

Answers

The questions, with the exception of those from past question papers, and all example answers that appear in this book were written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers which are contained in this publication.

Non-exact numerical answers should be given correct to three significant figures (or one decimal place for angles in degrees) unless a different level of accuracy is specified in the question. You should avoid rounding figures until reaching your final answer.

Chapter 1

? (Page 1)

See text that follows.

? (Page 2)

$$\cos^2 x + \sin^2 x = 1$$

? (Page 3)

$\cosh t$ is positive for all values of t , so only the branch for which x is positive is given.

Exercise 1A (Page 6)

$$2 \quad \cosh x = \sqrt{5}, \tanh x = \frac{2}{\sqrt{5}}$$

$$3 \quad \text{(ii)} \quad 3e^{2x} + 2e^x - 1 = 0$$

$$4 \quad \text{(i)} \quad \ln \frac{3}{4}, \ln 2 \quad \text{(ii)} \quad -\ln 2 \quad \text{(iii)} \quad \text{no real root}$$

$$5 \quad x = \ln 3, y = \ln 2$$

$$6 \quad 1.62\text{m}, 22.3^\circ$$

$$9 \quad \text{(i)} \quad a + b, a - b, c \text{ all have the same sign and } b^2 + c^2 > a^2$$

$$\text{(ii)} \quad a + b, a - b, c \text{ all have the same sign and } b^2 + c^2 = a^2 \text{ or } a + b \text{ and } a - b \text{ have opposite signs and } b^2 + c^2 > a^2$$

$$\text{(iii)} \quad b^2 + c^2 < a^2 \text{ or } a + b \text{ and } a - b \text{ have the same signs and } c \text{ has the opposite sign.}$$

Activity 1.1 (Page 12)

$\cosh^{-1} 2 = \ln(2 + \sqrt{3})$. This is one of the roots of the equation $\cosh x = 2$, but there is a second root which is the negative of this.

Exercise 1B (Page 13)

$$1 \quad \text{(i)} \quad \ln(3 + 2\sqrt{2})$$

$$\text{(ii)} \quad \ln(1 + \sqrt{2})$$

$$\text{(iii)} \quad \frac{1}{2} \ln 3$$

$$\text{(iv)} \quad \ln(-2 + \sqrt{5})$$

$$\text{(v)} \quad \ln 2$$

$$\text{(vi)} \quad -\frac{1}{2} \ln 5$$

$$2 \quad \text{(i)} \quad x = \ln(3 + \sqrt{10})$$

$$\text{(ii)} \quad x = \pm \ln(3 + 2\sqrt{2})$$

$$\text{(iii)} \quad x = \frac{1}{2} \ln \frac{3}{2}$$

$$3 \quad \text{(i)} \quad \ln(2x + \sqrt{4x^2 - 1})$$

$$\text{(ii)} \quad \ln((x + 3) + \sqrt{x^2 + 6x + 10})$$

$$\text{(iii)} \quad \frac{1}{2} \ln\left(\frac{2+x}{2-x}\right)$$

$$4 \quad \text{(i)} \quad x = \ln 2$$

$$\text{(ii)} \quad x = 0, x = \ln 5$$

$$\text{(iii)} \quad x = \pm \ln(2 + \sqrt{3}), \pm \ln\left(\frac{3 + \sqrt{5}}{2}\right)$$

$$\text{(iv)} \quad x = 0, \pm \ln(5 + \sqrt{24})$$

$$\text{(v)} \quad x = 0, \pm \ln(5 + 2\sqrt{6})$$

Chapter 2

Exercise 2A (Page 20)

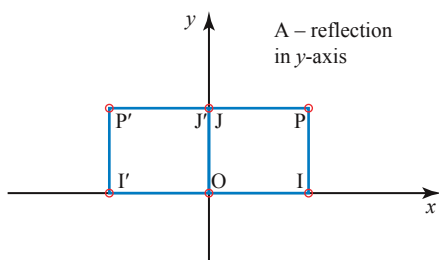
$$1 \quad x = -7, \frac{3}{2}$$

$$2 \quad \text{(i)} \quad \mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

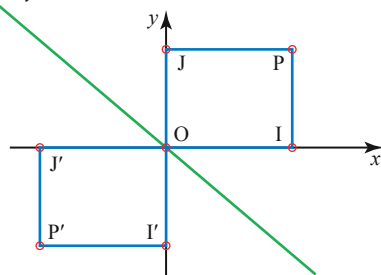
$$\text{(ii)} \quad \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \times 1 - 0 \times 0 = -1$$

$$\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = 0 \times 0 - (-1) \times (-1) = -1$$

(iii)



B – reflection in the line $y = -x$



3 (i) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

(ii) determinant = 1 so area is preserved.

4 (i) (11, -3)

(ii) $\frac{1}{20} \begin{pmatrix} 5 & 5 \\ -1 & 3 \end{pmatrix}$

(iii) (1, 0)

5 $k = 1, 6$

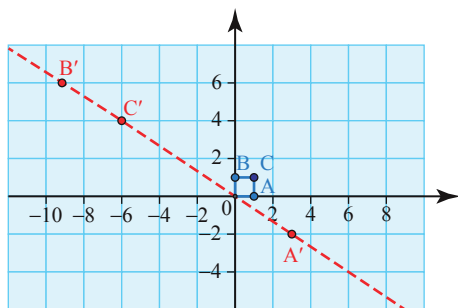
6 (i) $\det(\mathbf{M}) = -9, \det(\mathbf{N}) = -67$

(ii) $\mathbf{MN} = \begin{pmatrix} 19 & -17 \\ 50 & -13 \end{pmatrix}$

$\det(\mathbf{MN}) = 603$ and $603 = -9 \times -67$

7 $\begin{pmatrix} 3 & -1 & 3 & 6 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

8 (i)



(ii) The image of all points lie on the line $y = -\frac{2}{3}x$. The determinant of the matrix is zero which shows that the image will have zero area.

9 (i) $x = 2, y = -1$

(ii) $x = -3, y = 4$

10 $k = \pm 4$; $k = 4$ gives the same line so an infinite number of solutions.

$k = -4$ gives parallel lines so there are no solutions.

11 (i) $\mathbf{M}^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$
 $= \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix}$
 $= (a+d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+d)\mathbf{M}$

(ii) $\mathbf{M}^n = (a+d)^{n-1}\mathbf{M}$

(iii) True for $n = 1$: $\mathbf{M}^1 = (a+d)^0 \mathbf{M}$

Assume true for $= k$: $\mathbf{M}^k = (a+d)^{k-1} \mathbf{M}$

$\mathbf{M}^{k+1} = \mathbf{M}(\mathbf{M}^k) = \mathbf{M} \times (a+d)^{k-1} \mathbf{M}$

$= (a+d)^{k-1} \mathbf{M}^2 = (a+d)^k \mathbf{M}$

$= (a+d)^{(k+1)-1} \mathbf{M}$

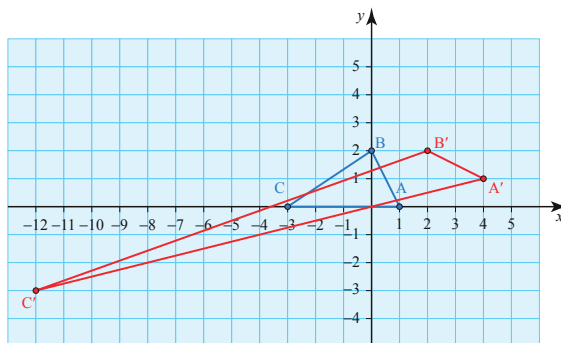
Thus also true for $n = k + 1$, and since it is true for $n = 1$, it must be true for all n .

12 $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \Rightarrow \begin{matrix} x' = ax + by \\ y' = cx + dy \end{matrix}$

Solving simultaneously and using the fact $ad - bc = 0$ gives the result.

13 (i) (4, 1), (2, 2) and (-12, -3)



- (ii) Area of $T = 4$; Area of $T' = 12$
Ratio 12:4 or 3:1. 3 is the determinant of $\mathbf{M}$.

(iii) $\mathbf{M}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$

14 (i) (a) $\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) $\mathbf{S}^{-1} = \frac{1}{2} \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$

(c) $\frac{1}{2} \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(ii) $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \mathbf{T}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{T}^{-1} \mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{I} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

so (x, y) is invariant under $\mathbf{T}^{-1}$.

15 (i) $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 3 & k \end{pmatrix}$

(ii) $k = -\frac{3}{2}$; $x = 2$, $y = 3$

(iii) There are no unique solutions.

(iv) (a) Lines intersect at a unique point.

(b) Lines are parallel.

(c) Lines are coincident.

16 (i) $\mathbf{M}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$, $\mathbf{N}^{-1} = \frac{1}{7} \begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}$

(ii) $\mathbf{MN} = \begin{pmatrix} 5 & -1 \\ 1 & 4 \end{pmatrix}$

$(\mathbf{MN})^{-1} = \frac{1}{21} \begin{pmatrix} 4 & 1 \\ -1 & 5 \end{pmatrix} = \mathbf{N}^{-1} \mathbf{M}^{-1}$

(iii) $(\mathbf{PQ})^{-1} \mathbf{PQ} = \mathbf{I}$

$(\mathbf{PQ})^{-1} \mathbf{PQQ}^{-1} = \mathbf{IQ}^{-1}$

$(\mathbf{PQ})^{-1} \mathbf{PI} = \mathbf{Q}^{-1}$

$(\mathbf{PQ})^{-1} \mathbf{PP}^{-1} = \mathbf{Q}^{-1} \mathbf{P}^{-1}$

$(\mathbf{PQ})^{-1} \mathbf{I} = \mathbf{Q}^{-1} \mathbf{P}^{-1}$

$(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1} \mathbf{P}^{-1}$

Exercise 2B (Page 27)

- 1 (i) (a) 5 (b) 5
(ii) (a) -5 (b) -5

Interchanging the rows and columns has not changed the determinant.

- (iii) (a) 0 (b) 0

If a matrix has a repeated row or column the determinant will be zero.

2 (i) $\frac{1}{3} \begin{pmatrix} 3 & 0 & -6 \\ -4 & 2 & 3 \\ 2 & -1 & 0 \end{pmatrix}$

(ii) Matrix is singular.

(iii) $\begin{pmatrix} -0.06 & -0.1 & -0.1 \\ 0.92 & 0.2 & 0.7 \\ 0.66 & 0.1 & 0.6 \end{pmatrix}$

(iv) $\frac{1}{21} \begin{pmatrix} 34 & 11 & 32 \\ 9 & 6 & 6 \\ -38 & -16 & -37 \end{pmatrix}$

3 $\frac{1}{7} \begin{pmatrix} 2 & 18 & -11 \\ 2 & 39 & -25 \\ 3 & 41 & -27 \end{pmatrix}$; $x = 8$, $y = 4$, $z = -3$

4 $\mathbf{M}^{-1} = \frac{1}{28-10k} \begin{pmatrix} 4 & -10 & 12 \\ -(4k+8) & 8 & -(4+2k) \\ -k & 7 & -3k \end{pmatrix}$;

$k = 2.8$

- 5 (i) The columns of the matrix have been moved one place to the right, with the final column moving to replace the first. This is called **cyclical interchange** of the columns.

(ii) $\det(\mathbf{A}) = \det(\mathbf{B}) = \det(\mathbf{C}) = -26$

Cyclical interchange of the columns leaves the determinant unchanged.

