3$$

$$A = \int_0^3 ((x-1) - (x^2 - 2x - 1)) dx$$

$$= \int_0^3 (-x^2 + 3x) dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3$$

$$= \left(-\frac{3^3}{3} + \frac{3 \times 3^2}{2} \right) - \left(-\frac{0^3}{3} + \frac{3 \times 0^2}{2} \right)$$

$$= 4.5$$

5 Finding points of intersection,

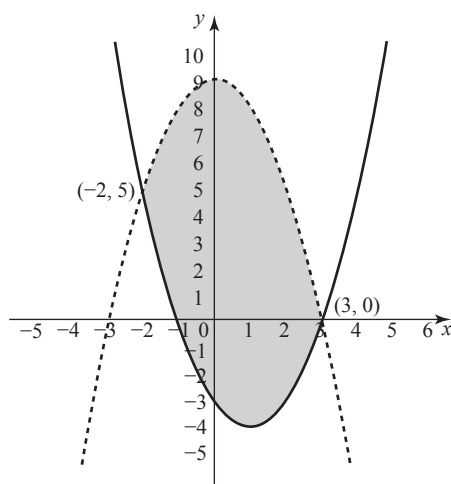
$$x^2 - 2x - 3 = 9 - x^2$$

$$2x^2 - 2x - 12 = 0$$

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x = -2 \text{ or } 3$$



$$\begin{aligned}
 A &= \int_{-2}^3 \left((9-x^2) - (x^2-2x-3) \right) dx \\
 &= \int_{-2}^3 (-2x^2 + 2x + 12) dx \\
 &= \left[-\frac{2x^3}{3} + x^2 + 12x \right]_{-2}^3 \\
 &= \left(-\frac{2 \times 3^3}{3} + 3^2 + 12 \times 3 \right) \\
 &\quad - \left(-\frac{2 \times (-2)^3}{3} + (-2)^2 + 12 \times -2 \right) \\
 &= 27 - \left(-14\frac{2}{3} \right) = 41\frac{2}{3}
 \end{aligned}$$

6 Area between curve and the x -axis:

$$\begin{aligned}
 A &= \int_1^2 (2x^2 + 1) dx \\
 &= \left[\frac{2x^3}{3} + x \right]_1^2 \\
 &= \left(\frac{2 \times 2^3}{3} + 2 \right) - \left(\frac{2 \times 1^3}{3} + 1 \right) \\
 &= 7\frac{1}{3} - \left(1\frac{2}{3} \right) = 5\frac{2}{3}
 \end{aligned}$$

Area between curve and the y -axis

$$= 9 \times 2 - 5\frac{2}{3} - 3 = 9\frac{1}{3}$$

$$\begin{aligned}
 7 \quad \int_1^k 3\sqrt{x} \, dx &= 14 \\
 \int_1^k 3x^{\frac{1}{2}} \, dx &= 14 \\
 \left[\frac{3x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^k &= 14 \\
 \left[2\sqrt{x^3} \right]_1^k &= 14 \\
 2\sqrt{k^3} - 2\sqrt{1^3} &= 14 \\
 2\sqrt{k^3} - 2 &= 14 \\
 2\sqrt{k^3} &= 16 \\
 \sqrt{k^3} &= 8 \\
 k^3 &= 64 \\
 k &= 4
 \end{aligned}$$

7.3 The reverse chain rule (Page 69)

$$\begin{aligned}
 1 \quad (i) \quad \int (x-2)^3 \, dx &= \frac{(x-2)^4}{4} + c \\
 (ii) \quad \int 2(1-6x)^9 \, dx &= 2 \frac{(1-6x)^{10}}{10} \times \frac{1}{-6} + c \\
 &= -\frac{(1-6x)^{10}}{30} + c \\
 (iii) \quad \int \left(\frac{x}{4} + 3 \right)^3 \, dx &= \frac{\left(\frac{x}{4} + 3 \right)^4}{\frac{1}{4}} \times \frac{1}{\frac{1}{4}} + c \\
 &= \left(\frac{x}{4} + 3 \right)^4 + c \\
 (iv) \quad \int (2x-1)^{\frac{1}{2}} \, dx &= \frac{(2x-1)^{\frac{3}{2}}}{\frac{3}{2}} \times \frac{1}{2} + c \\
 &= \frac{(2x-1)^{\frac{3}{2}}}{3} + c \\
 2 \quad (i) \quad \int \sqrt{1-x} \, dx &= \int (1-x)^{\frac{1}{2}} \, dx \\
 &= \frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}} \times \frac{1}{-1} + c \\
 &= -\frac{2(1-x)^{\frac{3}{2}}}{3} + c \\
 &= -\frac{2\sqrt{(1-x)^3}}{3} + c \\
 (ii) \quad \int \frac{2}{3(x-4)^5} \, dx &= \frac{2}{3} \int (x-4)^{-5} \, dx \\
 &= \frac{2}{3} \frac{(x-4)^{-4}}{-4} \times \frac{1}{1} + c \\
 &= -\frac{1}{6(x-4)^4} + c \\
 (iii) \quad \int \frac{3}{\sqrt{5-2x}} \, dx &= 3 \int (5-2x)^{-\frac{1}{2}} \, dx \\
 &= 3 \frac{(5-2x)^{\frac{1}{2}}}{\frac{1}{2}} \times \frac{1}{-2} + c \\
 &= -3\sqrt{5-2x} + c \\
 (iv) \quad \int \frac{12}{\sqrt[3]{1+\frac{x}{2}}} \, dx &= 12 \int \left(1 + \frac{x}{2} \right)^{-\frac{1}{3}} \, dx \\
 &= 12 \frac{\left(1 + \frac{x}{2} \right)^{\frac{2}{3}}}{\frac{2}{3}} \times \frac{1}{\frac{1}{2}} + c \\
 &= 36 \sqrt[3]{\left(1 + \frac{x}{2} \right)^2} + c
 \end{aligned}$$

7.4 Improper integrals (Page 70)

$$\begin{aligned}
 1 \quad \int_0^1 \frac{1}{(2x+1)^3} dx &= \int_0^1 (2x+1)^{-3} dx \\
 &= \left[\frac{(2x+1)^{-2}}{-2} \times \frac{1}{2} \right]_0^1 = \left[-\frac{1}{4(2x+1)^2} \right]_0^1 \\
 &= \left(-\frac{1}{4(2 \times 1 + 1)^2} \right) - \left(-\frac{1}{4(2 \times 0 + 1)^2} \right) \\
 &= -\frac{1}{36} - \left(-\frac{1}{4} \right) = \frac{2}{9}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \int_1^2 \frac{2}{\sqrt{x-1}} dx &= 2 \int_1^2 (x-1)^{-\frac{1}{2}} dx \\
 &= 2 \left[\frac{(x-1)^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^2 = 2 \left[2\sqrt{x-1} \right]_1^2 \\
 &= 2 \left[(2\sqrt{2-1}) - (2\sqrt{1-1}) \right] \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \int_0^\infty \frac{3}{(1+x)^2} dx &= 3 \int_0^\infty (1+x)^{-2} dx \\
 &= 3 \left[\frac{(1+x)^{-1}}{-1} \right]_0^\infty = 3 \left[-\frac{1}{1+x} \right]_0^\infty \\
 &= 3 \left[\left(-\frac{1}{1+\infty} \right) - \left(-\frac{1}{1+0} \right) \right] \\
 &= 3
 \end{aligned}$$

7.5 Finding volumes by integration (Page 71)

$$\begin{aligned}
 1 \quad (i) \quad V &= \int_2^\infty \pi \left(\frac{4}{x} \right)^2 dx = 16\pi \int_2^\infty x^{-2} dx \\
 &= 16\pi \left[\frac{x^{-1}}{-1} \right]_2^\infty \\
 &= 16\pi \left[-\frac{1}{x} \right]_2^\infty \\
 &= 16\pi \left[\left(-\frac{1}{\infty} \right) - \left(-\frac{1}{2} \right) \right] \\
 &= 16\pi \left[\frac{1}{2} \right] \\
 &= 8\pi = 25.1 \quad (3 \text{ s.f.})
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad V &= \int_3^4 \pi \left(\frac{4}{x} \right)^2 dx \\
 &= 16\pi \left[-\frac{1}{x} \right]_3^4 \\
 &= 16\pi \left[\left(-\frac{1}{4} \right) - \left(-\frac{1}{3} \right) \right] \\
 &= 16\pi \left[\frac{1}{12} \right] \\
 &= \frac{4}{3}\pi = 4.19 \quad (3 \text{ s.f.})
 \end{aligned}$$

$$\begin{aligned}
 2 \quad V &= \int_{-2}^1 \pi \left(\frac{3}{2-x} \right)^2 dx = 9\pi \int_{-2}^1 (2-x)^{-2} dx \\
 &= 9\pi \left[\frac{(2-x)^{-1}}{-1} \times \frac{1}{-1} \right]_{-2}^1 \\
 &= 9\pi \left[\frac{1}{2-x} \right]_{-2}^1 \\
 &= 9\pi \left[\left(\frac{1}{2-1} \right) - \left(\frac{1}{2-(-2)} \right) \right] \\
 &= 9\pi \left[\frac{3}{4} \right] \\
 &= \frac{27}{4}\pi
 \end{aligned}$$

$$\begin{aligned}
 3 \quad V(\text{solid}) &= V(\text{cylinder}) - \int_0^6 \pi x^2 dy \\
 V(\text{cylinder}) &= \pi \times 2^2 \times 6 = 24\pi \\
 y &= 2(x^2 - 1) \Rightarrow \frac{y}{2} = x^2 - 1 \Rightarrow x^2 = \frac{y}{2} + 1 \\
 \int_0^6 \pi x^2 dy &= \pi \int_0^6 \left(\frac{y}{2} + 1 \right) dy \\
 &= \pi \left[\frac{y^2}{4} + y \right]_0^6 \\
 &= \pi \left[\left(\frac{6^2}{4} + 6 \right) - \left(\frac{0^2}{4} + 0 \right) \right] \\
 &= 15\pi \\
 V(\text{solid}) &= 24\pi - 15\pi = 9\pi = 28.3 \quad (3 \text{ s.f.})
 \end{aligned}$$

$$\begin{aligned}
 4 \quad (i) \quad 4x + \frac{4}{x} &= 10 \\
 4x^2 + 4 &= 10x \\
 4x^2 - 10x + 4 &= 0 \\
 2x^2 - 5x + 2 &= 0 \\
 (2x-1)(x-2) &= 0 \\
 x &= \frac{1}{2} \quad \text{or} \quad x = 2
 \end{aligned}$$

A is $\left(\frac{1}{2}, 10\right)$, B is $(2, 10)$

To find M, find where $\frac{dy}{dx} = 0$

$$\begin{aligned}
 y &= 4x + \frac{4}{x} = 4x + 4x^{-1} \\
 \frac{dy}{dx} &= 4 - 4x^{-2} = 4 - \frac{4}{x^2} \\
 4 - \frac{4}{x^2} &= 0 \\
 4x^2 - 4 &= 0 \\
 4x^2 &= 4 \\
 x^2 &= 1 \\
 x &= 1 \quad (\text{since } x > 0)
 \end{aligned}$$

(ii) Volume of cylinder = $\pi \times 10^2 \times 1.5 = 150\pi$

Volume under curve

$$\begin{aligned}
 &= \int_{0.5}^2 \pi \left(4x + \frac{4}{x} \right)^2 dx \\
 &= \int_{0.5}^2 \pi \left(16x^2 + 32 + \frac{16}{x^2} \right) dx \\
 &= \int_{0.5}^2 \pi (16x^2 + 32 + 16x^{-2}) dx \\
 &= \pi \left[\frac{16x^3}{3} + 32x - 16x^{-1} \right]_{0.5}^2 \\
 &= \pi \left[\frac{16x^3}{3} + 32x - \frac{16}{x} \right]_{0.5}^2 \\
 &= \pi \left[\left(\frac{16 \times 2^3}{3} + 32 \times 2 + \frac{16}{2} \right) - \left(\frac{16 \times 0.5^3}{3} + 32 \times 0.5 + \frac{16}{0.5} \right) \right] \\
 &= \pi \left[\left(\frac{344}{3} \right) - \left(\frac{146}{3} \right) \right] \\
 &= 66\pi \\
 \text{Final volume} &= 150\pi - 66\pi = 84\pi \approx 264 \text{ units}^3
 \end{aligned}$$

