Full worked solutions

Target your revision (Statistics) (pages 1-5)

1 Understand the terms population and sample
To understand how people's friendships change as they age, a researcher interviews the same 100 people aged 60 and over every year for 10 years. For this study, identify:
i the sample ii the population.

i The sample is the 100 people interviewed.
ii The population is people aged 60 and over.

2 Use simple random sampling
All phone numbers in a town start with the same two digits (65). These are followed by four more digits to make the phone number. A researcher generates a 4-digit random number and then rings 65 followed by the random number. He does this 100 times. Will this give a simple random sample of the population in the town? Justify your answer.

No because each possible sample should be equally likely to be chosen for a simple random sample. There are some people who do not have landlines so they will not be chosen; people who do not answer the phone will not be chosen and only one person per household at most will be chosen.

3 Understand sampling methods: opportunity sampling, systematic sampling, stratified sampling, quota sampling, cluster sampling, self-selected samples
A group of market researchers interview people in different towns to ask about their shopping habits. Each researcher is told to interview the following numbers of people from different groups.

Age group	16–20	21–30	31–45	46–65	66 and over
Male number	10	20	35	40	25
Female number	10	20	35	40	30

Which sampling method is this?

The researchers choose the individuals so this is quota sampling (if the individuals were chosen using random sampling then this would be stratified sampling but that is not the case).

4 Recognise possible sources of bias when sampling
A researcher wants to investigate how many people have tried illegal drugs. She puts a link to an anonymous online questionnaire in a tweet and asks her twitter followers to pass the link on. Identify two possible sources of bias in her sampling method.

There are several possible sources of bias. You should identify two from the list below. The groups identified below will not be typical of the whole population.

- Only people with internet access will be sampled.
- Only people who follow the researcher on twitter (and their contacts) will be sampled.
- Only people who want to answer the survey will do so.
- The anonymity makes it difficult to check whether each person has only responded once.

5 Be aware of practicalities of implementation of sampling methods

A union leader wants to find out about working conditions for her union members. She chooses a random sample of 100 union members to send questionnaires to. She knows that some of the sample will not return the questionnaires. Which one of the following options would be the best course of action? Justify your answer.

A Give some questionnaires to local union members she knows to make up numbers.
B Work with the questionnaires that are returned and do not try to replace them.
C Use a larger random sample to allow for some questionnaires not being returned.

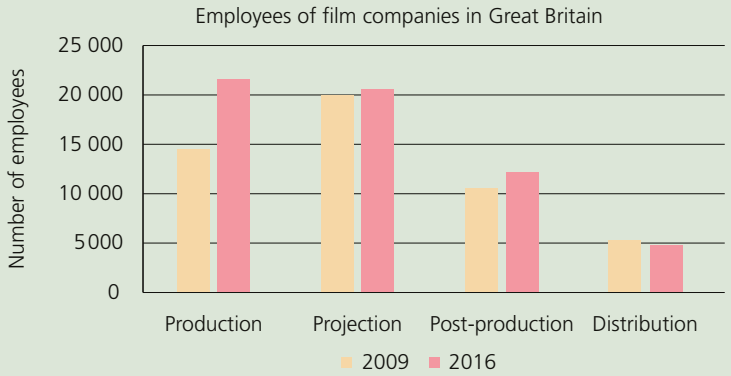
Option C is best. Option A will bias the sample as there will be too many sampled from the local area. Option B could result in a small sample. Although Option C is the best option, it is not without bias because people who do return the questionnaire may not be representative of the whole population.

6 Recognise categorical, discrete, continuous and ranked data

Which of the words categorical, discrete, continuous, ranked would apply to measurements of height?

The data are numerical and on a continuous scale so they are continuous.

7 Interpret a bar chart

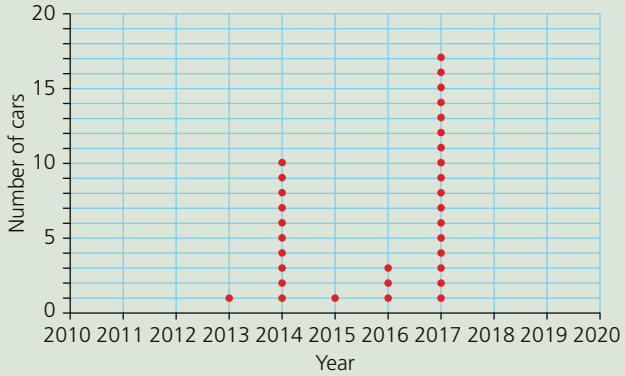


Show that the number of employees involved in production has risen by about 50% from 2009 to 2016.

There were about 14 000 production employees in 2009. 50% of 14 000 is 7000 so a 50% increase would result in 21 000 production employees and that is about what there are in 2016.

8 Interpret a dot plot or vertical line chart

The dot plot shows the year of first registration for each car on sale at a second-hand car dealer.



How many cars are on sale?

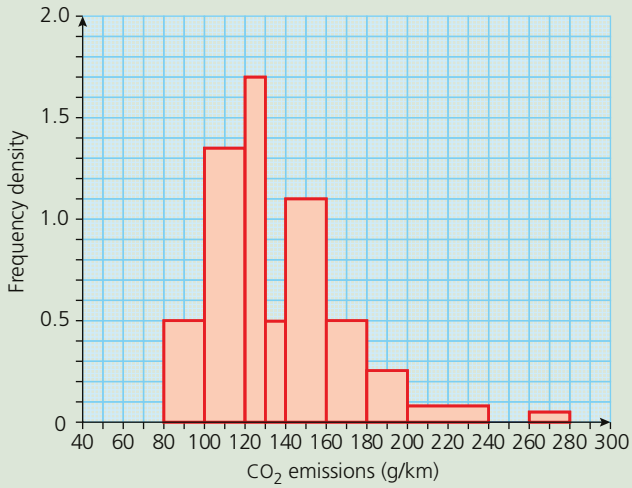
The frequency table for the data is as follows.