6 $x = \frac{-1 \pm \sqrt{41}}{2}$

7 $x = 1$, $x = 4$

8 $1 < k < 5$

- 9 (i) Let $\mathbf{X} = (\mathbf{PQ})^{-1}$ so $\mathbf{X}(\mathbf{PQ}) = \mathbf{I}$.
 $\Rightarrow \mathbf{X}(\mathbf{PQ})\mathbf{Q}^{-1} = \mathbf{IQ}^{-1} = \mathbf{Q}^{-1}$
 $\Rightarrow \mathbf{XP}(\mathbf{QQ}^{-1}) = \mathbf{XP} = \mathbf{Q}^{-1}$
 $\Rightarrow \mathbf{XPP}^{-1} = \mathbf{Q}^{-1} \mathbf{P}^{-1}$
 $\Rightarrow \mathbf{X} = \mathbf{Q}^{-1} \mathbf{P}^{-1}$

$$(ii) \quad \mathbf{P}^{-1} = \begin{pmatrix} -\frac{1}{9} & \frac{1}{6} & -\frac{4}{9} \\ \frac{2}{9} & \frac{1}{6} & -\frac{1}{9} \\ -\frac{1}{3} & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$$

$$\mathbf{Q}^{-1} = \begin{pmatrix} \frac{3}{2} & -4 & \frac{1}{2} \\ 1 & -2 & 0 \\ -\frac{3}{2} & 5 & -\frac{1}{2} \end{pmatrix}$$

$$(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} = \begin{pmatrix} -\frac{11}{9} & -\frac{1}{6} & -\frac{7}{18} \\ -\frac{5}{9} & -\frac{1}{6} & -\frac{2}{9} \\ \frac{13}{9} & \frac{1}{3} & \frac{5}{18} \end{pmatrix}$$

- 10** (ii) Multiplying only the first row by k equates to a stretch of scale factor k in one direction, so only multiplies the volume by k .
 (iii) Multiplying any row by k multiplies the determinant by k .
- 11** (i) $10 \times 43 = 430$
 (ii) $4 \times 5 \times -7 \times 43 = -6020$
 (iii) $x \times 2 \times y \times 43 = 86xy$
 (iv) $x^4 \times \frac{1}{2x} \times 4y \times 43 = 86x^3y$

Exercise 2C (Page 35)

- 1** (i) $\det \mathbf{M} = 20$, $\mathbf{M}^{-1} = \begin{pmatrix} 0.2 & 0.4 & -0.6 \\ -0.25 & 0.25 & -0.25 \\ -0.15 & -0.05 & 0.45 \end{pmatrix}$
- 2** (i) Planes π_1 and π_3 are parallel and the second is not parallel to either, so will cross through both to form two parallel straight lines.
 (ii) Planes π_1 and π_3 are parallel and π_2 is coincident to π_1 .
 (iii) All three planes are parallel.
- 3** (i) $\begin{pmatrix} 5 & 3 & -2 \\ 6 & 2 & 3 \\ 7 & 1 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ 12 \end{pmatrix}$

- (ii) Eliminating y gives the equations $-8x - 13z = -13$ and $-8x - 13z = -15$ which are inconsistent so the planes form a prism.

- 4** (i) Planes meet at the unique point $(3, -14, 8)$
 (ii) Inconsistent, the planes form a prism.
 (iii) Consistent, the planes form a sheaf.
 (iv) Planes meet at the unique point $(-15, 24, -1)$
 (v) Three coincident planes.
- 5** (i) The planes intersect in the unique point $(-0.8, 0.6, 1.5)$
 (ii) Inconsistent, the planes form a prism.
 (iii) Consistent, the planes form a sheaf.

- 6** (i) $\frac{1}{13k-65} \begin{pmatrix} 13 & -26 & -13 \\ 7 & -2k-4 & -3k+8 \\ -4 & 3k-7 & -2k+14 \end{pmatrix};$
 $k = 5$
 (ii) Unique point of intersection at:
 $\left(\frac{52-13p}{13}, \frac{20-7p}{13}, \frac{4p-17}{13} \right)$

- (iii) Form a sheaf of planes when $p = 4$

- 7** Any point on the line $(\lambda, 2\lambda - 4, 9 - 5\lambda)$

- 8** $\frac{1}{3}, 3;$
 $k = \frac{1}{3}$: no solution, the planes form a triangular prism.
 $k = 3$: any point on the line $(4.5 - 3.5\lambda, \lambda, 0.5\lambda - 1.5)$, the planes form a sheaf.

- 9** $a = -2, \frac{1}{2}$

- 10** $\det \begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & -2 \\ 3 & -3 & -4 \end{pmatrix} = 0$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

- 11** $a = -1$

- (i) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \\ 1.8 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

- (ii) The planes form a prism.

Activity 2.1 (Page 39)

- (i) (a) $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 10 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$

(e) $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ (f) $\begin{pmatrix} -6 \\ 1 \end{pmatrix}$

(ii) (a) $y = \frac{1}{4}x$ (b) $y = \frac{1}{2}x$

(c) $y = \frac{3}{2}x$ (d) $x = 0$

(e) $y = -x$ (f) $y = -\frac{1}{6}x$

Exercise 2D (Page 45)

1 (i) $2, k\begin{pmatrix} 1 \\ -1 \end{pmatrix}; 7, k\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ where $k \neq 0$; $2+7=5+4$

(ii) $-1, k\begin{pmatrix} 1 \\ -4 \end{pmatrix}; 4, k\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ where $k \neq 0$;
 $-1+4=7+(-4)$

(iii) $1-\sqrt{2}, k\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}; 1+\sqrt{2}, k\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$
 where $k \neq 0$; $1-\sqrt{2}+1+\sqrt{2}=1+1$.

(iv) 2 (repeated), $k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $k \neq 0$;
 $2+2=1+3$.

(v) $0.3, k\begin{pmatrix} 1 \\ 2 \end{pmatrix}; 1, k\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ where
 $k \neq 0$; $0.3+1=1.1+0.2$.

(vi) $p, k\begin{pmatrix} 1 \\ 0 \end{pmatrix}; q, k\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $k \neq 0$;
 $p+q=p+q$.

2 (i) $3, c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; 2, c\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; -1, c\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$ where $c \neq 0$;

$\sum_i \lambda_i = 3+2+(-1)=3+2+(-1)$

(ii) $-1, c\begin{pmatrix} 9 \\ -14 \\ -2 \end{pmatrix}; 3, c\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}; 4, c\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

where $c \neq 0$;

$\sum_i \lambda_i = -1+3+4=1+2+3$

(iii) $-3, c\begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix}; 0, c\begin{pmatrix} 5 \\ 9 \\ -7 \end{pmatrix}; 4, c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

where $c \neq 0$;

$\sum_i \lambda_i = -3+0+4=1+(-2)+2$

(iv) $1, c\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}; 2$ (repeated); $c\begin{pmatrix} p \\ 2p+4q \\ q \end{pmatrix}$

where $c \neq 0$ and p and q are not both zero.

$\sum_i \lambda_i = 1+2+2=0+7+(-2)$

(v) $-2, c\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; 2, c\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; 3, c\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

where $c \neq 0$.

$\sum_i \lambda_i = -2+2+3=0+3+0$

(vi) $1, c\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}; 4, c\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}; 9, c\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

where $c \neq 0$.

$\sum_i \lambda_i = 1+4+9=1+6+7$

3 $-1, c\begin{pmatrix} \tan \theta \\ -1 \end{pmatrix}; 1, c\begin{pmatrix} 1 \\ \tan \theta \end{pmatrix}$ where $c \neq 0$.

4 (i) $(\mathbf{A} + \mathbf{B})\mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{B}\mathbf{s} = \alpha\mathbf{s} + \beta\mathbf{s} = (\alpha + \beta)\mathbf{s}$;
 eigenvalue : $(\alpha + \beta)$

(ii) $\mathbf{A}\mathbf{B}\mathbf{s} = \mathbf{A}(\beta\mathbf{s}) = \beta\mathbf{A}\mathbf{s} = \beta\alpha\mathbf{s}$; eigenvalue: $\beta\alpha$.

6 (i) 6

(ii) $2, k\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; 3, k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

8 (i) (a) 2, 3 (b) 4, 9

(c) 32, 243 (d) $\frac{1}{2}, \frac{1}{3}$

(ii) (a) 1, 2, 3 (b) 1, 4, 9

(c) 1, 32, 243 (d) $1, \frac{1}{2}, \frac{1}{3}$.

9 $5, k\begin{pmatrix} -2 \\ 1 \end{pmatrix}; 1, k\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ and $k \neq 0$

$s = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

- 10 (i) $\mathbf{M}^n \mathbf{v}$ converges to 0
 (ii) if $\lambda_1 = 1$ $\mathbf{M}^n \mathbf{v}$ converges to $\mathbf{s}_1$; if $\lambda_1 = -1$, $\mathbf{M}^n \mathbf{v}$ eventually alternates between $\pm \mathbf{s}_1$
 (iii) The magnitude of $\mathbf{M}^n \mathbf{v}$ increases without limit; the direction of $\mathbf{M}^n \mathbf{v}$ becomes parallel to $\mathbf{s}_1$.

- 11 (i) $\mathbf{M} = \begin{pmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{pmatrix}$
 (ii) $\mathbf{M}^2 \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 76 \\ 124 \end{pmatrix}$, so 76 at Calgary and 124 at Vancouver.

(iii) $\mathbf{x} = \begin{pmatrix} 75 \\ 125 \end{pmatrix}$; $\mathbf{M}\mathbf{x} = \lambda\mathbf{x}$ with $\lambda = 1$.

(iv) 75 at Calgary and 125 at Vancouver.

- 12 (iv) 2; $c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; 4; $c \begin{pmatrix} 3 \\ 1 \end{pmatrix}$;

$$\mathbf{p} = -425e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 475e^{4t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

❓ (Page 52)

No. The suggested substitution is invalid since 'lambda' is a number and $\mathbf{M}$ is a matrix.

Exercise 2E (Page 53)

1 (i) $\mathbf{Q} = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 9 \end{pmatrix}$

(ii) $\mathbf{Q} = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(iii) $\mathbf{Q} = \begin{pmatrix} 5 & 1 \\ -3 & 1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 0.2 & 0 \\ 0 & 1 \end{pmatrix}$

2 $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.9 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$;
 $\begin{pmatrix} 4.0951 & -5.1585 \\ 2.0634 & -2.439 \end{pmatrix}$; tends to $\begin{pmatrix} 10 & -15 \\ 6 & -9 \end{pmatrix}$.

3 (i) $\begin{pmatrix} 876 & -1266 \\ 422 & -601 \end{pmatrix}$

(ii) $\begin{pmatrix} 524 & 800 & -523 & 576 \\ -523 & 576 & 524 & 800 \end{pmatrix}$

(iii) $\begin{pmatrix} 0.6667 & 0.3333 \\ 0.6666 & 0.3334 \end{pmatrix}$

4 (i) $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

5 $\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = \begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix} - 5 \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} + 6 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6 $\mathbf{M}^2 - (a+d)\mathbf{M} + (ad-bc)\mathbf{I}$
 $= \begin{pmatrix} a^2+bc & ac+cd \\ ab+bd & bc+d^2 \end{pmatrix} - (a+d) \begin{pmatrix} a & c \\ b & d \end{pmatrix} + (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

7 $-3, t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; 2, t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}; 5, t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\mathbf{P} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}; \mathbf{D} = \begin{pmatrix} -243 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 3125 \end{pmatrix}$

8 $\mathbf{A}\mathbf{B}\mathbf{e} = \mathbf{A}(\mu\mathbf{e}) = \mu\mathbf{A}\mathbf{e} = \mu\lambda\mathbf{e}$

$-1, t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; 1, t \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}; 2, t \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 1 \\ -6 & -3 & 4 \\ -9 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ is an eigenvector of $\mathbf{CD}$ with eigenvalue -2 .