Further practice (Page 73)

1 (i) $\int (2x - 5x^4) dx = \frac{2x^2}{2} - \frac{5x^5}{5} + c = x^2 - x^5 + c$

(ii) $\int \left(\frac{x}{2} + 2x^5 \right) dx = \frac{1}{2} \times \frac{x^2}{2} + \frac{2x^6}{6} + c$
 $= \frac{x^2}{4} + \frac{x^6}{3} + c$

(iii) $\int \left(\frac{1}{2x^2} + 6x \right) dx = \int \left(\frac{1}{2} x^{-2} + 6x \right) dx$
 $= \frac{1}{2} \frac{x^{-1}}{-1} + \frac{6x^2}{2} + c$
 $= -\frac{1}{2x} + 3x^2 + c$

(iv) $\int x(2x - 3\sqrt{x}) dx = \int (2x^2 - 3x^{\frac{3}{2}}) dx$
 $= \frac{2x^3}{3} - \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + c$
 $= \frac{2x^3}{3} - \frac{6\sqrt{x^5}}{5} + c$

(v) $\int (3x+1)^5 dx = \frac{(3x+1)^6}{6} \times \frac{1}{3} + c$
 $= \frac{(3x+1)^6}{18} + c$

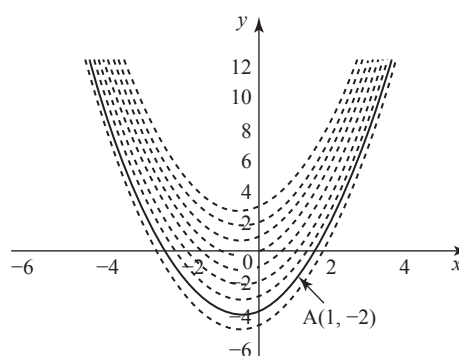
(vi) $\int \frac{4}{(2x+1)^3} dx = 4 \int (2x+1)^{-3} dx$
 $= 4 \frac{(2x+1)^{-2}}{-2} \times \frac{1}{2} + c$
 $= -\frac{1}{(2x+1)^2} + c$

2 If $\frac{dy}{dx} = 2x+1$ then $y = \int (2x+1) dx$
 $y = x^2 + x + c$

Substitute (1, -2) into the equation to find c:

$$\begin{aligned}
 -2 &= 1^2 + 1 + c \\
 c &= -4
 \end{aligned}$$

The curve is $y = x^2 + x - 4$



3 Points of intersection:

$$2 = 6 - x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$\begin{aligned}
 A &= \int_{-2}^2 \left((6 - x^2) - (2) \right) dx = 2 \int_0^2 (4 - x^2) dx \\
 &= 2 \left[4x - \frac{x^3}{3} \right]_0^2 \\
 &= 2 \left[\left(4 \times 2 - \frac{2^3}{3} \right) - \left(4 \times 0 - \frac{0^3}{3} \right) \right] \\
 &= 2 \times 5 \frac{1}{3} = 10 \frac{2}{3}
 \end{aligned}$$

4 Finding the x-intercept:

$$\frac{2}{\sqrt[3]{x}} - 2 = 0 \Rightarrow \frac{2}{\sqrt[3]{x}} = 2 \Rightarrow \sqrt[3]{x} = 1 \Rightarrow x = 1$$

Area between $x = 8$ and $x = 1$:

$$\begin{aligned}
 A &= \int_1^8 \left(\frac{2}{\sqrt[3]{x}} - 2 \right) dx = \int_1^8 (2x^{-\frac{1}{3}} - 2) dx \\
 &= \left[\frac{2x^{\frac{2}{3}}}{\frac{2}{3}} - 2x \right]_1^8 = \left[3\sqrt[3]{x^2} - 2x \right]_1^8 \\
 &= (3\sqrt[3]{8^2} - 2 \times 8) - (3\sqrt[3]{1^2} - 2 \times 1) \\
 &= -4 - 1 = -5
 \end{aligned}$$

Area between $x = 1$ and $x = 0$

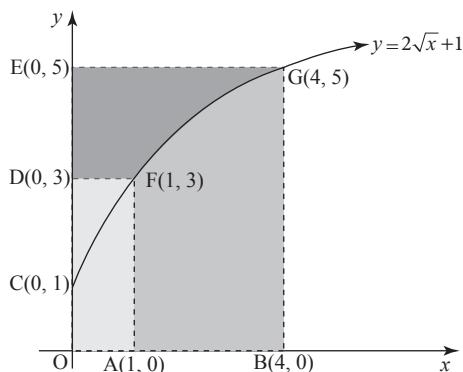
$$= \left[3\sqrt[3]{x^2} - 2x \right]_0^1$$

$$= (3\sqrt[3]{1^2} - 2 \times 1) - (3\sqrt[3]{0^2} - 2 \times 0)$$

$$= 1$$

So total area = $5 + 1 = 6$

- 5 To find this area, find the area ABGF between the curve and the x -axis between $x = 1$ and $x = 4$, find the area of the rectangle OAFD and subtract these areas from the area of the rectangle OBGE.



$$\text{Area of ABGF} = \int_1^4 (2\sqrt{x} + 1) dx$$

$$= \int_1^4 \left(2x^{\frac{1}{2}} + 1 \right) dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + x \right]_1^4$$

$$= \left[\frac{4\sqrt{x^3}}{3} + x \right]_1^4$$

$$= \left(\frac{4\sqrt{4^3}}{3} + 4 \right) - \left(\frac{4\sqrt{1^3}}{3} + 1 \right)$$

$$= 12\frac{1}{3}$$

$$\text{Area of DFGE} = \text{area of OBGE} - \text{area of ABGF} - \text{area of OAFD}$$

$$= (4 \times 5) - 12\frac{1}{3} - (3 \times 1)$$

$$= 4\frac{2}{3}$$

- 6 First, find the points of intersection of the curves.

$$x^2 = 16x \Rightarrow x^2 - 16x = 0 \Rightarrow x(x - 16) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 16$$

$$\text{Shaded area} = \int_0^{16} (4\sqrt{x} - x) dx$$

$$= \int_0^{16} \left(4x^{\frac{1}{2}} - x \right) dx$$

$$= \left[\frac{4x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} \right]_0^{16}$$

$$= \left[\frac{8\sqrt{x^3}}{3} - \frac{x^2}{2} \right]_0^{16}$$

$$= \left(\frac{8\sqrt{16^3}}{3} - \frac{16^2}{2} \right) - \left(\frac{8\sqrt{0^3}}{3} - \frac{0^2}{2} \right)$$

$$= 42\frac{2}{3} - 0$$

$$= 42\frac{2}{3}$$

7 (i) $\int_0^1 x^{-\frac{1}{2}} dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^1 = [2\sqrt{x}]_0^1 = 2\sqrt{1} - 2\sqrt{0} = 2$

(ii) $\int_1^r x^{-2} dx = \left[\frac{x^{-1}}{-1} \right]_1^r = \left[-\frac{1}{x} \right]_1^r$

$$= \left[-\frac{1}{r} - \left(-\frac{1}{1} \right) \right]$$

$$= -\frac{1}{r} + 1$$

$$\text{As } r \rightarrow \infty, -\frac{1}{r} \rightarrow 0$$

$$\text{So } \int_1^\infty x^{-2} dx \rightarrow 1$$

8 $V = \int_0^2 \pi(3x)^2 dx = \pi \int_0^2 9x^2 dx$

$$= 9\pi \int_0^2 x^2 dx$$

$$= 9\pi \left[\frac{x^3}{3} \right]_0^2$$

$$= 9\pi \left[\frac{2^3}{3} - \frac{0^3}{3} \right]$$

$$= 9\pi \left[\frac{8}{3} \right]$$

$$= \frac{72}{3}\pi = 75.4 \text{ (3 s.f.)}$$

$$V(\text{solid}) = \frac{1}{3}\pi \times 6^2 \times 2 = 24\pi$$

$$\begin{aligned}
 9 \quad \text{Volume} &= \int \pi y^2 dx \\
 &= \int_0^2 \pi \left(\frac{3}{2x+1} \right)^2 dx \\
 &= \pi \int_0^2 \frac{9}{(2x+1)^2} dx \\
 &= 9\pi \int_0^2 (2x+1)^{-2} dx \\
 &= 9\pi \left[\frac{(2x+1)^{-1}}{-1} \times \frac{1}{2} \right]_0^2 \\
 &= 9\pi \left[-\frac{1}{2(2x+1)} \right]_0^2 \\
 &= 9\pi \left[-\frac{1}{2(2 \times 2 + 1)} - \frac{-1}{2(2 \times 0 + 1)} \right] \\
 &= 9\pi \left[-\frac{1}{10} + \frac{1}{2} \right] \\
 &= 9\pi \left[\frac{2}{5} \right] = \frac{18}{5}\pi \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 10 \text{ (i)} \quad \text{Area under curve} &= \int_0^1 \sqrt{1-x} dx \\
 &= \int_0^1 (1-x)^{\frac{1}{2}} dx \\
 &= \left[-\frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 \\
 &= \left[-\frac{2\sqrt{(1-x)^3}}{3} \right]_0^1 \\
 &= -\frac{2\sqrt{(1-1)^3}}{3} - \left(-\frac{2\sqrt{(1-0)^3}}{3} \right) \\
 &= \left[0 - \left(-\frac{2}{3} \right) \right] = \frac{2}{3}
 \end{aligned}$$

$$\text{Area under line} = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

$$\begin{aligned}
 \text{Shaded area} &= \text{area under curve} - \text{area of triangle} \\
 &= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}
 \end{aligned}$$

(ii) (a) Volume of curve rotated around the x -axis:

$$\begin{aligned}
 V &= \int_0^1 \pi y^2 dx \\
 &= \pi \int_0^1 (\sqrt{1-x})^2 dx \\
 &= \pi \int_0^1 (1-x) dx \\
 &= \pi \left[x - \frac{x^2}{2} \right]_0^1 \\
 &= \pi \left[\left(1 - \frac{1^2}{2} \right) - 0 \right] = \frac{1}{2}\pi
 \end{aligned}$$

$$\text{Volume of the cone} = \frac{1}{3}\pi \times 1^2 \times 1 = \frac{1}{3}\pi$$

$$\text{Final volume} = \frac{1}{2}\pi - \frac{1}{3}\pi = \frac{1}{6}\pi$$

$$\begin{aligned}
 \text{OR } V &= \int_0^1 \pi \left[(\sqrt{1-x})^2 - (1-x)^2 \right] dx \\
 &= \pi \int_0^1 [(1-x) - (1-2x+x^2)] dx \\
 &= \pi \int_0^1 [x - x^2] dx \\
 &= \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\
 &= \pi \left[\left(\frac{1^2}{2} - \frac{1^3}{3} \right) - 0 \right] \\
 &= \frac{1}{6}\pi
 \end{aligned}$$

$$(b) \quad y = \sqrt{1-x} \Rightarrow y^2 = 1-x$$

$$x = 1 - y^2$$

$$x^2 = (1 - y^2)^2$$

$$x^2 = 1 - 2y^2 + y^4$$

$$\begin{aligned}
 V &= \int_0^1 \pi x^2 dy \\
 &= \int_0^1 \pi (1 - 2y^2 + y^4) dy \\
 &= \pi \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_0^1 \\
 &= \pi \left[\left(1 - \frac{2 \times 1^3}{3} + \frac{1^5}{5} \right) - 0 \right] \\
 &= \frac{8}{15}\pi
 \end{aligned}$$

$$\text{Volume of the cone} = \frac{1}{3}\pi \times 1^2 \times 1 = \frac{1}{3}\pi$$

$$\text{Final volume} = \frac{8}{15}\pi - \frac{1}{3}\pi = \frac{1}{5}\pi$$

$$\begin{aligned}
 \text{OR } y = 1 - x \Rightarrow x = 1 - y \Rightarrow x^2 &= 1 - 2y + y^2 \\
 V &= \pi \int_0^1 [(1 - 2y^2 + y^4) - (1 - 2y + y^2)] dy \\
 &= \pi \int_0^1 (2y - 3y^2 + y^4) dy \\
 &= \pi \left[y^2 - y^3 + \frac{y^5}{5} \right]_0^1 \\
 &= \frac{1}{5}\pi
 \end{aligned}$$

$$\begin{aligned}
 11 \text{ (i)} \quad V &= \int_0^1 \pi y^2 dx \\
 &= \int_0^1 \pi \left[(x)^2 - (x^3)^2 \right] dx \\
 &= \pi \int_0^1 [x^2 - x^6] dx \\
 &= \pi \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_0^1 \\
 &= \pi \left[\left(\frac{1^3}{3} - \frac{1^7}{7} \right) - 0 \right] \\
 &= \frac{4}{21} \pi
 \end{aligned}$$

$$\text{OR Volume of the cone} = \frac{1}{3} \pi \times 1^2 \times 1 = \frac{1}{3} \pi$$

Volume of $y = x^3$ rotated around the x -axis:

$$\begin{aligned}
 V &= \int_0^1 \pi (x^3)^2 dx \\
 &= \pi \int_0^1 (x^6) dx \\
 &= \pi \left[\frac{x^7}{7} \right]_0^1 \\
 &= \pi \left[\left(\frac{1^7}{7} \right) - 0 \right] \\
 &= \frac{1}{7} \pi
 \end{aligned}$$

$$\text{Final volume} = \frac{1}{3} \pi - \frac{1}{7} \pi = \frac{4}{21} \pi$$

$$(ii) \text{ If } y = x^3 \text{ then } x = \sqrt[3]{y} \Rightarrow x^2 = \sqrt[3]{y^2} = y^{\frac{2}{3}}$$

$$\text{If } y = x \text{ then } x = y \Rightarrow x^2 = y^2$$

$$\begin{aligned}
 \text{Volume} &= \int_0^1 \pi \left(y^{\frac{2}{3}} - y^2 \right) dy \\
 &= \pi \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}} - \frac{y^3}{3} \right]_0^1 \\
 &= \pi \left[\frac{3 \sqrt[3]{y^5}}{5} - \frac{y^3}{3} \right]_0^1 \\
 &= \pi \left[\left(\frac{3 \sqrt[3]{1^5}}{5} - \frac{1^3}{3} \right) - \left(\frac{3 \sqrt[3]{0^5}}{5} - \frac{0^3}{3} \right) \right] \\
 &= \pi \left[\left(\frac{3}{5} - \frac{1}{3} \right) - 0 \right] \\
 &= \frac{4\pi}{15} \text{ units}^3 \text{ (0.838 units}^3 \text{ to 3 s.f.)}
 \end{aligned}$$

$$\text{OR Volume of the cone} = \frac{1}{3} \pi \times 1^2 \times 1 = \frac{1}{3} \pi$$

Volume of $y = x^3$ rotated around the y -axis:

$$\begin{aligned}
 V &= \int_0^1 \pi \left(y^{\frac{2}{3}} \right) dy \\
 &= \pi \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}} \right]_0^1 \\
 &= \pi \left[\frac{3 \sqrt[3]{y^5}}{5} \right]_0^1 \\
 &= \pi \left[\left(\frac{3 \sqrt[3]{1^5}}{5} \right) - 0 \right] \\
 &= \frac{3}{5} \pi
 \end{aligned}$$

$$\text{Final volume} = \frac{3}{5} \pi - \frac{1}{3} \pi = \frac{4}{15} \pi \text{ units}^3$$

Past exam questions (Page 73)

$$\begin{aligned}
 1 \quad \text{Volume} &= \pi \int_1^4 (5-x)^2 dx - \pi \int_1^4 \left(\frac{4}{x} \right)^2 dx \\
 &= \pi \left[-\frac{(5-x)^3}{3} \right]_1^4 - \pi \left[-\frac{16}{x} \right]_1^4 \\
 &= \pi \left[\left(-\frac{(5-4)^3}{3} \right) - \left(-\frac{(5-1)^3}{3} \right) \right] \\
 &\quad - \pi \left[\left(-\frac{16}{4} \right) - \left(-\frac{16}{1} \right) \right] \\
 &= 9\pi \text{ (or 28.3)}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \text{Equation:} \quad y &= \int \frac{8}{(5-2x)^2} dx \\
 &= 8 \int (5-2x)^{-2} dx \\
 &= 8 \left[\frac{(5-2x)^{-1}}{-1} \times \frac{1}{-2} \right] + c \\
 &= \frac{4}{5-2x} + c \\
 7 &= \frac{4}{5-2 \times 2} + c \Rightarrow c = 3 \\
 \text{So } y &= \frac{4}{5-2x} + 3
 \end{aligned}$$

$$\begin{aligned}
 3 \text{ (i)} \quad A \text{ is where } 1-2x &= 0 \text{ or } (2x-1)^2 = 0 \\
 &\Rightarrow x = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) Area} &= \int_0^{\frac{1}{2}} \sqrt{1-2x} \, dx - \int_0^{\frac{1}{2}} (2x-1)^2 \, dx \\
 &= \left[\frac{(1-2x)^{\frac{3}{2}}}{\frac{3}{2}} \times \frac{1}{-2} \right]_0^{\frac{1}{2}} - \left[\frac{(2x-1)^3}{3} \times \frac{1}{2} \right]_0^{\frac{1}{2}} \\
 &= \left[-\frac{(1-2x)^{\frac{3}{2}}}{3} \right]_0^{\frac{1}{2}} - \left[\frac{(2x-1)^3}{6} \right]_0^{\frac{1}{2}} \\
 &= \left[0 - \left(-\frac{1}{3} \right) \right] - \left[0 - \left(-\frac{1}{6} \right) \right] \\
 &= \frac{1}{6}
 \end{aligned}$$

- 4 (i) Gradient of normal = $-\frac{1}{3}$ so gradient of tangent at P is 3.

$$\begin{aligned}
 \text{So } \frac{dy}{dx} &= \frac{12}{\sqrt{4 \times 2 + a}} = 3 \\
 \frac{12}{\sqrt{8+a}} &= 3 \\
 12 &= 3\sqrt{8+a} \\
 4 &= \sqrt{8+a} \\
 16 &= 8+a \\
 a &= 8
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } y &= \int \frac{12}{\sqrt{4x+8}} \, dx \\
 &= 12 \int (4x+8)^{-\frac{1}{2}} \, dx \\
 &= 12 \left[\frac{(4x+8)^{\frac{1}{2}}}{\frac{1}{2}} \times \frac{1}{4} \right] + c \\
 &= 6\sqrt{4x+8} + c \\
 14 &= 6\sqrt{4 \times 2 + 8} + c \Rightarrow c = -10 \\
 \text{So } y &= 6\sqrt{4x+8} - 10
 \end{aligned}$$

- 5 (i) $\frac{d^2y}{dx^2} = \frac{24}{x^3} - 4 = -1 < 0 \Rightarrow$ maximum

$$\begin{aligned}
 \frac{dy}{dx} &= \int (24x^{-3} - 4) \, dx \\
 &= -12x^{-2} - 4x + c \\
 &= -\frac{12}{x^2} - 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x &= 2, \frac{dy}{dx} = 0 \\
 0 &= -\frac{12}{2^2} - 4(2) + c \\
 c &= 11
 \end{aligned}$$

$$\text{So } \frac{dy}{dx} = -\frac{12}{x^2} - 4x + 11$$

$$\text{(iii) } \frac{dy}{dx} = -12x^{-2} - 4x + 11$$

$$y = \int (-12x^{-2} - 4x + 11) \, dx$$

$$y = \frac{12}{x} - 2x^2 + 11x + c$$

$$13 = 12 - 2 + 11 + c$$

$$c = -8$$

When $x = 2$,

$$y = \frac{12}{2} - 2(2)^2 + 11(2) - 8 = 12$$

P is (2, 12)

- 6 (i) Intersection is where $\frac{8}{x^3} = 9 - x^3$

$$8 = 9x^3 - x^6$$

$$x^6 - 9x^3 + 8 = 0$$

Let $t = x^3$. Then the equation becomes

$$t^2 - 9t + 8 = 0$$

$$(t-1)(t-8) = 0$$

$$t = 1 \text{ or } 8$$

$$x^3 = 1 \text{ or } x^3 = 8$$

$$x = 1 \text{ or } x = 2$$

So $a = 1, b = 2$

$$\begin{aligned}
 \text{(ii) } A &= \int_1^2 \left((9-x^3) - \left(\frac{8}{x^3} \right) \right) \, dx \\
 &= \left[9x - \frac{x^4}{4} + \frac{4}{x^2} \right]_1^2 \\
 &= \left(9 \times 2 - \frac{2^4}{4} + \frac{4}{2^2} \right) - \left(9 \times 1 - \frac{1^4}{4} + \frac{4}{1^2} \right) \\
 &= 15 - 12.75 = 2.25
 \end{aligned}$$

$$\text{(iii) } \frac{dy}{dx} = -3x^2$$

$$y = \frac{8}{x^3} = 8x^{-3} \Rightarrow \frac{dy}{dx} = -24x^{-4} = -\frac{24}{x^4}$$

If the tangents are parallel, the gradients are the same so

$$-3x^2 = -\frac{24}{x^4}$$

$$x^6 = 8$$

$$x = \sqrt[6]{8} = \sqrt{2} \approx 1.41$$

So $c = \sqrt[6]{8} = \sqrt{2} \approx 1.41$

Stretch and challenge (Page 75)