9 $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
 $\lambda = -2$

$$\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix};$$

$$\mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & 2 \end{pmatrix}$$

10 $(\mathbf{A} + \mathbf{B})\mathbf{e} = (\lambda + \mu)\mathbf{e}$

Eigenvalues of $\mathbf{A}$: $-1, 1, 3$, eigenvalues of $\mathbf{M}$: $-8, -2, 2$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} -32768 & 0 & 0 \\ 0 & -32 & 0 \\ 0 & 0 & 32 \end{pmatrix}$$

$$\mathbf{S} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 2 \\ -1 & -1 & 2 \\ 2 & 0 & -2 \end{pmatrix}$$

11 (i) $\mathbf{A}\mathbf{e} \neq 0 \Rightarrow \lambda\mathbf{e} \neq 0$

(ii) $\mathbf{A}^{-1}\mathbf{e} = \lambda^{-1}\mathbf{e}$

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 25 \\ 0 & 0 & 20 \end{pmatrix}; \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$$

12 Eigenvalue = 5

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}^{-1}$$

13 (i) $-2, 3$

(ii) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

(iii) $p = 2, q = 5, r = -6$

Chapter 3

❓ (Page 61)

A polynomial of degree n has at most $n-1$ stationary points, whereas $\sin x$ is periodic and has infinitely many.

Exercise 3A (Page 63)

1 $\sin^{-1}$ has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^{-1}$ has domain $[-1, 1]$ and range $[0, \pi]$

$\tan^{-1}$ has domain $\mathbb{R}$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

2 (i) $\frac{3}{\sqrt{1-9x^2}}$

(ii) $-\frac{1}{\sqrt{4-x^2}}$

(iii) $\frac{5}{25x^2+1}$

(iv) $\frac{6x}{\sqrt{1-9x^4}}$

(v) $\frac{e^x}{e^{2x}+1}$

(vi) $-\frac{6x}{x^4-2x^2+2}$

3 (i) $4 \cosh 4x$

(ii) $2x \sinh x^2$

(iii) $2 \cosh x \sinh x$

(iv) $\cosh^2 x + \sinh^2 x$

4 $\left(-\frac{1}{2} \ln 3, -\frac{1}{3\sqrt{3}}\right)$

5 (i) $\frac{3}{\sqrt{1+9x^2}}$

(ii) $\frac{4}{\sqrt{16x^2-1}}$

(iii) $\frac{4x}{\sqrt{1+4x^4}}$

(iv) $\frac{1}{\sqrt{x^2+x}}$

6 (i) $\frac{1+x(x^2-1)^{-\frac{1}{2}}}{x+\sqrt{x^2-1}}$

8 (ii) $\tan h^{-1}x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$

$$10 \quad \sin^{-1}(x^2) + \frac{2x^2}{\sqrt{1-x^4}}$$

Activity 3.3 (Page 66)

Gradients tend to zero and to infinity on the two branches.

Activity 3.4 (Page 67)

$$\frac{dy}{dx} = -e^{-x} \sin x - e^{-x} \cos x, \quad \frac{d^2y}{dx^2} = 2e^{-x} \sin x$$

Activity 3.5 (Page 68)

$$y^2 = 4ax$$

$$\frac{dy}{dx} = \frac{2a}{y}, \quad \frac{1}{t} \frac{d^2y}{dx^2} = \frac{4a^2}{y^3} = \frac{1}{2at^3}$$

Exercise 3B (Page 68)

$$1 \quad (i) \quad \frac{-3x^2 - 3y^2 \left(\frac{dy}{dx} \right)^2}{y^3}$$

$$(ii) \quad \frac{4 \frac{dy}{dx} - 1 - \left(\frac{dy}{dx} \right)^2}{y - 2x}$$

$$(iii) \quad \frac{8 \frac{dy}{dx} - 6x - 6y \left(\frac{dy}{dx} \right)^2}{(3y^2 - 4x)}$$

$$(iv) \quad -\frac{2}{x} \left(\frac{dy}{dx} \right) - \left(\frac{dy}{dx} \right)^2$$

$$(v) \quad \left(\frac{\sin y + \cos y}{\cos y - \sin y} \right) \left(\frac{dy}{dx} \right)^2$$

$$2 \quad (i) \quad \frac{-2}{9} \operatorname{cosec}^3 \theta$$

$$(ii) \quad \frac{1}{8t^3}$$

$$(iii) \quad -\frac{1}{16} \operatorname{cosec}^3 \theta$$

$$(iv) \quad \frac{3 - t^2}{4t^5}$$

$$(v) \quad -e^{-2t} (\cos t + \sin t)$$

$$3 \quad (i) \quad (-1, -1) \text{ as } \frac{dy}{dx} \text{ is not defined at } (0, 0).$$

$$(ii) \quad \frac{d^2y}{dx^2} = 2 \text{ so minimum point.}$$

$$4 \quad (i) \quad \frac{dy}{dx} = \coth t, \quad \frac{d^2y}{dx^2} = -\operatorname{cosech}^3 t$$

$$(ii) \quad \coth t \neq 0 \text{ for all values of } t.$$

$$(iii) \quad x^2 - y^2 = 1$$

$$5 \quad (i) \quad \text{For } \pi < x < 2\pi, \cos 2y > 1$$

$$(ii) \quad \frac{dy}{dx} = \frac{\cos x}{2 \sin 2y},$$

$$\frac{d^2y}{dx^2} = -\frac{\sin x + 4 \cos 2y \left(\frac{dy}{dx} \right)^2}{2 \sin 2y}$$

$$(iii) \quad \left(\frac{\pi}{2}, \pm \frac{\pi}{4} \right)$$

$$(iv) \quad \left(\frac{\pi}{2}, \frac{\pi}{4} \right) \frac{d^2y}{dx^2} = -\frac{1}{2} \text{ so maximum.}$$

$$\left(\frac{\pi}{2}, -\frac{\pi}{4} \right) \frac{d^2y}{dx^2} = -\frac{1}{2} \text{ so minimum.}$$

$$6 \quad \frac{1}{16t^3(2-t)^{3/2}}(4-3t)$$

$$8 \quad (ii) \quad \frac{d^2y}{dx^2} = \frac{1}{2} \sec^2 \frac{1}{2} \theta \frac{1}{1 + \cos \theta} = \frac{1}{4} \sec^4 \frac{1}{2} \theta$$

$$10 \quad y = x^2, \left(2^{\frac{1}{3}}, 2^{\frac{2}{3}} \right) \text{ maximum.}$$

Activity 3.6 (Page 71)

$$(ii) \quad p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

Activity 3.7 (Page 71)

$$\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \right) \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \right)$$

$$= 1 - \frac{1}{12}x^4 - \frac{1}{36}x^6$$

If more terms were used, the terms in x^4 and x^6 and higher terms would cancel out, leaving 1.

? (Page 72)

It is often the case that if the terms of a series converge, the sum of the series also converges. However, it is not always the case: for example, the terms of the series $u_n = \frac{1}{n}$ converge to zero but the sum of the terms does not converge.

Activity 3.8 (Page 72)

Order = 7

Maclaurin approximation up to term in x^{10} is 7.388994709, percentage error = -0.00083%.

Activity 3.9 (Page 74)

Using the Binomial expansion

$$\begin{aligned}(1-x)^{-1} &= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 \\ &\quad + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \dots \\ &= 1 + x + x^2 + x^3 + \dots\end{aligned}$$

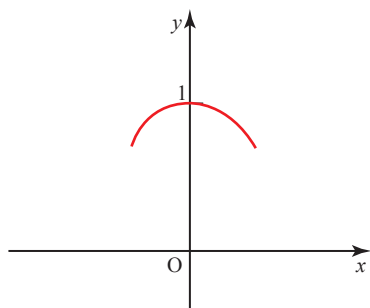
Sum to infinity of the sequence $1 + x + x^2 + x^3 + \dots$ with $a = 1, r = x$

$$S = \frac{a}{1-r} = \frac{1}{1-x} = (1-x)^{-1}$$

Notice Maclaurin expansion, Binomial expansion and the geometric sequence give the same result.

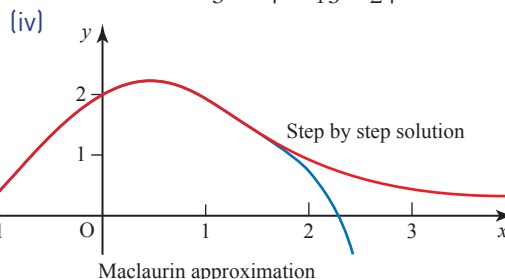
Exercise 3C (Page 77)

- 1 (ii) 0.995004 (6 sf)
(iii) $1.4 \times 10^{-7}\%$
- 2 (i) 0.60653, 7 terms
(ii) $2.40 \times 10^{-4}\%$
- 3 (i) $\sin x = x - \frac{x^3}{3!} + \dots$
(ii) $x^3 + 6x^2 - 6x = 0$
- 4 (i) $\ln x$ is not defined at $x = 0$
- 5 $f'(0) = 0, f''(0) = -3, f^{(3)}(0) = 15$



- 6 $\tan x = x + \frac{1}{3}x^3 + \dots$
- 7 (iii) 0.242%
- 8 (i) $a_0 = 2$

$$(iii) \quad 2 + x - x^2 - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} - \frac{x^6}{24}$$



$$9 \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$10 \quad (i) \quad (a) \quad 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$(b) \quad \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$(c) \quad 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$$(ii) \quad 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$$11 \quad (ii) \quad a_1 = 1, a_2 = 0$$

$$(iv) \quad x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \dots$$

$$n^{\text{th}} \text{ entry} = \begin{cases} \frac{[(n-2)(n-4)(n-6)\dots(1)]^2}{n!} x^n, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Activity 3.10 (Page 79)

- (i) In each case, you get the series for the derivative of the original function.
- (ii) $\sin x$ has only odd powers of x . The graph of $y = \sin x$ has rotational symmetry about the origin, as do all graphs of the form $y = x^n$ for odd n .
 $\cos x$ has only even powers of x . The graph of $y = \cos x$ is symmetrical about the y -axis, as are all graphs of the form $y = x^n$ for even n .

Activity 3.11 (Page 80)

- (i) because $\ln(1+0) = 0$
 (ii) terms need to be discarded as they are incomplete – other parts arise from higher powers in the series themselves.
 (iv) $1 + y$ is the Maclaurin expansion for $\cos x$

Exercise 3D (Page 80)

- 1** (i) $1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$
 (ii) $1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \dots$
- 2** (i) $3x - \frac{9}{2}x^3$
 (ii) $2x - 2x^2 + \frac{8}{3}x^3 - 4x^4$
 (iii) $1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4$
- 3** (i) $x^2 - \frac{1}{3}x^4$
 (ii) $x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4$
 (iii) $x - x^2 + \frac{1}{3}x^3$
 (iv) $1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4$
- 4** (i) $\frac{1}{2}\sin^{-1}2x + c$
 (ii) $2x + \frac{4}{3}x^3 + \frac{12}{5}x^5 + \frac{40}{7}x^7 + \dots$
 (iii) Using $x = 0.25$ gives $0.5235258557\dots$, percentage error = -0.0139%
- 5** (i) 0.6456 (4 dp)
 (ii) 0.6911 (4 dp)
 (iii) $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$, $x = \frac{1}{3}$ gives $\ln 2 = 0.6930$
- 6** (i) $1 + x + x^2 + x^3 + \dots$
 (ii) $\frac{1}{(1-x)^2}$
- 7** (i) $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6$
 (ii) $x + x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^5 - \frac{1}{36}x^6$, agrees with (i) as far as the term in x^4
 (iii) $1 + x - \frac{1}{3}x^3$; the x^4 term of the product is not correct.