$$\begin{aligned}
 1 \quad \int_{-2}^0 ((2x+k)-(x+2)^2) dx &= \frac{10}{3} \\
 \int_{-2}^0 ((2x+k)-(x^2+4x+4)) dx &= \frac{10}{3} \\
 \int_{-2}^0 (-x^2-2x+k-4) dx &= \frac{10}{3} \\
 \left[-\frac{x^3}{3} - x^2 + (k-4)x \right]_{-2}^0 &= \frac{10}{3} \\
 0 - \left(-\frac{(-2)^3}{3} - (-2)^2 + (k-4) \times -2 \right) &= \frac{10}{3} \\
 -\frac{8}{3} + 4 + 2k - 8 &= \frac{10}{3} \\
 -\frac{20}{3} + 2k &= \frac{10}{3} \\
 2k &= 10 \\
 k &= 5
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \text{Gradient of the sloping line is } m &= \frac{h}{R-r} \\
 \text{Equation is } y = mx + c \Rightarrow 0 &= \frac{h}{R-r} \times r + c \Rightarrow c = -\frac{hr}{R-r} \\
 y &= \frac{h}{R-r}x - \frac{hr}{R-r} \\
 y + \frac{hr}{R-r} &= \frac{h}{R-r}x \\
 x &= \frac{R-r}{h}y + r \\
 x^2 &= \left(\frac{R-r}{h} \right)^2 y^2 + \frac{2r(R-r)}{h}y + r^2 \\
 V &= \int_0^h \pi x^2 dy \\
 &= \pi \int_0^h \left(\left(\frac{R-r}{h} \right)^2 y^2 + \frac{2r(R-r)}{h}y + r^2 \right) dy \\
 &= \pi \left[\left(\frac{R-r}{h} \right)^2 \frac{y^3}{3} + \frac{2r(R-r)}{h} \frac{y^2}{2} + r^2 y \right]_0^h \\
 &= \pi \left[\left(\frac{R-r}{h} \right)^2 \frac{h^3}{3} + \frac{2r(R-r)}{h} \frac{h^2}{2} + r^2 h - 0 \right] \\
 &= \pi \left[\frac{h(R-r)^2}{3} + hr(R-r) + r^2 h \right] \\
 &= \pi \left[\frac{h(R^2 - 2Rr + r^2)}{3} + hr(R-r) + r^2 h \right] \\
 &= \frac{\pi h}{3} [R^2 - 2Rr + r^2 + 3rR - 3r^2 + 3r^2] \\
 &= \frac{\pi h}{3} [R^2 + Rr + r^2]
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \text{Area} &= \int_k^{k+1} 2x^2 dx = \frac{109}{6} \\
 \left[\frac{2x^3}{3} \right]_k^{k+1} &= \frac{109}{6} \\
 \left[\frac{2(k+1)^3}{3} - \frac{2k^3}{3} \right] &= \frac{109}{6} \\
 \left[\frac{2(k^3 + 3k^2 + 3k + 1) - 2k^3}{3} \right] &= \frac{109}{6} \\
 \left[\frac{2k^3 + 6k^2 + 6k + 2 - 2k^3}{3} \right] &= \frac{109}{6} \\
 \left[\frac{6k^2 + 6k + 2}{3} \right] &= \frac{109}{6} \\
 2(6k^2 + 6k + 2) &= 109 \\
 12k^2 + 12k + 4 &= 109 \\
 12k^2 + 12k - 105 &= 0
 \end{aligned}$$

$$k = \frac{5}{2} \text{ or } -\frac{7}{2}$$

$$\text{Since } x > 0, k = \frac{5}{2}$$

$$\begin{aligned}
 4 \quad x^2 + y^2 &= R^2 \Rightarrow x^2 = R^2 - y^2 \\
 \text{Also } \left(\frac{w}{2} \right)^2 + r^2 &= R^2 \Rightarrow r^2 = R^2 - \frac{w^2}{4} \\
 V &= \int_{-w/2}^{w/2} \pi x^2 dy - V(\text{cylinder}) \\
 &= 2 \int_0^{w/2} \pi x^2 dy - \pi r^2 w \\
 &= 2 \int_0^{w/2} \pi [R^2 - y^2] dy - \pi r^2 w \\
 &= 2\pi \left[R^2 y - \frac{y^3}{3} \right]_0^{w/2} - \pi r^2 w \\
 &= 2\pi \left[R^2 \left(\frac{w}{2} \right) - \frac{\left(\frac{w}{2} \right)^3}{3} \right] - \pi r^2 w \\
 &= 2\pi \left[\frac{R^2 w}{2} - \frac{w^3}{24} \right] - \pi r^2 w \\
 &= \pi R^2 w - \frac{\pi w^3}{12} - \pi r^2 w \\
 &= \pi R^2 w - \frac{\pi w^3}{12} - \pi \left(R^2 - \frac{w^2}{4} \right) w \\
 &= \pi R^2 w - \frac{\pi w^3}{12} - \pi R^2 w + \frac{\pi w^3}{4} \\
 &= \frac{1}{6} \pi w^3
 \end{aligned}$$

Since the volume does not depend on R , the claim is true.

8 Trigonometry

8.1 Trigonometrical functions (Page 76)

1 (i) $\frac{d}{\sin 95^\circ} = \frac{8.5}{\sin 50^\circ}$

$$d = \frac{8.5}{\sin 50^\circ} \times \sin 95^\circ = 11.1 \text{ cm}$$

(ii) $\frac{\sin \beta}{35} = \frac{\sin 53^\circ}{32}$

$$\sin \beta = \frac{\sin 53^\circ}{32} \times 35$$

$$\beta = 60.9^\circ$$

$$\theta = 180 - 60.9 - 53 = 66.1^\circ$$

(iii) $x^2 = 15^2 + 12^2 - 2 \times 15 \times 12 \times \cos 82^\circ$

$$x^2 = 318.897...$$

$$x = 17.9 \text{ cm}$$

(iv) $\cos \theta = \frac{4.1^2 + 2.8^2 - 5.6^2}{2 \times 4.1 \times 2.8} = -0.2922...$

$$\theta = \cos^{-1}(-0.2922...) = 107^\circ$$

2 (Using rounded answers from Q1)

(i) Area = $\frac{1}{2} \times 8.5 \times 11.1 \times \sin 35^\circ = 26.9 \text{ cm}^2$

(ii) Area = $\frac{1}{2} \times 32 \times 35 \times \sin 66.1^\circ = 512 \text{ m}^2$

(iii) Area = $\frac{1}{2} \times 15 \times 12 \times \sin 82^\circ = 89.1 \text{ cm}^2$

(iv) Area = $\frac{1}{2} \times 4.1 \times 2.8 \times \sin 107^\circ = 5.49 \text{ m}^2$

3 (i) $\frac{x}{7} = \frac{8}{5} \Rightarrow x = \frac{8}{5} \times 7 = 11.2 \text{ cm}$

(ii) $\frac{y}{4} = \frac{y+9}{10}$

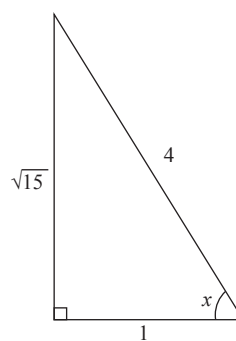
$$10y = 4(y+9)$$

$$10y = 4y + 36$$

$$6y = 36$$

$$y = 6 \text{ cm}$$

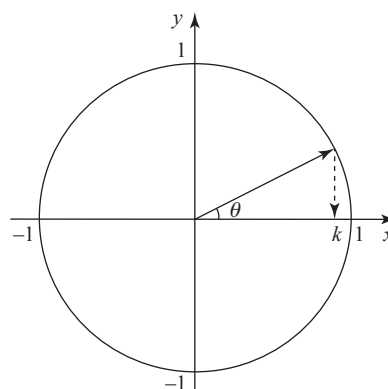
2 Using Pythagoras' theorem gives a length of $\sqrt{15}$ for the other side.



(i) $\sin x = \frac{\sqrt{15}}{4}$

(ii) $\tan x = \sqrt{15}$
 $\Rightarrow \tan^2 x = 15$

3 Using symmetry from the unit circle,



(i) $\cos(-\theta) = k$

(ii) $\cos(\pi - \theta) = -k$

(iii) $\sin \theta = \sqrt{1 - k^2}$

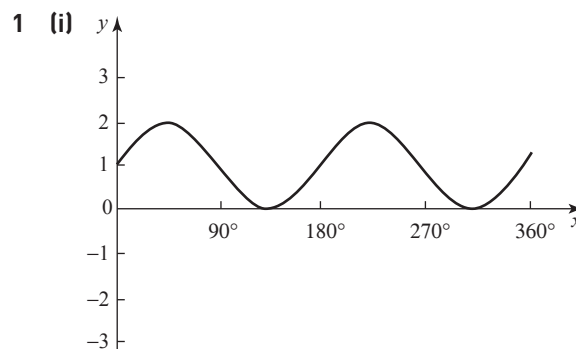
(iv) $\cos(\pi + \theta) = -k$

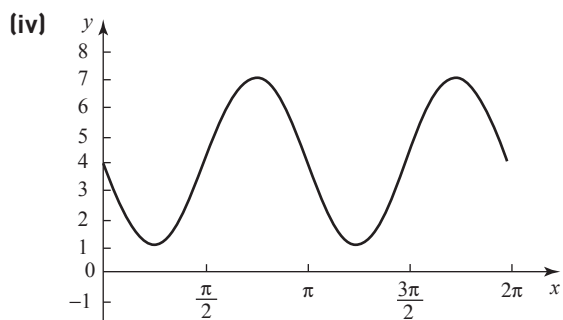
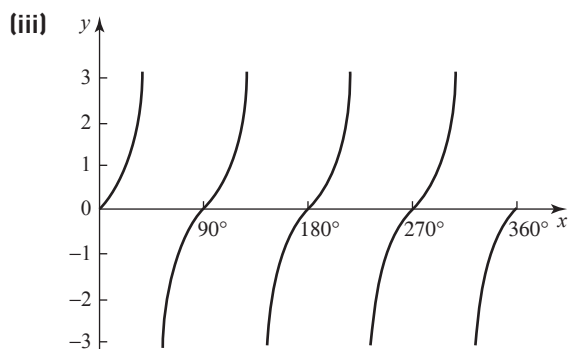
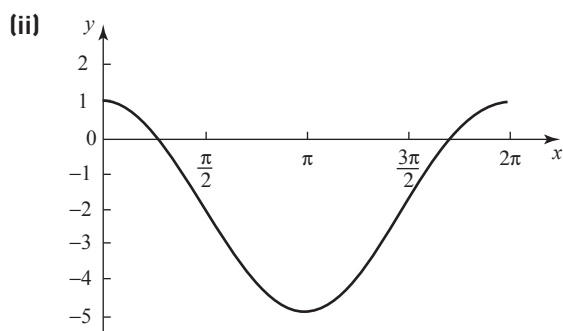
(v) $\tan \theta = \frac{\sqrt{1 - k^2}}{k}$

8.2 Trigonometrical functions for angles of any size (Page 77)

First ratio	Other ratio that has the same value	Exact value
$\sin 30^\circ$	$\sin 150^\circ$	$\frac{1}{2}$
$\sin 210^\circ$	$\sin 330^\circ$	$-\frac{1}{2}$
$\cos 60^\circ$	$\cos 300^\circ$	$\frac{1}{2}$
$\tan 30^\circ$	$\tan 210^\circ$	$\frac{1}{\sqrt{3}}$
$\cos 150^\circ$	$\cos 210^\circ$	$-\frac{\sqrt{3}}{2}$
$\tan 120^\circ$	$\tan 300^\circ$	$-\sqrt{3}$

8.3 Trigonometrical graphs (Page 78)





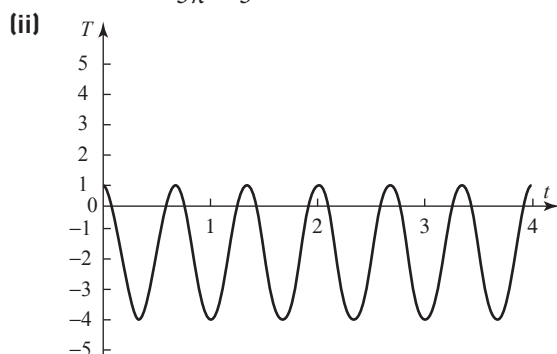
2 $A = 4, B = 1, C = -2$

3 $A = 5, B = 2, C = -1$

4 $A = 2, B = 3, C = -2$

5 $A = 2, B = -1$

6 (i) Period = $\frac{2\pi}{3\pi} = \frac{2}{3}$

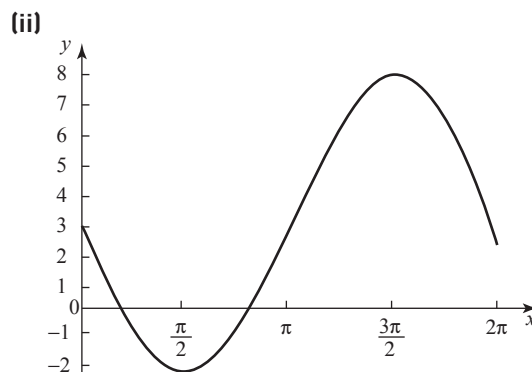


(iii) From the graph, the highest points occur at

$$t = 0.7, 1.3, 2, 2.7, 3.3 \text{ and } 4 \text{ (1 d.p.)}$$

$$t = 12.40 \text{ a.m., } 1.20 \text{ a.m., } 2 \text{ a.m., } 2.40 \text{ a.m., } 3.20 \text{ a.m., } 4 \text{ a.m.}$$

7 (i) Since the amplitude is half the range,
 $b = \frac{1}{2}(8 - -2) = 5$. Also, 3 is halfway between these two values so $a = 3$.



8.4 Identities (Page 80)

1 (i) $\sin x \tan x \equiv \frac{1}{\cos x} - \cos x$

$$\begin{aligned} LHS &= \sin x \times \frac{\sin x}{\cos x} \\ &= \frac{\sin^2 x}{\cos x} \\ &= \frac{1 - \cos^2 x}{\cos x} \\ &= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \\ &= \frac{1}{\cos x} - \cos x \\ &= RHS \end{aligned}$$

(ii) $\frac{1 + \cos x}{\sin x} \equiv \frac{\sin x}{1 - \cos x}$

$$\begin{aligned} RHS &= \frac{\sin x}{1 - \cos x} \times \frac{1 + \cos x}{1 + \cos x} \\ &= \frac{\sin x(1 + \cos x)}{1 - \cos^2 x} \\ &= \frac{\sin x(1 + \cos x)}{\sin^2 x} \\ &= \frac{1 + \cos x}{\sin x} \\ &= LHS \end{aligned}$$

(iii) $\tan^2 x - \sin^2 x \equiv \tan^2 x \sin^2 x$

$$\begin{aligned} LHS &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x \\ &= \frac{\sin^2 x - \sin^2 x \cos^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x(1 - \cos^2 x)}{\cos^2 x} \\ &= \frac{\sin^2 x \times \sin^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x}{\cos^2 x} \times \sin^2 x \\ &= \tan^2 x \sin^2 x \\ &= RHS \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \tan x + \frac{1}{\cos x} &\equiv \frac{\cos x}{1 - \sin x} \\
 \text{LHS} &= \frac{\sin x}{\cos x} + \frac{1}{\cos x} \\
 &= \frac{\sin x + 1}{\cos x} \\
 &= \frac{\sin x + 1}{\cos x} \times \frac{\cos x}{\cos x} \\
 &= \frac{(\sin x + 1)\cos x}{\cos^2 x} \\
 &= \frac{(\sin x + 1)\cos x}{1 - \sin^2 x} \\
 &= \frac{(\sin x + 1)\cos x}{(1 - \sin x)(1 + \sin x)} \\
 &= \frac{\cos x}{1 - \sin x} \\
 &= \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 2 \text{ (i)} \quad \frac{1}{1 - \cos x} + \frac{1}{1 + \cos x} &\equiv \frac{2}{\sin^2 x} \\
 \text{LHS} &= \frac{(1 + \cos x) + (1 - \cos x)}{(1 - \cos x)(1 + \cos x)} \\
 &= \frac{2}{1 - \cos^2 x} \\
 &= \frac{2}{\sin^2 x} \\
 &= \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \frac{\cos x}{1 - \cos x} - \frac{\cos x}{1 + \cos x} &\equiv \frac{2}{\tan^2 x} \\
 \text{LHS} &= \frac{\cos x}{1 - \cos x} - \frac{\cos x}{1 + \cos x} \\
 &= \frac{\cos x(1 + \cos x) - \cos x(1 - \cos x)}{(1 - \cos x)(1 + \cos x)} \\
 &= \frac{\cos x + \cos^2 x - \cos x + \cos^2 x}{1 - \cos^2 x} \\
 &= \frac{2\cos^2 x}{\sin^2 x} \\
 &= \frac{2}{\frac{\sin^2 x}{\cos^2 x}} \\
 &= \frac{2}{\tan^2 x} \\
 &= \text{RHS}
 \end{aligned}$$

8.5 Trigonometrical equations (Page 81)

$$1 \text{ (i)} \quad 2 \sin x = -1$$

$$\sin x = -\frac{1}{2}$$

$$x = \sin^{-1}\left(-\frac{1}{2}\right) = -30^\circ$$

$$x = 210^\circ \text{ or } 330^\circ$$

$$\text{(ii)} \quad \cos 2x = \frac{1}{2}$$

$$2x = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \text{ or } 300^\circ \text{ or } 420^\circ \text{ or } 660^\circ$$

$$x = 30^\circ \text{ or } 150^\circ \text{ or } 210^\circ \text{ or } 330^\circ$$

$$2 \quad 2\cos 3x - \sin 3x = 0$$

$$\frac{2\cos 3x}{\cos 3x} - \frac{\sin 3x}{\cos 3x} = \frac{0}{\cos 3x}$$

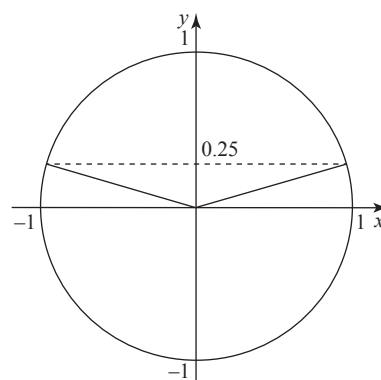
$$2 - \tan 3x = 0$$

$$\tan 3x = 2$$

$$3x = \tan^{-1}(2) = 63.4^\circ \text{ or } 243.4^\circ \text{ or } 423.4^\circ \dots$$

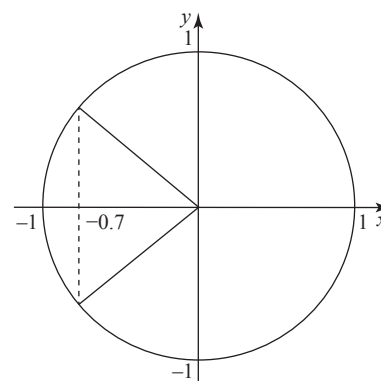
$$x = 21.1^\circ \text{ or } 81.1^\circ \text{ or } 141.1^\circ \text{ (1 d.p.)}$$

$$3 \text{ (i)}$$



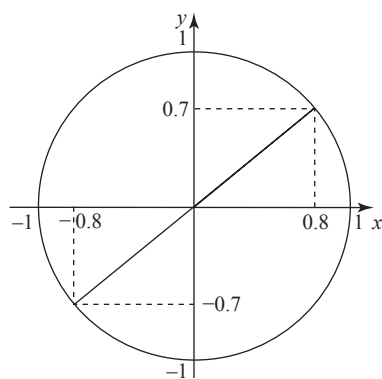
$$x = 14.5^\circ \text{ or } 165.5^\circ \text{ (1 d.p.)}$$

$$\text{(ii)}$$



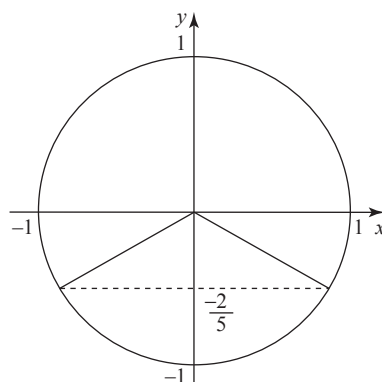
$$x = 134.4^\circ \text{ or } 225.6^\circ$$

(iii)



$$x = 41.2^\circ \text{ or } 221.2^\circ$$

(iv)



$$x = 203.6^\circ \text{ or } 336.4^\circ$$

$$4 \text{ (i)} \quad 2\sin x = -\sqrt{3}$$

$$\sin x = \frac{-\sqrt{3}}{2}$$

$$x = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$x = \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$

$$(ii) \quad \cos 3x = -1$$

$$3x = \cos^{-1}(-1) = \pi \text{ or } 3\pi \text{ or } 5\pi \dots$$

$$x = \frac{\pi}{3} \text{ or } \pi \text{ or } \frac{5\pi}{3}$$

$$5 \text{ (i)} \quad \text{Period} = \frac{2\pi}{\frac{\pi}{20}} = 40 \text{ seconds}$$

$$(ii) \quad \text{Maximum height is when } \sin\left(\frac{\pi}{20}t\right) = 1$$

i.e. $54 + 53 = 107 \text{ m}$

$$(iii) \quad h = 54 + 53\sin\left(\frac{\pi}{20}t\right) = 30$$

$$53\sin\left(\frac{\pi}{20}t\right) = -24$$

$$\sin\left(\frac{\pi}{20}t\right) = \frac{-24}{53}$$

$$\left(\frac{\pi}{20}t\right) = \sin^{-1}\left(\frac{-24}{53}\right)$$

$$\left(\frac{\pi}{20}t\right) = -0.47 \text{ or } 3.61$$

$$t = 3.61 \times \frac{20}{\pi} = 23 \text{ seconds}$$

$$6 \text{ (i)} \quad \left(\frac{1}{\cos\theta} + \tan\theta\right)^2 \equiv \frac{1+\sin\theta}{1-\sin\theta}$$

$$\begin{aligned} LHS &= \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)^2 \\ &= \left(\frac{1+\sin\theta}{\cos\theta}\right)^2 \\ &= \frac{(1+\sin\theta)^2}{\cos^2\theta} \\ &= \frac{(1+\sin\theta)^2}{1-\sin^2\theta} \\ &= \frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)} \\ &= \frac{1+\sin\theta}{1-\sin\theta} \\ &= RHS \end{aligned}$$

$$(ii) \quad \left(\frac{1}{\cos\theta} + \tan\theta\right)^2 = \frac{3}{7} \Rightarrow \frac{1+\sin\theta}{1-\sin\theta} = \frac{3}{7}$$

$$\frac{1+\sin\theta}{1-\sin\theta} = \frac{3}{7}$$

$$7(1+\sin\theta) = 3(1-\sin\theta)$$

$$7 + 7\sin\theta = 3 - 3\sin\theta$$

$$10\sin\theta = -4$$

$$\sin\theta = -\frac{4}{10} = -\frac{2}{5}$$

$$\theta = \sin^{-1}\left(-\frac{2}{5}\right) = -23.6^\circ$$

$$\theta = 203.6^\circ \text{ or } 336.4^\circ$$

$$7 \text{ (i)} \quad 2\cos^2x = \cos x$$

$$2\cos^2x - \cos x = 0$$

$$\cos x(2\cos x - 1) = 0$$

$$\cos x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$x = 90^\circ, 270^\circ, 60^\circ, 300^\circ$$

$$(ii) \quad 2\tan x = \tan^2x$$

$$0 = \tan^2x - 2\tan x$$

$$0 = \tan x(\tan x - 2)$$

$$\tan x = 0 \text{ or } \tan x = 2$$

$$x = 0, \pi, 2\pi, 1.11, 4.25$$

$$(iii) \quad \cos x = -3\tan x$$

$$\cos x = -3\frac{\sin x}{\cos x}$$

$$\cos^2x = -3\sin x$$

$$1 - \sin^2x = -3\sin x$$

$$0 = \sin^2x - 3\sin x - 1$$

$$\sin x = \frac{3 \pm \sqrt{3^2 - 4 \times 1 \times -1}}{2 \times 1}$$

$$= \frac{3 \pm \sqrt{13}}{2}$$

$$= 3.30 \text{ or } -0.303$$

$$x = -0.308 \text{ or } -2.83$$

$$\begin{aligned}
 \text{(iv)} \quad 4 + 5 \cos x &= 2 \sin^2 x \\
 4 + 5 \cos x &= 2(1 - \cos^2 x) \\
 4 + 5 \cos x &= 2 - 2 \cos^2 x \\
 2 \cos^2 x + 5 \cos x + 2 &= 0 \\
 (2 \cos x + 1)(\cos x + 2) &= 0 \\
 \cos x &= -\frac{1}{2} \text{ or } \cos x = -2 \\
 x &= 120^\circ \text{ or } -120^\circ
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{(i)} \quad 4 \cos^2 x + 7 \sin x - 2 &= 0 \\
 4(1 - \sin^2 x) + 7 \sin x - 2 &= 0 \\
 4 - 4 \sin^2 x + 7 \sin x - 2 &= 0 \\
 -4 \sin^2 x + 7 \sin x + 2 &= 0 \\
 4 \sin^2 x - 7 \sin x - 2 &= 0 \\
 (4 \sin x + 1)(\sin x - 2) &= 0 \\
 \sin x &= -\frac{1}{4} \text{ or } \sin x = 2 \\
 \text{Since } \sin x \text{ is never greater than } 1, \sin x &= -\frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \sin(\theta - 20^\circ) &= -\frac{1}{4} \\
 \theta - 20^\circ &= \sin^{-1}\left(-\frac{1}{4}\right) \\
 \theta - 20^\circ &= -14.5^\circ, 194.5^\circ, \dots \\
 \theta &= 5.5^\circ, 214.5^\circ
 \end{aligned}$$