- 8** (ii) $f^{(3)}(x) = 4f''(x) - 13f'(x)$
 $f^{(4)}(x) = 4f^{(3)}(x) - 13f''(x)$
 (iii) $f(x) = 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + \dots$

- 10** Given $0 < r < 1$, the area of the square

$$= \left(\frac{1}{1-r}\right)^2 = 1 + 2r + 3r^2 + 4r^3 + 5r^4 + \dots$$

Also the side of the square

$$= (1-r)(1 + 2r + 3r^2 + 4r^3 + 5r^4 + \dots)$$

$$= (1-r)\left(\frac{1}{1-r}\right)^2 = \frac{1}{1-r}$$

Chapter 4

? (Page 85)

Estimate the area of the photograph and the number of birds per unit area.

Exercise 4A (Page 88)

- 1** $\sin^{-1}\frac{x}{5} + c$
2 0.615 (3 s.f.)
3 $\sin^{-1}\frac{x}{2} + c$
4 (i) $\frac{\pi}{4}$
 (ii) $\frac{\pi}{12}$
5 (i) $\sin^{-1}\left(\frac{x-3}{4}\right) + c$
 (ii) $\frac{1}{3}\sin^{-1}\left(\frac{3x-1}{\sqrt{3}}\right) + c$
 (iii) $\sin^{-1}\left(\frac{x+1}{2}\right) + c$
 (iv) $\frac{1}{\sqrt{2}}\sin^{-1}(4x-1) + c$
6 (i) $\frac{1}{2}\left(x\sqrt{9-x^2} + 9\sin^{-1}\left(\frac{x}{3}\right)\right) + c$
 (ii) $\frac{1}{2}x\sqrt{4-9x^2} + \frac{2}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$
 (iii) $\frac{x}{\sqrt{1-x^2}} + c$

Exercise 4B (Page 91)

- 1** (i) $\cosh^{-1}\frac{x}{2} + c$
 (ii) $\sinh^{-1}\frac{x}{2} + c$

2 (i) $\ln\left(\frac{3(2+\sqrt{3})}{4+\sqrt{7}}\right)$

(ii) $\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$

3 (i) $\frac{1}{3}\cosh^{-1} 3x + c$

(ii) $\frac{1}{3}\sinh^{-1} 3x + c$

(iii) $\frac{1}{2}\sinh^{-1} \frac{2x}{3} + c$

(iv) $\frac{1}{2}\cosh^{-1} \frac{2x}{3} + c$

4 (i) $\frac{1}{5}\ln\left(\frac{1}{4}(5+\sqrt{21})\right)$

(ii) $\frac{1}{3}\ln(\sqrt{10}+3)$

5 (ii) $\frac{1}{2}x\sqrt{x^2+4} + 2\sinh^{-1} \frac{x}{2} + c$

6 (ii) $\frac{1}{2}x\sqrt{x^2-9} + \frac{9}{2}\cosh^{-1} \frac{x}{3} + c$

9 (i) $x\sinh^{-1} x - \sqrt{x^2+1}$

(ii) $x\tanh^{-1} 2x + \frac{1}{4}\ln(1-4x^2)$

10 (i) $4\left(x + \frac{3}{2}\right)^2 - 49$

(ii) 0.322

11 Constant of proportionality $\frac{1}{2}a^2$.

Exercise 4C (Page 96)

1 $\frac{3\pi}{16}, \frac{8}{15}$

2 $6e^{-1} - 2$

3 $\frac{1}{2}\pi^3 - 12\pi + 24$

4 $\frac{1}{4} + \frac{3}{32}\pi$

Activity 4.3 (Page 99)

0.8099815 and 0.7599815 for 10 rectangles and 0.797794 and 0.772794 for 20 rectangles. For 20 rectangles, both round to 0.8, so this would be an answer accurate to 1 significant figure. 0.8 is 1.9% error and mid-value between bounds give 0.78 to 2 significant figures and 0.7% error.

Exercise 4D (Page 100)

1 (i) 0.5165 and 0.5320

(ii) $\frac{\pi}{6}$

(iii) $3.099 < \pi < 3.192$

2 0.5336 and 0.7171

3 (i) 6.1404

(ii) (a) 6.1056

(b) 6.1344.

(iii) The turning point in the interval means that it is not clear that either is definitely an underestimate and the other an overestimate.

5 0.4282

6 (i) $\log 2 + \log 3 + \dots + \log 10$

$= \log(2 \times 3 \times \dots \times 10)$

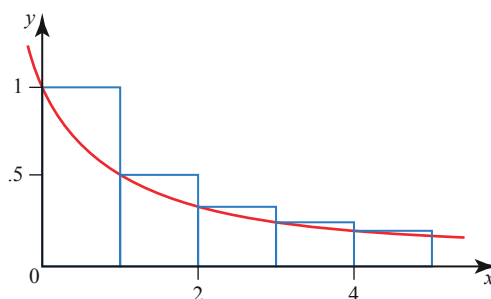
$= \log 10!$

(ii) $\frac{1}{\ln 10}(10 \ln 10 - 9)$

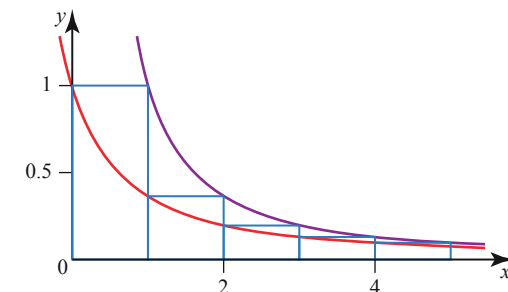
(iii) $10! > 1234098.041$

7 $\sum_{r=1}^n \frac{1}{r} > \int_0^n \frac{1}{x+1} dx = [\ln(x+1)]_0^n = \ln(n+1)$

As $n \rightarrow \infty$, $\ln(n+1) \rightarrow \infty$



8

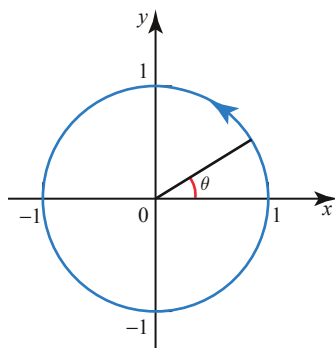


$$\frac{-2}{\sqrt{1+n}} + \frac{2}{\sqrt{2}} < \sum_{r=1}^n \frac{1}{r^{3/2}} < 3 - \frac{2}{\sqrt{n}} \text{ so}$$

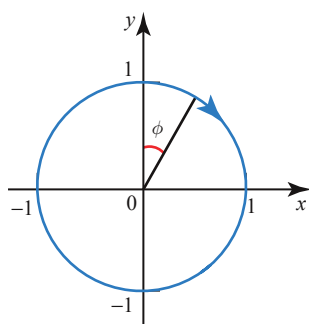
$$\sqrt{2} < \sum_{r=1}^n \frac{1}{r^{3/2}} < 3$$

Activity 4.4 (Page 103)

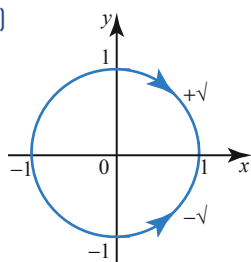
(i)



(ii)



(iii)

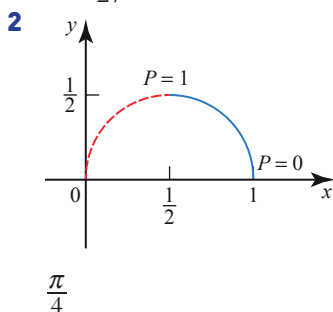


❓ (Page 107)

PQ = δs , QK = $r\delta\theta$, KP = $r\delta\theta$. Use Pythagoras on 'triangle' PQQ.

Exercise 4E (Page 107)

1 $\frac{8(10^{3/2} - 1)}{27} \approx 9.07$



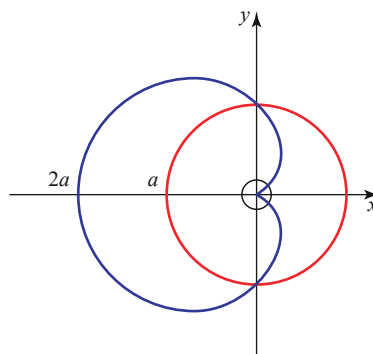
3 $c \sinh\left(\frac{X}{c}\right)$

6 $24a$

7 $\frac{\sqrt{1+k^2}}{k}$

8 $8a$

11



$\left(a, \frac{\pi}{2}\right) \left(a, \frac{3\pi}{2}\right)$

12 $\frac{5}{16}\pi - \frac{1}{2} - \frac{\sqrt{2}}{3}$

Exercise 4F (Page 112)

1 (i) 45π
(ii) $4\pi a^2$
(iii) $2\pi ac + \pi c^2 \sinh\left(\frac{2a}{c}\right)$

(iv) $\frac{8\pi a^2(5^{3/2} - 2^{3/2})}{3}$

(v) $\frac{64\pi a^2}{3}$

(vi) $\frac{12\pi a^2}{5}$

(vii) $(2 - \sqrt{2})\pi a^2$

(viii) $\frac{32\pi a^2}{5}$

2 (i) $2\pi a^2(1 - \cos\alpha)$

(ii) $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$

4 (i) $4\pi a^2$

(ii) $2.03 \times 10^8 \text{ km}^2$

6 (i) $\frac{4}{3}$

(ii) $\frac{11}{9}\pi$

7 24π

8 $s = 2 - \frac{1}{4}\pi \quad S = 4\pi\sqrt{2}$

Chapter 5

? (Page 118)

Similarity – both use r and θ .

Difference – Modulus argument form is used to represent a complex number on the Argand diagram, polar coordinates are used to represent a real number on the coordinate plane.

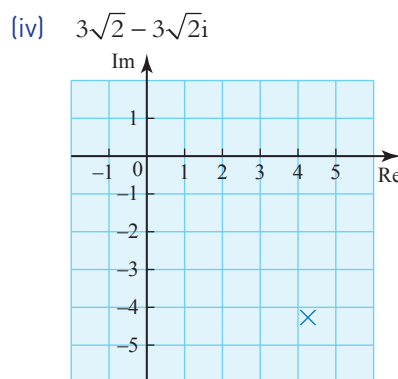
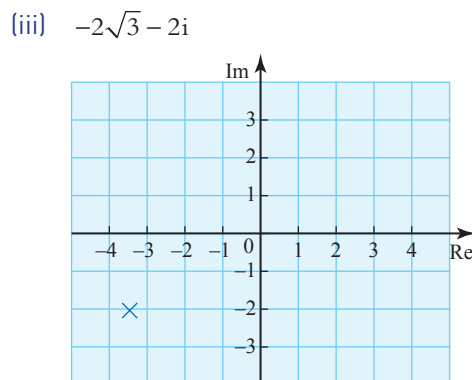
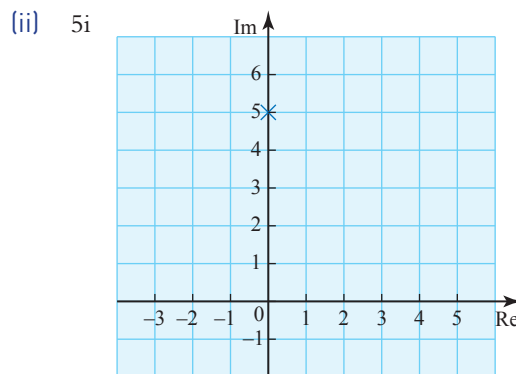
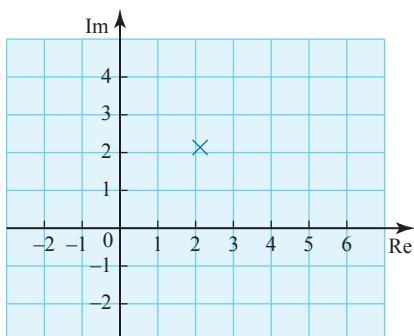
Activity 5.1 (Page 119)

- (i) $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}; -2 = 2(\cos \pi + i \sin \pi)$
- (ii) (a) Anticlockwise rotation of the vector z through $\frac{\pi}{2}$ radians.
- (b) Anticlockwise rotation of the vector z through π radians (which is equivalent to two successive rotations through $\frac{\pi}{2}$ radians) and enlargement by scale factor 2.