8.6 Circular measure (Page 84)

1	Degrees	Radians
	120°	$\frac{2\pi}{3}$
	36°	$\frac{\pi}{5}$
	330°	$\frac{11\pi}{6}$
	150°	$\frac{5\pi}{6}$
	240°	$\frac{4\pi}{3}$
	54°	$\frac{3\pi}{10}$
	225°	$\frac{5\pi}{4}$
	315°	$\frac{7\pi}{4}$

2	Degrees	Radians
	12°	0.209
	10.9°	0.190
	145°	2.53
	169.0°	2.95
	235°	4.10
	288.8°	5.04
	342.5°	5.98
	78.5°	1.37

$$3 \quad \text{(i)} \quad P = 7 + 7 + 7 \times 1.2 = 22.4 \text{ cm}$$

$$A = \frac{1}{2} \times 7^2 \times 1.2 = 29.4 \text{ cm}^2$$

$$\text{(ii)} \quad \text{Angle of sector is } \theta = 360^\circ - 130^\circ = 230^\circ$$

Converting to radians,

$$230 \times \frac{\pi}{180} = \frac{23\pi}{18} = 4.014\dots$$

$$P = 12.5 + 12.5 + (12.5 \times 4.014\dots) = 75.2 \text{ cm (3 s.f.)}$$

$$A = \frac{1}{2} \times 12.5^2 \times 4.014\dots = 314 \text{ cm}^2 \text{ (3 s.f.)}$$

$$4 \quad 12 = 7\theta$$

$$\theta = \frac{12}{7} = 1.71 = 98.2^\circ$$

5 Perimeter:

$$\text{Length of arc AB} = 5 \times \frac{2\pi}{3} = \frac{10\pi}{3} \approx 10.47\dots$$

By joining OC create two right-angled triangles AOC and BOC.

$$\begin{aligned}
 \tan \frac{\pi}{3} &= \frac{AC}{5} \Rightarrow AC = 5 \times \tan \frac{\pi}{3} \\
 &= 5\sqrt{3} \approx 8.66\dots
 \end{aligned}$$

$$\begin{aligned}
 P &= \frac{10\pi}{3} + 5\sqrt{3} + 5\sqrt{3} \\
 &= \frac{10\pi}{3} + 10\sqrt{3} = 27.8 \text{ cm (3 s.f.)}
 \end{aligned}$$

Area:

$$\text{Area of triangle AOC} = \frac{1}{2} \times 5 \times 5\sqrt{3} = \frac{25\sqrt{3}}{2}$$

$$\text{Area of sector AOB} = \frac{1}{2} \times 5^2 \times \frac{2\pi}{3} = \frac{25\pi}{3}$$

$$\begin{aligned}
 \text{Shaded area} &= 2 \times \frac{25\sqrt{3}}{2} - \frac{25\pi}{3} \\
 &= 25\sqrt{3} - \frac{25\pi}{3} \\
 &= 17.1 \text{ cm}^2
 \end{aligned}$$

$$6 \quad \text{(i)} \quad \text{Length of arc, } s = 10 \times 0.8 = 8 \text{ cm}$$

$$\sin 0.8 = \frac{RP}{10}$$

$$RP = 10 \times \sin 0.8 = 7.1735\dots \text{ cm}$$

$$\cos 0.8 = \frac{OP}{10}$$

$$OP = 10 \times \cos 0.8 = 6.967\dots \text{ cm}$$

$$\text{so } QP = 10 - 6.967\dots = 3.0329\dots \text{ cm}$$

$$\text{Perimeter} = 8 + 7.1735 + 3.0329$$

$$= 18.2 \text{ cm (3 s.f.)}$$

(ii) Area of shaded region

$$= \text{area of sector} - \text{area of triangle OPR}$$

$$= \frac{1}{2} r^2 \theta - \frac{1}{2} \times OP \times PR$$

$$= \frac{1}{2} \times 10^2 \times 0.8 - \frac{1}{2} \times 6.967\dots \times 7.173\dots$$

$$= 40 - 24.989\dots$$

$$= 15.0 \text{ cm}^2 \text{ (3 s.f.)}$$

$$7 \text{ (i)} \quad \frac{1}{2} \times 6^2 \times \theta - \frac{1}{2} \times 6 \times 6 \times \sin \theta = 36$$

$$18\theta - 18 \sin \theta = 36$$

$$18(\theta - \sin \theta) = 36$$

$$\theta - \sin \theta = 2$$

$$\theta = \sin \theta + 2$$

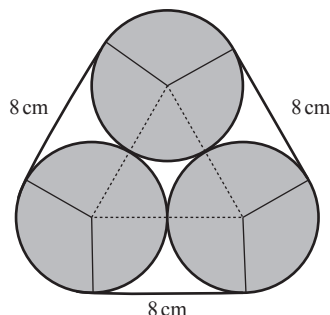
$$\text{(ii)} \quad 6 + 6 + 6\theta = 27.3$$

$$12 + 6\theta = 27.3$$

$$6\theta = 15.3$$

$$\theta = 2.55$$

8



Since each part of the band around a circle is one-third the length of the circumference, the total length is

$$(2 \times \pi \times 4) + 8 + 8 + 8 = 49.1 \text{ cm (3 s.f.)}$$

$$9 \text{ (i)} \quad \tan \theta = \frac{AC}{8} \Rightarrow AC = 8 \tan \theta$$

Shaded area

$$= \text{area of triangle OAC} - \text{area of sector OAB}$$

$$= \frac{1}{2} \times 8 \tan \theta \times 8 - \frac{1}{2} \times 8^2 \times \theta$$

$$= 32 \tan \theta - 32\theta$$

$$= 32(\tan \theta - \theta)$$

$$\text{(ii)} \quad \tan \frac{\pi}{3} = \frac{AC}{8} \Rightarrow AC = 8 \tan \frac{\pi}{3}$$

$$= 8\sqrt{3} \approx 13.86 \text{ cm}$$

$$\cos \frac{\pi}{3} = \frac{8}{OC} \Rightarrow OC = \frac{8}{\cos \frac{\pi}{3}} = 16 \text{ cm}$$

$$\text{So } BC = 16 - 8 = 8 \text{ cm}$$

$$\text{Length of arc AB} = 8 \times \frac{\pi}{3} = \frac{8\pi}{3}$$

$$\text{Perimeter} = \frac{8\pi}{3} + 8 + 8\sqrt{3} \approx 30.2 \text{ cm (3 s.f.)}$$

$$10 \text{ (i)} \quad \text{Since } \triangle OCD \text{ is equilateral, } \angle OCD = \frac{\pi}{3}.$$

$$\text{So } \angle ACD = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$$

$$\text{(ii)} \quad \text{Perimeter} = AD + BD + AB$$

$$= 8 \times \frac{2}{3}\pi + 8 + 16 \times \frac{1}{3}\pi$$

$$= \frac{32}{3}\pi + 8 \text{ cm}$$

$$\text{(iii)} \quad \text{Shaded area}$$

$$= \text{area OAB} - \text{area OCD} - \text{area ACD}$$

$$= \frac{1}{2} \times 16^2 \times \frac{1}{3}\pi - \frac{1}{2} \times 8^2 \times \sin \frac{1}{3}\pi - \frac{1}{2} \times 8^2 \times \frac{2}{3}\pi$$

$$= \frac{128}{3}\pi - 32 \times \frac{\sqrt{3}}{2} - \frac{64}{3}\pi$$

$$= \frac{64}{3}\pi - 16\sqrt{3}$$

Further practice (Page 88)

1 (i) The amplitude of the graph is 2 and it repeats 3 times in 360° (the period is 120°) so the equation is $y = 2 \sin 3x$.

(ii) The graph repeats twice in 2π so $c = 2$.

The range is $-2 \leq y \leq 6$ and the amplitude is half of the total range so $b = 4$.

The graph has been shifted up 2 units so $a = 2$.

The negative sign in front of the cos term means the graph is upside-down.

The equation is $y = 2 - 4 \cos 2x$.

$$2 \text{ (i)} \quad \tan x(1 - \sin^2 x) \equiv \sin x \cos x$$

$$LHS = \tan x \times \cos^2 x$$

$$= \frac{\sin x}{\cos x} \times \cos^2 x$$

$$= \sin x \cos x$$

$$= RHS$$

$$\text{(ii)} \quad \sin^4 x - \cos^4 x \equiv \sin^2 x - \cos^2 x$$

$$LHS = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$$

$$= (\sin^2 x - \cos^2 x)(1)$$

$$= \sin^2 x - \cos^2 x$$

$$= RHS$$

$$\text{(iii)} \quad \frac{\cos \theta}{\tan \theta(1 + \sin \theta)} \equiv \frac{1}{\sin \theta} - 1$$

$$LHS = \frac{\cos \theta}{\frac{\sin \theta}{\cos \theta}(1 + \sin \theta)}$$

$$= \frac{\cos^2 \theta}{\sin \theta(1 + \sin \theta)}$$

$$= \frac{(1 - \sin^2 \theta)}{\sin \theta(1 + \sin \theta)}$$

$$= \frac{(1 - \sin \theta)(1 + \sin \theta)}{\sin \theta(1 + \sin \theta)}$$

$$= \frac{1 - \sin \theta}{\sin \theta}$$

$$= \frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta}$$

$$= \frac{1}{\sin \theta} - 1$$

$$= RHS$$

3 (i) $\tan x + 1 = 3$

$$\tan x = 2$$

$$x = \tan^{-1}(2) = 63.4^\circ \text{ or } 243.4^\circ \text{ (1 d.p.)}$$

(ii) $\sin(x - 30^\circ) = 1$

$$x - 30^\circ = \sin^{-1}(1)$$

$$x - 30^\circ = 90^\circ$$

$$x = 120^\circ$$

4 (i) $2 \tan x - 1 = 1$

$$2 \tan x = 2$$

$$\tan x = 1$$

$$x = \tan^{-1}(1) = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

(ii) $\frac{1}{2} \sin(2x + 1) = 0.1$

$$\sin(2x + 1) = 0.2$$

$$2x + 1 = \sin^{-1}(0.2)$$

$$2x + 1 = 0.2014 \text{ or } 2.940$$

$$\text{or } 6.485 \text{ or } 9.223 \text{ or } 12.77$$

$$x = 0.0507 \text{ or } 1.47 \text{ or } 2.86 \text{ or } 6.38 \text{ (3 s.f.)}$$

(iii) $4 \sin^2 x + 4 \sin x - 3 = 0$

$$(2 \sin x + 3)(2 \sin x - 1) = 0$$

$$\sin x = -\frac{3}{2} \text{ or } \sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

(iv) $2 \sin^2 x = 3 - 3 \cos x$

$$2(1 - \cos^2 x) = 3 - 3 \cos x$$

$$2 - 2 \cos^2 x = 3 - 3 \cos x$$

$$0 = 2 \cos^2 x - 3 \cos x + 1$$

$$0 = (2 \cos x - 1)(\cos x - 1)$$

$$\cos x = \frac{1}{2} \text{ or } \cos x = 1$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}, 0, 2\pi$$

(v) $3 \sin^2 x + 5 \cos^2 x = 9 \sin x$

$$3 \sin^2 x + 5(1 - \sin^2 x) = 9 \sin x$$

$$3 \sin^2 x + 5 - 5 \sin^2 x = 9 \sin x$$

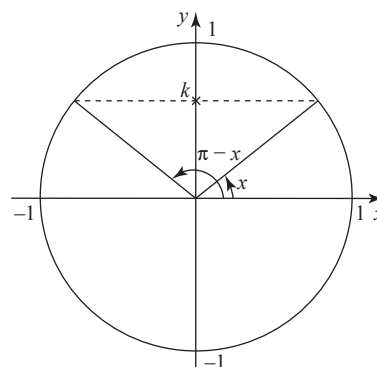
$$0 = 2 \sin^2 x + 9 \sin x - 5$$

$$0 = (2 \sin x - 1)(\sin x + 5)$$

$$\sin x = \frac{1}{2} \text{ or } \sin x = -5$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

- 5 (i) The diagram shows the relationship between $\sin x$ and $\sin(\pi - x)$:



$$\sin x = \sin(\pi - x) = k$$

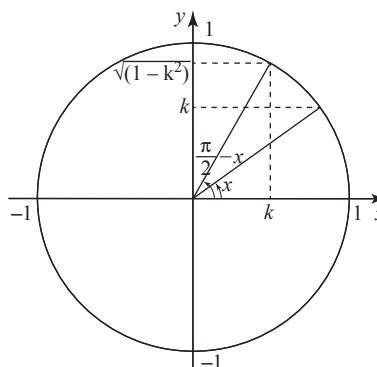
- (ii) Looking at the right-angled triangle formed in the unit circle, we can calculate the length of the unknown side with Pythagoras' theorem:

$$x^2 + k^2 = 1^2$$

$$x = \sqrt{1 - k^2}$$

$$\text{Hence } \cos x = \sqrt{1 - k^2}$$

- (iii) Looking at the diagram, we can calculate the value of $\tan x$ from the answers to the first two parts of the question.



$$\tan x = \frac{k}{\sqrt{1 - k^2}}$$

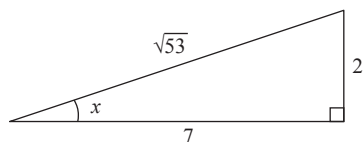
Looking at $\tan\left(\frac{\pi}{2} - x\right)$ we can see that the values of $\sin x$ and $\cos x$ have swapped so

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sqrt{1 - k^2}}{k}$$

OR the values can be found directly by looking at the unit circle:

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\cos x}{\sin x} = \frac{\sqrt{1 - k^2}}{k}$$

- 6 Pythagoras' theorem gives the length of the other side as $\sqrt{53}$.



- (i) From the triangle, $\cos x = \frac{7}{\sqrt{53}}$, but if $90^\circ \leq x \leq 180^\circ$, then the value of $\cos x$ is negative, so $\cos x = -\frac{7}{\sqrt{53}}$

(ii) $\sin^2 x = \left(\frac{2}{\sqrt{53}}\right)^2 = \frac{4}{53}$

7 $LHS = 1 + \tan^2 \theta$

$$\begin{aligned} &= 1 + \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} \\ &= \frac{1}{\cos^2 \theta} \\ &= RHS \end{aligned}$$

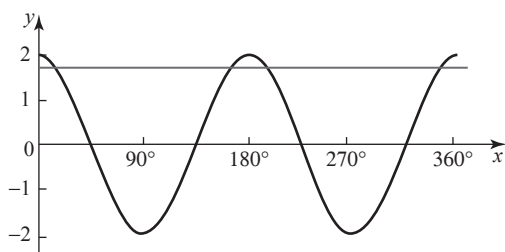
8 $LHS = \frac{\sin x}{\cos x} + \frac{1}{\frac{\sin x}{\cos x}}$

$$\begin{aligned} &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\ &= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\ &= \frac{1}{\sin x \cos x} \\ &= RHS \end{aligned}$$

9 $\frac{1 - \cos x}{\sin x} + \frac{\sin x}{1 - \cos x} \equiv \frac{2}{\sin x}$

$$\begin{aligned} LHS &= \frac{(1 - \cos x)(1 - \cos x) + \sin^2 x}{\sin x(1 - \cos x)} \\ &= \frac{(1 - 2\cos x + \cos^2 x) + \sin^2 x}{\sin x(1 - \cos x)} \\ &= \frac{2 - 2\cos x}{\sin x(1 - \cos x)} \\ &= \frac{2(1 - \cos x)}{\sin x(1 - \cos x)} \\ &= \frac{2}{\sin x} \\ &= RHS \end{aligned}$$

- 10 (i) From the graphs of $y = 2\cos 2x$ and $y = \sqrt{3}$:



Clearly there are four solutions.

$$\cos 2x = \frac{\sqrt{3}}{2}$$

$$2x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$2x = 30^\circ \text{ or } 330^\circ \text{ or } 390^\circ \text{ or } 690^\circ$$

$$x = 15^\circ \text{ or } 165^\circ \text{ or } 195^\circ \text{ or } 345^\circ$$

- (ii) Whenever there is an equation with a mixture of $\sin$ and $\cos$ terms, you need to use either $\cos^2 x = 1 - \sin^2 x$ or $\sin^2 x = 1 - \cos^2 x$ so that the equation can be written as a quadratic with only $\sin x$ or $\cos x$ terms.

$$\cos^2 x - 1 = \sin x$$

$$(1 - \sin^2 x) - 1 = \sin x$$

$$0 = \sin^2 x + \sin x$$

$$0 = \sin x(\sin x + 1)$$

$$\sin x = 0 \text{ or } \sin x = -1$$

$$x = 0 \text{ or } -\frac{\pi}{2}$$

If the domain for the solutions was $-\pi \leq x \leq \pi$, then you would have included $x = \pi$ or $-\pi$.

- (iii) $2 \sin x \tan x = 3$

$$2 \sin x \frac{\sin x}{\cos x} = 3$$

$$\frac{2 \sin^2 x}{\cos x} = 3$$

$$2 \sin^2 x = 3 \cos x$$

$$2(1 - \cos^2 x) = 3 \cos x$$

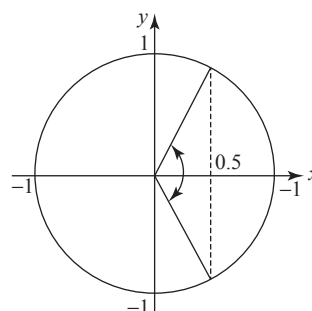
$$2 - 2 \cos^2 x = 3 \cos x$$

$$0 = 2 \cos^2 x + 3 \cos x - 2$$

$$0 = (2 \cos x - 1)(\cos x + 2)$$

$$\cos x = \frac{1}{2} \text{ or } -2$$

$$x = \frac{\pi}{3}, -\frac{\pi}{3}$$



11 Area of segment = area of sector – area of triangle

$$\begin{aligned}
 &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \\
 &= \frac{1}{2} \times 8^2 \times \frac{\pi}{3} - \frac{1}{2} \times 8^2 \times \sin\frac{\pi}{3} \\
 &= 5.80 \text{ cm}^2 \text{ (3 s.f.)}
 \end{aligned}$$

12 Shaded area = area of triangle OAB – area of sector OAC

$$\begin{aligned}
 &= \frac{1}{2} \times OA \times AB - \frac{1}{2}r^2\theta \\
 &= \frac{1}{2} \times 10 \times AB - \frac{1}{2} \times 10^2 \times \frac{\pi}{3}
 \end{aligned}$$

$$\tan\frac{\pi}{3} = \frac{AB}{10} \text{ so } AB = 10 \times \tan\frac{\pi}{3} = 10\sqrt{3}$$

$$\begin{aligned}
 \text{So shaded area} &= \frac{1}{2} \times 10 \times 10\sqrt{3} - \frac{1}{2} \times 10^2 \times \frac{\pi}{3} \\
 &= \left(50\sqrt{3} - \frac{50}{3}\pi\right) \text{ cm}^2
 \end{aligned}$$

To find the perimeter of the shaded region, we need to find the length of OB.

$$\cos\frac{\pi}{3} = \frac{10}{OB} \Rightarrow OB = \frac{10}{\cos\frac{\pi}{3}} = 20 \text{ cm}$$

$$BC = OB - OC = 20 \text{ cm} - 10 \text{ cm} = 10 \text{ cm}$$

$$\begin{aligned}
 \text{Perimeter of shaded region} &= \text{arc AC} + AB + BC \\
 &= 10 \times \frac{\pi}{3} + 10\sqrt{3} + 10 \\
 &= \frac{10\pi}{3} + 10\sqrt{3} + 10
 \end{aligned}$$

13 (i) $s = r\theta \Rightarrow 10 = 6\theta \Rightarrow \theta = \frac{10}{6} = \frac{5}{3}$

(ii) Angle ODE is 90° so triangle ODE is a right-angled triangle.

$$\text{Angle DOE} = \frac{1}{2} \times \text{angle DOF} = \frac{1}{2} \times \frac{5}{3} = \frac{5}{6}$$

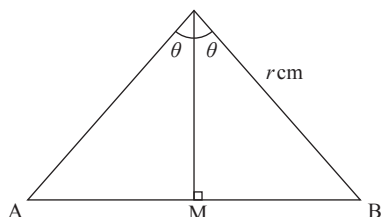
$$\tan\frac{5}{6} = \frac{DE}{6} \Rightarrow DE = 6 \times \tan\frac{5}{6} = 6.60 \text{ (3 s.f.)}$$

(iii) Shaded area = $2 \times$ area triangle ODE – area of sector DOF

$$\begin{aligned}
 &= 2 \times \left(\frac{1}{2} \times 6.60 \times 6\right) - \frac{1}{2} \times 6^2 \times \frac{5}{3} \\
 &= 9.63 \text{ cm}^2 \text{ (3 s.f.)}
 \end{aligned}$$

Past exam questions (Page 90)

1 (i) Length DC = length AB



$$\sin\theta = \frac{BM}{r} \Rightarrow BM = r \sin\theta$$

$$\text{So } AB = 2r \sin\theta$$

OR

$$AB^2 = r^2 + r^2 - 2r^2 \cos 2\theta$$

$$\begin{aligned}
 AB &= \sqrt{2r^2 - 2r^2 \cos 2\theta} \\
 &= \sqrt{2r^2(1 - \cos 2\theta)}
 \end{aligned}$$

$$\text{Arc AB} = 2r\theta$$

$$\begin{aligned}
 \text{Perimeter} &= 2r + 2r \sin\theta + 2r\theta \\
 &= 2r(1 + \sin\theta + \theta)
 \end{aligned}$$