Exercise 5A (Page 120)

- 1 (i) $3\sqrt{2} - 3\sqrt{2}i$ (ii) $\frac{3\sqrt{3}}{2} - \frac{3}{2}i$
- (iii) $-\sqrt{2} + \sqrt{2}i$ (iv) $-\frac{7\sqrt{3}}{2} - \frac{7}{2}i$
- 2 (i) $4 \quad \frac{\pi}{6} \quad 4\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$
- (ii) $4 \quad \frac{5\pi}{6} \quad 4\left(\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)\right)$
- (iii) $4 \quad -\frac{\pi}{6} \quad 4\left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right)$
- (iv) $4 \quad -\frac{5\pi}{6} \quad 4\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)\right)$

- 3 (i) $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$



- 4 (i) $15\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$
- (ii) $\frac{3}{5}\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)\right)$
- (iii) $\frac{5}{3}\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$
- (iv) $\frac{1}{5}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$
- 5 (i) Enlargement scale factor $5\sqrt{2}$ and a clockwise rotation through $\frac{\pi}{4}$ radians.
- (ii) Enlargement scale factor $\frac{1}{2}\sqrt{2}$ and a clockwise rotation through $\frac{2\pi}{3}$ radians.
- 6 (i) $1 + \sqrt{3}i$
- (ii) $-\frac{1}{6} - \frac{1}{6}\sqrt{3}i$

Activity 5.2 (Page 121)

$$z^2 = 4(\cos 0.2 + i \sin 0.2)$$

$$z^3 = 8(\cos 0.3 + i \sin 0.3)$$

$$z^4 = 16(\cos 0.4 + i \sin 0.4)$$

$$z^5 = 32(\cos 0.5 + i \sin 0.5)$$

$$z^n = 2^n(\cos 0.1n + i \sin 0.1n)$$

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

? (Page 122)

The lines will continue all the way round. What happens then depends on whether or not $\frac{\theta}{\pi}$ is a rational number. If it is rational, a limited number of lines get repeated over and over again as $n \rightarrow \infty$. If it is not rational the number of lines $\rightarrow \infty$ but no line ever gets repeated.

Activity 5.3 (Page 124)

Using the two results given.

$$\cos(-\phi) + i \sin(-\phi) = \cos \phi - i \sin \phi$$

$$\Rightarrow [\cos(-\phi) + i \sin(-\phi)]^n = [\cos \phi - i \sin \phi]^n$$

$$\Rightarrow \cos(-n\phi) + i \sin(-n\phi) = [\cos \phi - i \sin \phi]^n$$

Using de Moivre's theorem on the left-hand side.

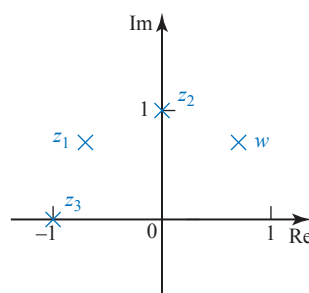
$$\Rightarrow \cos(n\phi) - i \sin(n\phi) = [\cos \phi - i \sin \phi]^n$$

Using the two results given in the question.

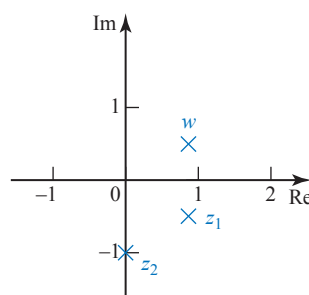
Exercise 5B (Page 124)

- 1 (i) (a) $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$
 (b) $\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right)$
 (c) $\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)$
 (ii) (a) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
 (b) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
 (c) $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$

- 2 (i) $z_1 = w^3$ (ii) $z_2 = w^2$ (iii) $z_3 = w^4$



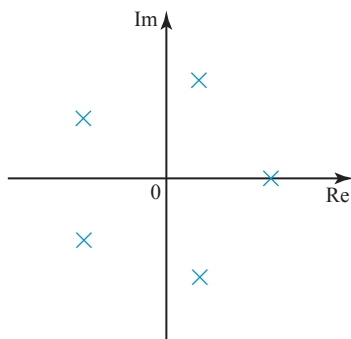
- 3 (i) $\cos(-3\theta) + i \sin(-3\theta)$
 (ii) $\cos\left(\frac{5\pi}{12}\right) + i \sin\left(\frac{5\pi}{12}\right)$
 (iii) -1
- 4 (i) $z_1 = w^{-1}$
 (ii) $z_2 = w^{-3}$



- 5 (i) $-8 + 8\sqrt{3}i$
 (ii) $-1024 - 1024i$
 (iii) -46656

- 6 $64(\sqrt{3} + i)$
- 7 (i) $81(\cos 8\theta + i \sin 8\theta)$
 (ii) $i \cos 15\theta - \sin 15\theta$
 (iii) $\frac{1}{8}(i \cos(-21\theta) - \sin(-21\theta))$
- 8 $k = 41472$
- 9 (i) $z_1 = \cos 0 + i \sin 0$, $z_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$,
 $z_3 = \cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right)$

(iii)



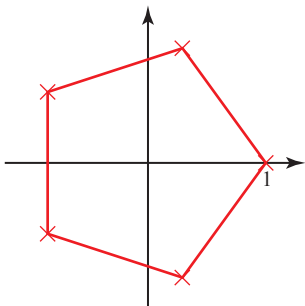
$$\cos 0 + i \sin 0, \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}, \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}, \cos \left(-\frac{2\pi}{5}\right) + i \sin \left(-\frac{2\pi}{5}\right), \cos \left(-\frac{4\pi}{5}\right) + i \sin \left(-\frac{4\pi}{5}\right),$$

Activity 5.4 (Page 128)

$$\begin{aligned} n=2 & \quad 1 + (-1) = 0 \\ n=3 & \quad 1 + \frac{-1 + \sqrt{3}i}{2} + \frac{-1 - \sqrt{3}i}{2} = 0 \\ n=4 & \quad 1 + (-1) + i + (-i) = 0 \end{aligned}$$

Exercise 5C (Page 132)

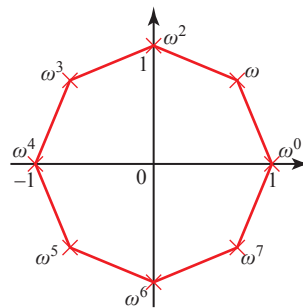
$$\begin{aligned} 1 \quad \omega^0 &= 1, \omega^1 = 0.309 + 0.951i, \\ \omega^2 &= -0.809 + 0.588i, \omega^3 = -0.809 - 0.588i, \\ \omega^4 &= 0.309 - 0.951i \end{aligned}$$



The points form a regular pentagon with one vertex at the point 1.

$$\begin{aligned} 2 \quad \omega^0 &= 1, \omega = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \omega^2 = i, \\ \omega^3 &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \omega^4 = -1, \\ \omega^5 &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \omega^6 = -i, \end{aligned}$$

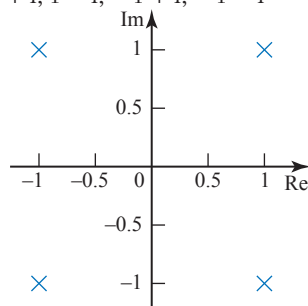
$$\omega^7 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$



The points form a regular octagon with one vertex at the point 1.

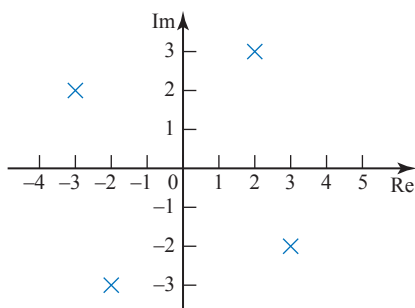
$$3 \quad 0.90 + 2.79i, -0.90 - 2.79i$$

$$4 \quad 1 + i, 1 - i, -1 + i, -1 - i$$



- 5 (i) $2^{\frac{1}{6}} \left(\cos \left(-\frac{\pi}{12} \right) + i \sin \left(-\frac{\pi}{12} \right) \right);$
 $2^{\frac{1}{6}} \left(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12} \right);$
 $2^{\frac{1}{6}} \left(\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right)$
- (ii) $1.38(\cos 0.25 + i \sin 0.25);$
 $1.38(\cos 1.82 + i \sin 1.82);$
 $1.38(\cos(-2.90) + i \sin(-2.90));$
 $1.38(\cos(-1.33) + i \sin(-1.33))$
- (iii) $1.38(\cos 0.44 + i \sin 0.44);$
 $1.38(\cos 1.70 + i \sin 1.70);$
 $1.38(\cos 2.96 + i \sin 2.96);$
 $1.38(\cos(-2.07) + i \sin(-2.07));$
 $1.38(\cos(-0.81) + i \sin(-0.81))$
- 6 The fifth roots give alternate tenth roots and their negatives (given by a half turn about the origin) fill the gaps.

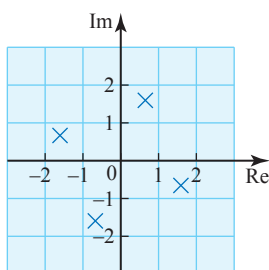
7 $w = -119 - 120i$
 $-3 + 2i, -2 - 3i, 3 - 2i$



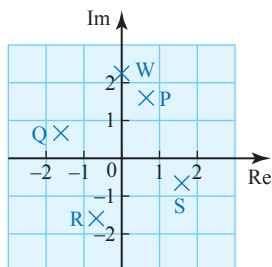
8 $2\left(\cos\left(-\frac{\pi}{18}\right) + i\sin\left(-\frac{\pi}{18}\right)\right);$
 $2\left(\cos\left(-\frac{13\pi}{18}\right) + i\sin\left(-\frac{13\pi}{18}\right)\right);$
 $2\left(\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}\right)$

9 $(z + 1 - 3i)^7 = 2187$

10 (i) $\sqrt{3}\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right), \sqrt{3}\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right),$
 $\sqrt{3}\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right),$
 $\sqrt{3}\left(\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}\right)$



(ii) $w = \frac{9}{4}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$



12 $\cos\frac{1}{5}\pi + \cos\frac{3}{5}\pi = \frac{1}{2}; \cos\frac{1}{5}\pi \times \cos\frac{3}{5}\pi = -\frac{1}{4}$
 $4x^2 - 2x - 1 = 0; \cos\frac{1}{5}\pi = \frac{1}{4}(1 + \sqrt{5})$

Activity 5.5 (Page 133)

$$\begin{aligned}\sin 5\theta &= 5c^4s - 10c^2s^3 + s^5 \\ &= 5s(1-s^2)^2 - 10s^3(1-s^2) + s^5 \\ &= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5 \\ \sin 5\theta &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\end{aligned}$$

Activity 5.6 (Page 134)

$$\begin{aligned}2i\sin\theta &= z - z^{-1} \\ \Rightarrow (2i\sin\theta)^5 &= (z - z^{-1})^5 \\ \Rightarrow 32i\sin^5\theta &= z^5 - 5z^3 + 10z - 10z^{-1} \\ &\quad + 5z^{-3} - z^{-5} \\ &= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) \\ &\quad + 10(z - z^{-1}) \\ &= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \\ \Rightarrow \sin^5\theta &= \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}\end{aligned}$$

Exercise 5D (Page 135)

- 1 (ii) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$
- 2 (i) $z^3 = \cos 3\theta + i\sin 3\theta,$
 $z^{-3} = \cos 3\theta - i\sin 3\theta$
- 3 (i) $z^{-1} = \cos\theta - i\sin\theta$
 (ii) (b) $\frac{\cos 4\theta + 4\cos 2\theta + 3}{8}$
 (iii) (b) $\frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}$
- 4 $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
 $\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$
- 5 $\sin^6\theta = \frac{-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10}{32}$
 $\int \sin^6\theta d\theta = -\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta$
 $\quad - \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta + c$
- 6 $\cos^4\theta \sin^3\theta = \frac{-\sin 7\theta - \sin 5\theta + 3\sin 3\theta + 3\sin\theta}{64}$
 $\int_0^\pi \cos^4\theta \sin^3\theta d\theta = \frac{4}{35}$
- 7 $\frac{203}{480}$

8 (ii) $\cos^2 \theta = \frac{5 \pm \sqrt{5}}{8}$

(iii) $\theta = 18^\circ \Rightarrow \cos 5\theta = 0$

$$\cos \theta = \pm \left(\frac{5 + \sqrt{5}}{8} \right)^{\frac{1}{2}} \text{ or } \pm \left(\frac{5 - \sqrt{5}}{8} \right)^{\frac{1}{2}}$$

$$\cos 18^\circ \text{ is close to } 1 \text{ so } \cos 18^\circ = \left(\frac{5 + \sqrt{5}}{8} \right)^{\frac{1}{2}}$$

and using $\cos^2 \theta + \sin^2 \theta \equiv 1$ gives

$$\sin 18^\circ = \left(\frac{3 - \sqrt{5}}{8} \right)^{\frac{1}{2}}$$

9 $a = \frac{1}{2} z^{-(2n-1)}, r = z^2, 2n \text{ terms}$

10 (i) $z^n = \cos n\theta + i \sin n\theta$

$$\frac{1}{z^n} = \cos n\theta - i \sin n\theta$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

(ii) $p = \frac{1}{16}, q = -\frac{1}{32}, r = -\frac{1}{16}, s = \frac{1}{32}$

11 $1 + z = 1 + \cos \theta + i \sin \theta$

$$= 2 \cos^2 \frac{1}{2} \theta + 2i \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta$$

$$= 2 \cos \frac{1}{2} \theta \left(\cos \frac{1}{2} \theta + i \sin \frac{1}{2} \theta \right)$$

$$\left(\begin{matrix} n \\ 1 \end{matrix} \right) \sin \theta + \left(\begin{matrix} n \\ 2 \end{matrix} \right) \sin 2\theta + \dots + \left(\begin{matrix} n \\ n \end{matrix} \right) \sin n\theta$$

$$= 2^n \cos^n \frac{1}{2} \theta \sin \frac{1}{2} n\theta$$

13 $\sin^5 \theta = \frac{1}{16} \sin 5\theta - \frac{5}{16} \sin 3\theta + \frac{10}{16} \sin \theta;$

$$p = \frac{1}{16}, q = -\frac{5}{16}, r = \frac{5}{8}$$

? (Page 138)

$$\cos(-\theta) = \cos \theta \text{ and } \sin(-\theta) = -\sin \theta$$

so

$$r(\cos \theta - i \sin \theta) = r(\cos(-\theta) + i \sin(-\theta))$$

Therefore

$$r(\cos \theta - i \sin \theta) = re^{-i\theta}$$

Activity 5.7 (Page 138)

(i) $e^{x+yi} = e^x e^{yi} = e^x (\cos y + i \sin y)$

(ii) $e^{z+2\pi ni} = e^z e^{2\pi ni} = e^z (\cos 2\pi n + i \sin 2\pi n)$
 $= e^z \times 1 = e^z$

(iii) Using (i) with $x = 0, y = \pi$ gives
 $e^0 (\cos \pi + i \sin \pi) = -1$

? (Page 139)

The earlier results $\cos \theta = \frac{z+z^{-1}}{2}$ and

$$\sin \theta = \frac{z-z^{-1}}{2i}, \text{ where } z = \cos \theta + i \sin \theta, \text{ can}$$

be rewritten with $z = e^{i\theta}$, so $z^{-1} = e^{-i\theta}$, giving

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Activity 5.8 (Page 141)

$$S = 2^n \cos^n \frac{\theta}{2} \sin \frac{n\theta}{2}$$

Exercise 5E (Page 141)

1 (i) $4e^{\frac{\pi}{3}}$ (ii) $\sqrt{3}e^{-\frac{5\pi}{6}i}$ (iii) $5e^{-\frac{\pi}{2}i}$

(iv) $3\sqrt{2}e^{\frac{3\pi}{4}i}$ (v) $2e^{\frac{\pi}{6}i}$

2 (i) -5 (ii) $-1 + i$

(iii) $-1 - i$ (iv) $\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

3 $zw = 6e^{-\frac{11}{12}\pi i}$

$$\frac{z}{w} = \frac{2}{3}e^{\frac{5\pi}{12}i}$$

4 (i) $w = 32e^{\frac{\pi}{2}i}$

(ii) $2e^{\frac{\pi}{10}i}, 2e^{\frac{\pi}{2}i}, 2e^{\frac{9\pi}{10}i}, 2e^{-\frac{3\pi}{10}i}, 2e^{-\frac{7\pi}{10}i}$

5 (i) $C + iS = 1 + e^{i\theta} + e^{2i\theta} + \dots + e^{(n-1)i\theta}$

(iii) $\frac{\sin \theta + \sin(n-1)\theta - \sin n\theta}{2 - 2\cos \theta}$

6 (ii) $S = 2^n \cos^n \theta \sin n\theta$

7 (i) $a = 16, b = -20, c = 5$

$$\begin{aligned}
 8 \quad z^* &= a - bi \text{ so} \\
 e^{z^*} &= e^{a-bi} = e^a (\cos(-b) + i \sin(-b)), \\
 &= e^a (\cos b - i \sin b) \\
 e^z &= e^{a+bi} = e^a (\cos b + i \sin b) \text{ so} \\
 (e^z)^* &= e^a (\cos b - i \sin b)
 \end{aligned}$$

$$\begin{aligned}
 9 \quad (i) \quad C &= \frac{e^{3x} (3 \cos 2x + 2 \sin 2x)}{13} + c_1 \\
 S &= \frac{e^{3x} (-2 \cos 2x + 3 \sin 2x)}{13} + c_2
 \end{aligned}$$

$$10 \quad C = \frac{2 \cos \theta + 1}{5 + 4 \cos \theta} \quad S = \frac{2 \sin \theta}{5 + 4 \cos \theta}$$

Chapter 6

? (Page 145)

Winds of moderate speed caused large oscillations in the bridge, which eventually collapsed into the river below.

Activity 6.1 (Page 147)

- (i) (a) (ii) (a) $(P = -x^2, Q = x^2)$,
 (b) $(P = x, Q = x^2)$,
 (c) $(P = -(x+1), Q = x^2 + x)$

Exercise 6A (Page 151)

- 1 (i) $e^{\frac{1}{3}x^3}$
 (ii) $e^{-\cos x}$
 (iii) $x^{-\frac{1}{4}}$
 (iv) x
 (v) e^{7x}
 (vi) $\sec x$

- 2 (i) $\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2}$
 (ii) x
 (iii) $x \frac{dy}{dx} + y = \frac{1}{x}$
 (iv) $\frac{d}{dx}(xy) = \frac{1}{x}$
 (v) $xy = \ln x + c$
 (vi) $y = \frac{1}{x} \ln x + \frac{c}{x}$

$$(vii) \quad c = 0$$

$$(viii) \quad y = \frac{1}{x} \ln x$$

$$3 \quad (i) \quad y = \frac{x^2}{4} - \frac{1}{4x^2}$$

$$(ii) \quad y = 4 - 2e^{\frac{1}{2}x^2}$$

$$(iii) \quad y = 3e^{-3(x^2-1)}$$

$$(iv) \quad x = \frac{1}{2}(3e^{t^2} - 1)$$

$$(v) \quad y = x^3 - x^2$$

$$(vi) \quad v = \frac{1}{3}(1 - 4e^{-t^3})$$

$$4 \quad (i) \quad v = 25 + Ae^{-0.4t}$$

$$(ii)-(iii) \quad v = 25(1 - e^{-0.4t})$$

(iv) The method of separation of variables is usually preferred as it involves less work.

$$5 \quad (i) \quad k = \frac{1}{3}$$

$$(ii) \quad \frac{dv}{dt} = 10 - \frac{1}{3}v$$

$$(iii) \quad v = 30 + Ae^{-\frac{1}{3}t}$$

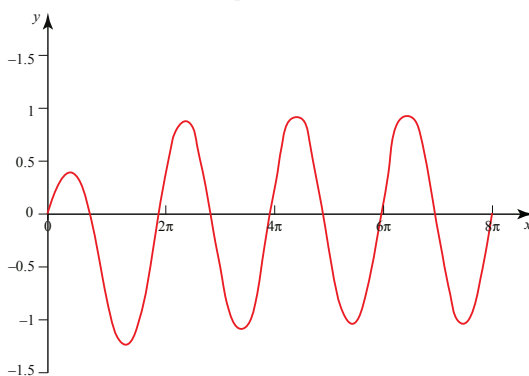
$$(iv) \quad v = 30 \left(1 + e^{-\frac{1}{3}t} \right)$$

$$6 \quad (i) \quad y = \sin x + \frac{\cos x}{x} + \frac{A}{x}$$

$$(ii) \quad y = \sin x + \frac{\cos x - 1}{x}$$

$$(iii) \quad y = \sin x$$

$$(iv) \quad \text{As } x \rightarrow 0, \quad y \rightarrow \sin x$$



$$\begin{aligned}
 7 \quad (i) \quad y &= Ae^{-k_2 t} + \frac{k_1 a}{k_2 - k_1} e^{-k_1 t} \\
 (ii) \quad y &= \frac{k_1 a}{k_2 - k_1} (e^{-k_1 t} - e^{-k_2 t})
 \end{aligned}$$

(iii) $\frac{dx}{dt} = -k_1x$

(iv) $x = ae^{-k_1t}$

The amount y of Th-234 is affected by both its own decay (the k_2y part of the differential equation) and the rate at which U-238 decays into it – given by k , x i.e. $ak_1e^{-k_1t}$.