(ii) Shaded area = area rectangle – area segment

$$\begin{aligned}
 \text{Area rectangle} &= r \times 2r \sin\theta \\
 &= 5 \times 2 \times 5 \times \sin\frac{1}{6}\pi \\
 &= 25
 \end{aligned}$$

$$\begin{aligned}
 \text{Area segment} &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta \\
 &= \frac{1}{2} \times 5^2 \times \frac{1}{3}\pi - \frac{1}{2} \times 5^2 \times \sin\frac{1}{3}\pi \\
 &= 2.26
 \end{aligned}$$

$$\text{Shaded area} = 25 - 2.26 = 22.7 \text{ (3 s.f.)}$$

2 (i) $\cos^4 x = (1 - \sin^2 x)^2$

$$= 1 - 2\sin^2 x + \sin^4 x$$

(ii) $8\sin^4 x + (1 - 2\sin^2 x + \sin^4 x) = 2(1 - \sin^2 x)$

$$9\sin^4 x = 1$$

$$\sin^4 x = \frac{1}{9}$$

$$\sin x = \pm\sqrt[4]{\frac{1}{9}}$$

$$x = 35.3^\circ, 144.7^\circ, 215.3^\circ, 324.7^\circ$$

3 (i) $\tan\frac{\pi}{3} = \frac{AC}{2x}$

$$AC = 2x \times \tan\frac{\pi}{3} = 2x \times \sqrt{3} = 2\sqrt{3}x$$

$$\begin{aligned}
 AM^2 &= AC^2 + MC^2 \\
 &= (2\sqrt{3}x)^2 + x^2 \\
 &= 12x^2 + x^2 \\
 &= 13x^2
 \end{aligned}$$

$$AM = \sqrt{13}x$$

$$\begin{aligned} \text{(ii)} \quad \tan \text{MAC} &= \frac{x}{2\sqrt{3}x} = \frac{1}{2\sqrt{3}} \\ \angle \text{MAC} &= \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) \\ \angle \text{BAC} &= \frac{1}{6}\pi \\ \theta &= \frac{1}{6}\pi - \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) \end{aligned}$$

$$\begin{aligned} 4 \quad \text{(i)} \quad \text{OC} &= r \cos \alpha, \text{AC} = r \sin \alpha \\ \text{Area OAC} &= \frac{1}{2} \times r \cos \alpha \times r \sin \alpha \\ \frac{1}{2} \left(\frac{1}{2} r^2 \alpha \right) &= \frac{1}{2} \times r \cos \alpha \times r \sin \alpha \\ \frac{1}{2} r^2 \alpha &= r^2 \cos \alpha \sin \alpha \\ \frac{1}{2} \alpha &= \cos \alpha \sin \alpha \\ \sin \alpha \cos \alpha &= \frac{1}{2} \alpha \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \text{Perimeter OAC} &= r + r \cos \alpha + r \sin \alpha \\ &= r + r \cos 0.9477 + r \sin 0.9477 \\ &= 2.396r \end{aligned}$$

$$\begin{aligned} \text{Perimeter ACB} &= r\alpha + r \sin \alpha + (r - r \cos \alpha) \\ &= r \times 0.9477 + r \sin 0.9477 + (r - r \cos 0.9477) \\ &= 2.176r \end{aligned}$$

$$\begin{aligned} \text{Ratio is } 2.396r : 2.176r \\ 2.396 : 2.176 \\ \frac{2.396}{2.176} : 1 \\ 1.1 : 1 \end{aligned}$$

$$\text{(iii)} \quad \angle \text{AOB} = 0.9477 \times \frac{180}{\pi} = 54.3^\circ$$

$$\begin{aligned} 5 \quad \text{(i)} \quad \frac{1}{\cos \theta} + 3 \sin \theta \tan \theta + 4 &= 0 \\ \frac{1}{\cos \theta} + 3 \sin \theta \frac{\sin \theta}{\cos \theta} + 4 &= 0 \\ \frac{1}{\cos \theta} + \frac{3 \sin^2 \theta}{\cos \theta} + 4 &= 0 \\ 1 + 3 \sin^2 \theta + 4 \cos \theta &= 0 \\ 1 + 3(1 - \cos^2 \theta) + 4 \cos \theta &= 0 \\ 1 + 3 - 3 \cos^2 \theta + 4 \cos \theta &= 0 \\ 0 &= 3 \cos^2 \theta - 4 \cos \theta - 4 \\ 3 \cos^2 \theta - 4 \cos \theta - 4 &= 0 \\ (3 \cos \theta + 2)(\cos \theta - 2) &= 0 \\ \cos \theta &= -\frac{2}{3} \text{ or } \cos \theta = 2 \\ \theta &= \cos^{-1}\left(-\frac{2}{3}\right) = 131.8^\circ \text{ or } 228.2^\circ \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \text{At C, } y &= 0 \text{ so} \\ a \cos x - b &= 0 \\ \cos x &= \frac{b}{a} \\ x &= \cos^{-1}\left(\frac{b}{a}\right) \end{aligned}$$

$$\text{Since } x = \cos^{-1} c, c = \cos x = \frac{b}{a}$$

$$\begin{aligned} \text{At D, } x &= 0 \text{ so} \\ d &= a \cos 0 - b = a - b \end{aligned}$$

$$6 \quad \text{(i)} \quad \tan(\text{BAO}) = \frac{5}{12} \Rightarrow \text{BAO} = 0.3948$$

$$\text{(ii)} \quad \text{Angle ABO} = \frac{\pi}{2} - 0.3948 \approx 1.176$$

$$\begin{aligned} \text{Shaded area} &= \text{area sector APQ} + \text{area sector} \\ &\quad \text{BOQ} - \text{area triangle ABO} \\ &= \frac{1}{2} \times 12^2 \times 0.3948 + \frac{1}{2} \times 5^2 \times 1.176 - \frac{1}{2} \times 12 \times 5 \\ &= 13.1 \text{ cm}^2 \end{aligned}$$

$$7 \quad \text{(i)} \quad \left(\frac{1}{\sin \theta} - \frac{1}{\tan \theta} \right)^2 = \frac{1 - \cos \theta}{1 + \cos \theta}$$

$$\begin{aligned} LHS &= \left(\frac{1}{\sin \theta} - \frac{1}{\tan \theta} \right)^2 \\ &= \left(\frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \right)^2 \\ &= \left(\frac{1 - \cos \theta}{\sin \theta} \right)^2 \\ &= \frac{(1 - \cos \theta)^2}{\sin^2 \theta} \\ &= \frac{(1 - \cos \theta)^2}{(1 - \cos^2 \theta)} \\ &= \frac{(1 - \cos \theta)(1 - \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} \\ &= \frac{1 - \cos \theta}{1 + \cos \theta} \\ &= RHS \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \frac{1 - \cos \theta}{1 + \cos \theta} &= \frac{2}{5} \\ 5(1 - \cos \theta) &= 2(1 + \cos \theta) \\ 5 - 5 \cos \theta &= 2 + 2 \cos \theta \\ 3 &= 7 \cos \theta \\ \frac{3}{7} &= \cos \theta \\ \theta &= 64.6^\circ \text{ or } 295.4^\circ \end{aligned}$$

8 (i) Distance from D to AB is $6\sin\frac{\pi}{3} = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

Distance from E to AB is $10\sin\theta$

Equating these distances gives

$$10\sin\theta = 3\sqrt{3}$$

$$\sin\theta = \frac{3\sqrt{3}}{10}$$

$$\theta = \sin^{-1}\left(\frac{3\sqrt{3}}{10}\right)$$

(ii) $\theta = 31.3^\circ$ (1 d.p.) or 0.546 radians

$$AB = 6 + 10 = 16 \text{ cm}$$

$$DE = 16 - 6\cos\frac{\pi}{3} - 10\cos\theta$$

$$= 16 - 6 \times \frac{1}{2} - 10\cos(31.3^\circ)$$

$$= 4.46 \text{ cm}$$

$$P = \text{arc DX} + \text{arc EX} + DE$$

$$= 6 \times \frac{1}{3}\pi + 10 \times 0.546\dots + 4.46\dots$$

$$= 16.20 \text{ cm (4 s.f.)}$$

(ii) Period $= \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{2}} = 4$ hours

(iii) 1 a.m.

3 $10.5 = 9 - 3\cos\left(\frac{\pi}{8}t\right)$

$$3\cos\left(\frac{\pi}{8}t\right) = -1.5$$

$$\cos\left(\frac{\pi}{8}t\right) = -\frac{1}{2}$$

$$\frac{\pi}{8}t = \cos^{-1}\left(-\frac{1}{2}\right)$$

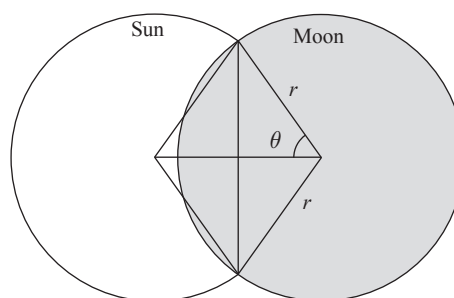
$$\frac{\pi}{8}t = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$t = \frac{16}{3}, \frac{32}{3}$$

$$t = 5\frac{1}{3}, 10\frac{2}{3}$$

Since t is hours after 9.30 a.m., refloating can occur between $5\frac{1}{3}$ and $10\frac{2}{3}$ after 9.30 a.m., so between 2.50 p.m. and 8.10 p.m.

4



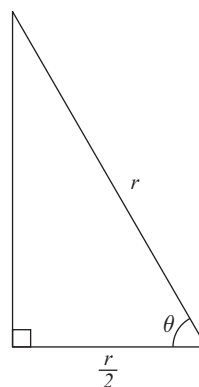
(i) $A = 2\left(\frac{1}{2}r^2 2\theta - \frac{1}{2}r^2 \sin 2\theta\right)$

$$\frac{1}{2}\pi r^2 = r^2 2\theta - r^2 \sin 2\theta$$

$$\pi = 4\theta - 2\sin 2\theta$$

$$2\sin 2\theta = 4\theta - \pi$$

(ii)



$$\cos\theta = \frac{\frac{r}{2}}{r} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ = \frac{\pi}{3}$$

Stretch and challenge (Page 92)

1 $3\sin 3\theta + 3 = 2\cos^2 3\theta$

$$3\sin 3\theta + 3 = 2(1 - \sin^2 3\theta)$$

$$3\sin 3\theta + 3 = 2 - 2\sin^2 3\theta$$

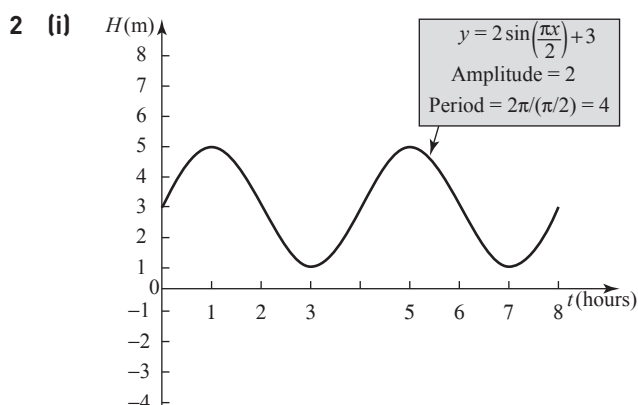
$$2\sin^2 3\theta + 3\sin 3\theta + 1 = 0$$

$$(2\sin 3\theta + 1)(\sin 3\theta + 1) = 0$$

$$\sin 3\theta = -\frac{1}{2} \text{ or } \sin 3\theta = -1$$

$$3\theta = -30^\circ, -150^\circ, -390^\circ, -510^\circ, 210^\circ, 330^\circ, -90^\circ, 270^\circ, -450^\circ$$

$$\theta = -170^\circ, -150^\circ, -130^\circ, -50^\circ, -30^\circ, -10^\circ, 70^\circ, 90^\circ, 110^\circ$$



$$(iii) \quad A = 2\left(\frac{1}{2}r^2 2\theta - \frac{1}{2}r^2 \sin 2\theta\right)$$

$$= 2r^2\theta - r^2 \sin 2\theta$$

$$= 2r^2 \frac{\pi}{3} - r^2 \sin \frac{2\pi}{3}$$

$$= \frac{2\pi}{3}r^2 - \frac{\sqrt{3}}{2}r^2$$

$$= \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2$$

$$\% \text{ covered} = \frac{\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2}{\pi r^2} \times 100\%$$

$$= \frac{\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)}{\pi} \times 100\%$$

$$= 39.1\%$$