(v) $y = kate^{-kt}$

(vi) $\frac{dz}{dt} = -ky, z = a(kt+1)e^{-kt} - a$

8 (i) $y = \frac{1-x}{1+x} \left(A + x + \frac{x^2}{2} \right)$

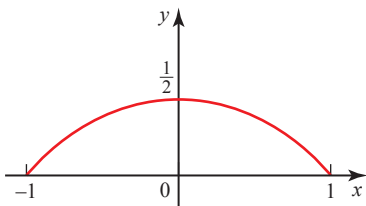
(ii) (a) $y = \frac{1-x}{1+x} \left(x + \frac{x^2}{2} \right)$

As $x \rightarrow -1, y \rightarrow -\infty$

(b) $y = \frac{1-x}{1+x} \left(1 + x + \frac{x^2}{2} \right)$

As $x \rightarrow -1, y \rightarrow \infty$

(iii) $y = \frac{1}{2}(1-x^2)$



9 (i) $x^2y = A + \frac{1}{3}(1+x^2)^{\frac{3}{2}}$

(ii) $x^2y = \left(1 - \frac{2\sqrt{2}}{3} \right) + \frac{1}{3}(1+x^2)^{\frac{3}{2}}$
 $y \rightarrow \infty$ as $x \rightarrow 0$

(iii) $1 + \frac{3x^2}{2} + \frac{3x^4}{8}$

(iv) $y \approx \frac{A}{x^2} + \frac{1}{3x^2} \left(1 + \frac{3x^2}{2} + \frac{3x^4}{8} + \dots \right)$

Take $A = -\frac{1}{3} \Rightarrow y \approx -\frac{1}{3x^2} + \frac{1}{3x^2}(1+x^2)^{\frac{3}{2}}$

Activity 6.2 (Page 157)

(i) (a), (c) and (d) are linear, (a), (b), (d) and (e) have constant coefficients.

(ii) (a) Can be solved by any of the three methods,
 $y = Ae^{17x}$

(b) Separation of variables, $y = \frac{1}{c-x}$

(c) Separation of variables or integrating factor,
 $y = Ae^{-\frac{1}{2}x^2}$

(d) Can be solved by any of the three methods,
 $y = Ae^{3x}$

(e) After division by y , any of the three methods can be used, $y = Ae^x$

Activity 6.3 (Page 158)

$$y = Ae^{3x} + Be^{2x}; \frac{dy}{dx} = 3Ae^{3x} + 2Be^{2x}; \frac{d^2y}{dx^2} =$$

$$9Ae^{3x} + 4Be^{2x}$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = (9A - 15A + 6A)e^{3x} +$$

$$(4B - 10B + 6B)e^{2x} = 0$$

Exercise 6B (Page 164)

1 (i) $y = Ae^{3x}$

(ii) $y = Ae^{-7x}$

(iii) $x = Ae^{-t}$

(iv) $p = Ae^{0.02t}$

(v) $z = Ae^{0.2t}$

2 (i) $y = 3e^{-2x}$

(ii) $y = e^{\frac{5}{2}x}$

(iii) $x = 2e^{\frac{1}{3}(1-t)}$

(iv) $P = P_0e^{kt}$

(v) $m = m_0e^{-kt}$

3 (i) $y = Ae^{4x} + Be^{-4x}$

(ii) $y = Ae^{1.79x} + Be^{-2.79x}$

(iii) $x = Ae^{-1.71t} + Be^{-0.29t}$

(iv) $y = A + Be^{-\frac{2}{7}x}$

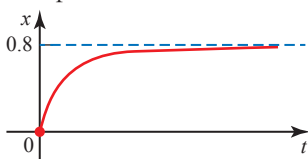
(v) $y = e^{\frac{2}{3}x}(Ax + B)$

- 4 (i) $\lambda^2 - 3\lambda + 2 = 0$
 (ii) $\lambda = 1, 2$
 (iii) $y = Ae^x + Be^{2x}$
 (iv) $\frac{dy}{dx} = Ae^x + 2Be^{2x}$
 (v) $y = 2e^x - e^{2x}$

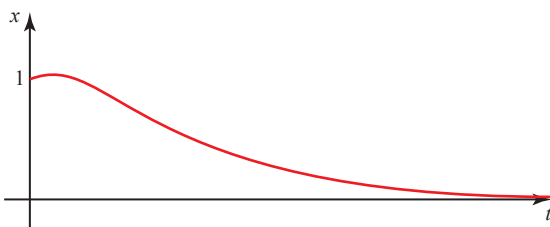
- 5 (i) $\lambda^2 - 8\lambda + 16 = 0, \lambda = 4$
 (ii) $y = Ae^{4x} + Bxe^{4x}$
 (iii) $y = e^{4x} - 5xe^{4x}$

- 6 (i) $x = \frac{4}{5}(1 - e^{-5t})$

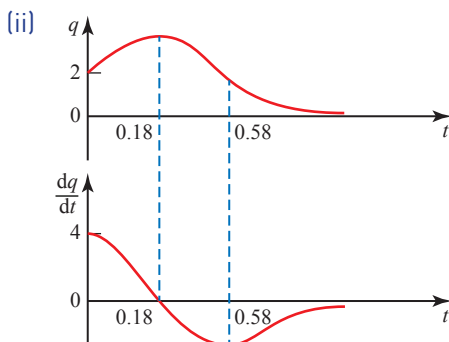
Initially moves quickly towards its limiting position $x = 0.8$



- (ii) $x = (1 + t(\sqrt{e} - 1))e^{-\frac{1}{2}t}$ Initially increases briefly then decays to zero.

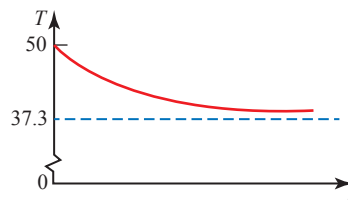


- 7 (i) $q = (2 + 9t)e^{-2.5t}$



- (iii) $q \rightarrow 0$ and $\frac{dq}{dt} \rightarrow 0$ as $t \rightarrow \infty$

- 8 (i) $T = 37.3 + 12.7e^{-0.5t}$
 (ii) $T = 42.0^\circ\text{C}$
 (iii)



- (iv) 37.3°C

- 9 (i) $T = Ae^{2x} + Be^{-2x}$
 (ii) $T = -52.3e^{2x} + 152.3e^{-2x}$
 (iii) 87.9°C

- 10 (ii) $x = 2 - 2e^{-50t}$

- (iii) 49.9 seconds, 50 km h^{-1}

- (iv) The car takes an infinite amount of time to stop.

- 11 (i) $y = Ax + B$

- (ii) $f(x) = Ax + B$

- (iii) $y = (Ax + B)e^{-kx}$

Activity 6.4 (Page 168)

$$x = a \sin(\omega t + \epsilon) = a \sin \epsilon \cos \omega t + a \cos \epsilon \sin \omega t$$

$$\equiv P \cos \omega t + Q \sin \omega t$$

$$\Rightarrow P = a \sin \epsilon \text{ and } Q = a \cos \epsilon$$

$$\Rightarrow P^2 + Q^2 = a^2 (\sin^2 \epsilon + \cos^2 \epsilon) = a^2$$

$$\Rightarrow \frac{P}{Q} = \frac{a \sin \epsilon}{a \cos \epsilon} = \tan \epsilon$$

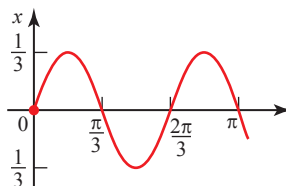
$$\frac{P}{Q} = \frac{a \sin \epsilon}{a \cos \epsilon} = \tan \epsilon$$

Exercise 6C (Page 175)

- 1 (i) $x = A \sin 3t + B \cos 3t$

- (ii) $x = \frac{1}{3} \sin 3t$

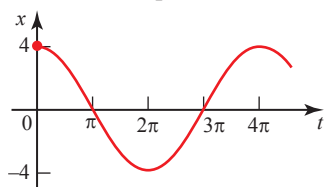
- (iii) Period = $\frac{2\pi}{3}$, amplitude = $\frac{1}{3}$



2 (i) $x = A \sin \frac{1}{2}t + B \cos \frac{1}{2}t$

(ii) $x = 4 \cos \frac{1}{2}t$

(iii) Period = 4π , amplitude = 4



3 (i) $y = e^{2x}(A \sin x + B \cos x)$

(ii) $y = e^x(A \sin 2x + B \cos 2x)$

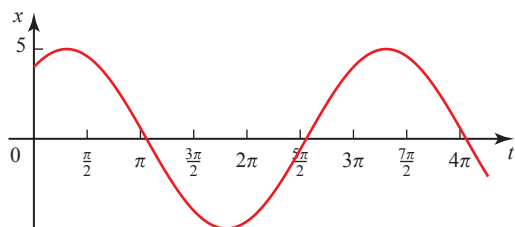
(iii) $x = e^{-t}(A \sin \sqrt{3}t + B \cos \sqrt{3}t)$

(iv) $x = e^{-0.5t}(A \sin t + B \cos t)$

4 (i) $x = A \sin \frac{2}{3}t + B \cos \frac{2}{3}t$

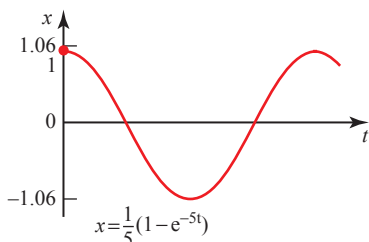
(ii) $x = 5 \sin\left(\frac{2}{3}t + 0.927\right)$

(iii)



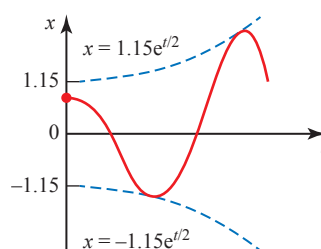
5 (i) $x = \frac{\sqrt{2}}{4} \sin 2\sqrt{2}t + \cos 2\sqrt{2}t$
 $= \frac{3\sqrt{2}}{4} \sin\left(2\sqrt{2}t + 1.23\right)$

Oscillations with constant amplitude



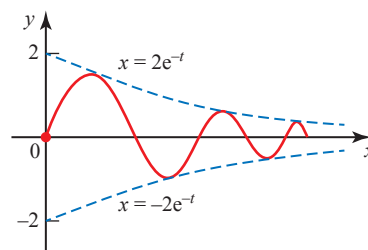
(ii) $x = e^{\frac{t}{2}} \left(\cos \frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t \right)$
 $= \frac{2}{\sqrt{3}} e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}\right)$

Exponentially increasing oscillations



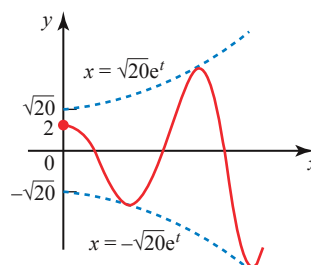
(iii) $x = 2e^{-t} \sin t$

Decaying oscillations



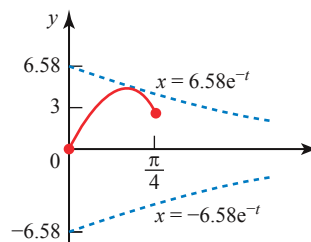
(iv) $x = e^t \left(2 \cos \frac{1}{2}t - 4 \sin \frac{1}{2}t \right)$
 $= \sqrt{20}e^t \cos\left(\frac{1}{2}t + 1.11\right)$

Increasing oscillations



(v) $x = 3e^{-t+\frac{\pi}{4}} \sin 2t = 6.58e^{-t} \sin 2t$

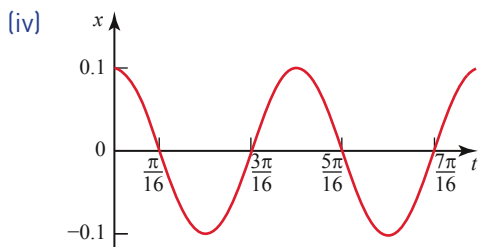
Decaying oscillations



6 (i) $x = A \sin 8t + B \cos 8t$

(ii) $x = 0.1 \cos 8t$

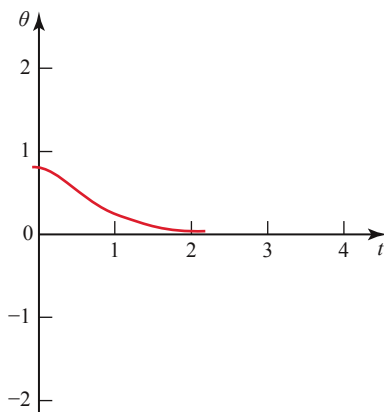
(iii) Period = $\frac{\pi}{4}$, amplitude = 0.1



7 (i) $\theta = e^{-2t}(A \sin t + B \cos t)$ or
 $\theta = e^{-2t} A \cos(t + \epsilon)$

(ii) $\theta = \frac{\pi}{4} e^{-2t}(2 \sin t + \cos t)$ or
 $\theta = 1.76 e^{-2t} \cos(t - 1.11)$

(iii) The motion effectively ends at about $t = 2.5$. Although θ subsequently does become negative, the amplitude of the oscillation is very small and decreases rapidly.



(iv) $\theta \rightarrow 0$ and $t \rightarrow \infty$

8 (i) $k = \sqrt{20}$

- (ii) [a] Underdamped.
 [b] Overdamped.

10 $y = Ae^{2t} + Be^{-2t} + C \sin 2t + D \cos 2t$

Activity 6.6 (Page 178)

(i) $\frac{dy}{dx} = -2Ae^{-2x}$; $\frac{dy}{dx} + 2y = -2Ae^{-2x} + 2Ae^{-2x} = 0$

(ii) $\frac{dy}{dx} = -2Ae^{-2x} + \frac{3}{2}$; $\frac{dy}{dx} + 2y$
 $= -2Ae^{-2x} + \frac{3}{2} + 2Ae^{-2x} + 3x - \frac{5}{2}$
 $= 3x - 1$

(iii) $\frac{dy}{dx} = -2Ae^{-2x} + \frac{3}{5}e^{3x}$; $\frac{dy}{dx} + 2y$
 $= -2Ae^{-2x} + \frac{3}{5}e^{3x} + 2Ae^{-2x} + \frac{2}{5}e^{3x} = e^{3x}$

(iv) $\frac{dy}{dx} = -2Ae^{-2x} + \frac{1}{5} \sin x + \frac{2}{5} \cos x$;
 $\frac{dy}{dx} + 2y = -2Ae^{-2x} + \frac{1}{5} \sin x$
 $+ \frac{2}{5} \cos x + 2Ae^{-2x} - \frac{2}{5} \cos x$
 $+ \frac{4}{5} \sin x = \sin x$

Exercise 6D (Page 182)

1 (i) $y = -2x - 5$

(ii) $x = \frac{1}{4}t + \frac{1}{2}$

(iii) $y = -0.08 \cos 3x + 0.06 \sin 3x$

(iv) $x = \frac{3}{4}e^{-2t}$

2 (i) $y = A \cos 2x + B \sin 2x + \frac{1}{8}e^{2x}$

(ii) $x = Ae^{3t} + Be^{-t} + e^{-2t}$

(iii) $y = e^{-x}(A \cos 2x + B \sin 2x) + 0.1 \sin x + 0.2 \cos x$

(iv) $y = e^{-x}(A \cos 2x + B \sin 2x) + 0.6x + 0.16$

3 (i) $x = A \cos t + B \sin t - \frac{1}{2}t \cos t$

(ii) $y = Ae^{-4x} + Be^x + \frac{1}{5}xe^x$

(iii) $x = Ae^t + Be^{3t} + \frac{1}{2}te^{3t}$

(iv) $x = (At + B)e^{3t} + 2t^2e^{3t}$

4 (i) $y = e^{2x} - 6e^{3x} + 6x + 5$

(ii) $x = -2 \cos 3t - \frac{1}{3} \sin 3t + 2e^t$

(iii) $y = e^{-x}(-\cos 2x + 1)$

(iv) $y = -e^{-2t} + 4e^t - \cos 2t - 3 \sin 2t$

(v) $x = -\frac{3}{2}e^t + \frac{7}{6}e^{2t} + \frac{1}{2} - \frac{1}{6}e^{-t}$

(vi) $y = 2 \sin 2x - 3x \cos 2x$

(vii) $y = e^{-2x}(2 \cos x + 3 \sin x) - \cos x + \sin x$

(viii) $y = \frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x} + x - 1$

- 5 (i) $P = Ae^{-t}$
 (ii) $P = 100 - 25(\cos t - \sin t)$
 (iii) $P = 100 - 25(\cos t - \sin t) - 55e^{-t}$
 (iv) 100
 (v) $25\sqrt{2}$

- 6 (i) $x = e^{-1.5t} \left(A \cos \frac{\sqrt{31}}{2} t + B \sin \frac{\sqrt{31}}{2} t \right) + 0.05$
 (ii) $x = e^{-1.5t} \left(0.05 \cos \frac{\sqrt{31}}{2} t + \frac{0.15}{\sqrt{31}} \sin \frac{\sqrt{31}}{2} t \right) + 0.05$

$$x \rightarrow 0.05 \text{ as } t \rightarrow \infty$$

- (iii) 1.16
 7 (i) $e^{-50t} (A \sin 50\sqrt{3}t + B \cos 50\sqrt{3}t)$

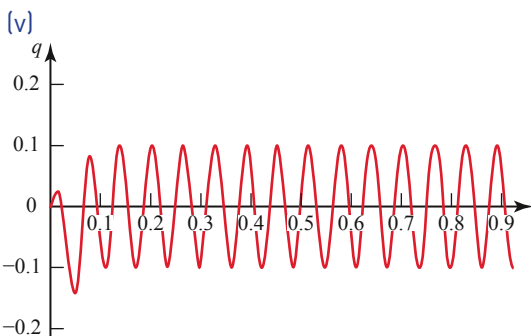
(ii) Underdamped.

(iii) $-0.1 \cos 100t$

$$q = e^{-50t} (A \sin 50\sqrt{3}t + B \cos 50\sqrt{3}t)$$

$$-0.1 \cos 100t$$

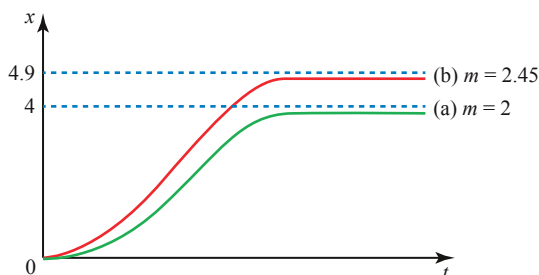
(iv) $q = e^{-50t} \left(\frac{1}{10\sqrt{3}} \sin 50\sqrt{3}t + \frac{1}{10} \cos 50\sqrt{3}t \right)$
 $-0.1 \cos 100t$



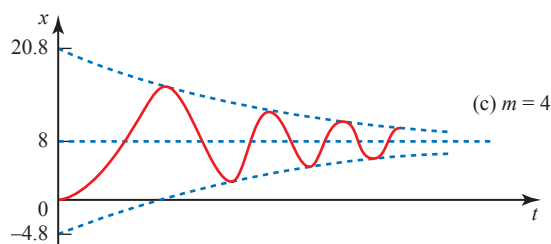
The oscillations quickly settle to almost constant amplitude.

- 8 (i) (a) $x = 4 - \frac{20}{3}e^{-t} + \frac{8}{3}e^{-2.5t}$
 (b) $x = 4.9 - (7.01t + 4.9)e^{-1.43t}$
 (c) $x = 8 - e^{-0.875t} (10.0 \sin 0.70t + 8 \cos 0.70t)$

(ii) (a) (b)



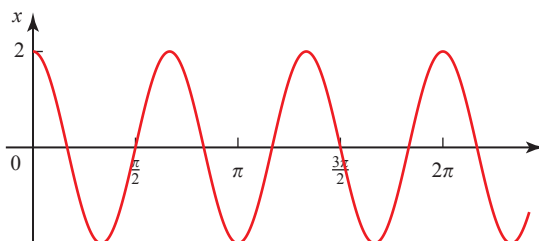
(c)



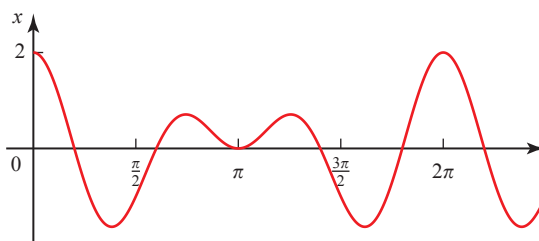
9 (i) (a) (b) $y = Ae^{kt} + \frac{1}{p-k} e^{pt}$

(ii) (a) (b) $y = Ae^{kt} + te^{kt}$

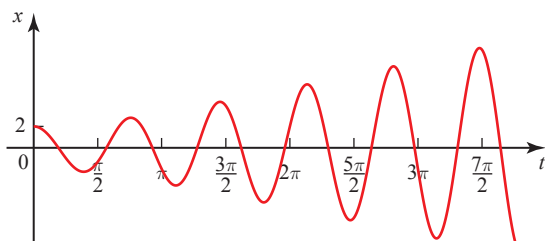
10 (i) $x = 2 \cos 3t$



(ii)



(iii) $x = 2 \cos 3t + \frac{5}{6}t \sin 3t$



The machine will shake itself to bits (this situation is described as **resonance**).

11 $x = (A + Bt)e^{-2t} + 1 + t - \frac{1}{2}t^2$

12 General solution

$$x = e^{-3t}(A \cos 2t + B \sin 2t) + 3 \cos 2t + 4 \sin 2t;$$

Particular solution

$$x = e^{-3t}(2 \cos 2t - \sin 2t) + 3 \cos 2t + 4 \sin 2t$$

As $t \rightarrow \infty$, $x \approx 3 \cos 2t + 4 \sin 2t = 5 \cos(2t - \phi)$

Exercise 6E (Page 191)

1 $y = x(2 + \ln x)$

2 $2x = (x - y)(\ln x - \frac{2}{3})$

3 $\ln(y^2 + x^2) + 2 \tan^{-1}\left(\frac{y}{x}\right) = 0$

4 $xy^2 = A(y - x)^2$

5 $y + \sqrt{x^2 + y^2} = Ax^2$

6 (i) $\frac{dz}{dx} = \frac{2}{1 - 2z}$

(ii) $(x + y)^2 + (x - y) = A$

7 $y = \frac{1}{2}(x^3 - 2x^2 + x)$

8 $y = -2e^\pi \sin(\ln x)$

9 $y = \left[A + B \ln x + \frac{1}{2}(\ln x)^2\right]x^3 + \frac{2}{3}$

10 $y = \frac{1}{8}x^2 + A(\ln x)^2 + B \ln x + C$

11 $y = \tan^{-1} x$

12 $y = \ln\left(\frac{x}{\sqrt{1+x^2}}\right) + C \tan^{-1} x + D$

13 $y = \frac{2}{(1-x)^2}$

14 $\frac{1}{3}y^3 + Cy + D = 2x$

15 $y = x \cos x$

16 $y = \sqrt[3]{Ae^{3x} + Be^{-5x} - \frac{3}{2}e^{-x}}$

17 (ii) $y = \frac{1}{Ax - x^2}$

18 (ii) 31.6 m s^{-1}

(iii) 92.0 m