Year	Frequency
2013	1
2014	10
2015	1
2016	3
2017	17
TOTAL	32

So 32 cars in total.

9 Interpret a histogram or frequency chart

The histogram below shows CO_2 emissions for each of a random sample of 100 cars.



Describe the distribution of CO_2 emissions. Make two distinct comments about the information shown in the histogram.

Two distinct correct comments – here are some examples.

- There are no cars with CO_2 emissions below 80 g/km.
- There seem to be two groups of cars – one group with lower emissions and another group with higher emissions.
- The distribution is bimodal.
- The distribution is positively skewed.
- The cars with the highest CO_2 emissions emit over three times as much CO_2 as the cars with the lowest emissions.

10 Calculate frequencies or proportions from a histogram

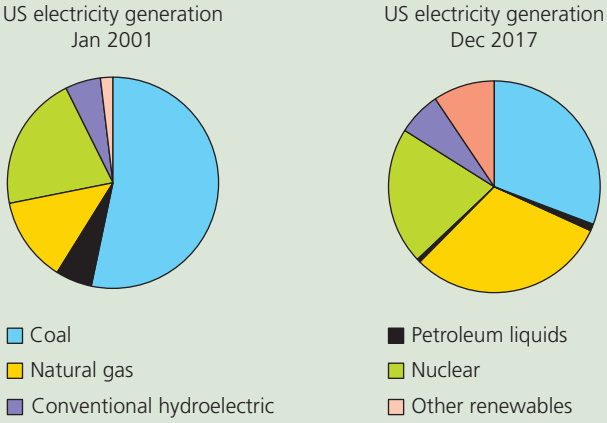
For the histogram in question 9, how many cars have CO_2 emissions between 120 and 130 g/km?

The frequency density is 1.7, and the class width is 10.

Frequency = $1.7 \times 10 = 17$.

11 Interpret a pie chart

The pie charts below show the proportions of electricity generated in the US from different sources in January 2001 and December 2017.



Make two comments about changes in sources for electricity generation.

Two correct comments comparing proportions (not amounts) of electricity from different sources, as in examples given here.

Compared to January 2001:

- In December 2017 a lower proportion of electricity was generated using coal.
- In December 2017 a higher proportion of electricity was generated using natural gas.
- In December 2017 a lower proportion of electricity was generated using petroleum liquids.
- In December 2017 about the same proportion of electricity was generated using nuclear.

12 Interpret a stem-and-leaf diagram, use and interpret median and quartiles

The following stem-and-leaf diagram shows the hand lengths of a sample of 28 boys aged 10.

13	2	8										
14	2	2	4	4	5	6	6	7	7	8	9	9
15	0	2	3	4	5	6	6	8	9	9		
16	0	0	0	5								

Key 13|2 means 13.2 cm

Find:

i the median

ii the interquartile range.

i The data values are in order in the stem-and-leaf diagram. There are 14 values above the red line and 14 below.

13	2	8										
14	2	2	4	4	5	6	6	7	7	8	9	9
15	0	2	3	4	5	6	6	8	9	9		
16	0	0	0	5								

The middle two measurements are 14.9 cm and 15.0 cm so the median is halfway between these.

$$\text{Median} = \frac{14.9 + 15.0}{2} = 14.95 \text{ cm.}$$

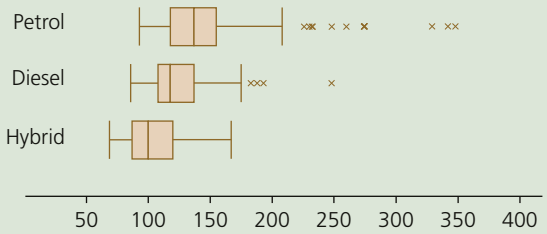
ii For the first 14 data values (the ones below the median), there are 7 each side of the lower quartile. The lower quartile is between 14.5 cm and 14.6 cm hence $LQ = 14.55$ cm.

13	2 8
14	2 2 4 4 5 | 6 6 7 7 8 9 9
15	0 2 3 4 5 6 6 | 8 9 9
16	0 0 0 5

Similarly, the upper quartile is the median of the 14 highest data values, so it is between 15.6 cm and 15.8 cm, so that there are 7 data values above the upper quartile.
$UQ = 15.7$ cm.
Interquartile range = 15.7 − 14.55 = 1.15 cm

13 Interpret box plots

The box plots below show the CO_2 emission in g/km for three types of car.



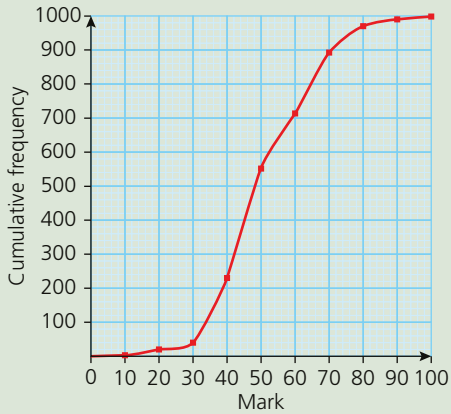
Compare CO_2 emissions for the three types of car. Make two distinct comments about the information shown by the box plots.

Two distinct correct comments, examples given here.

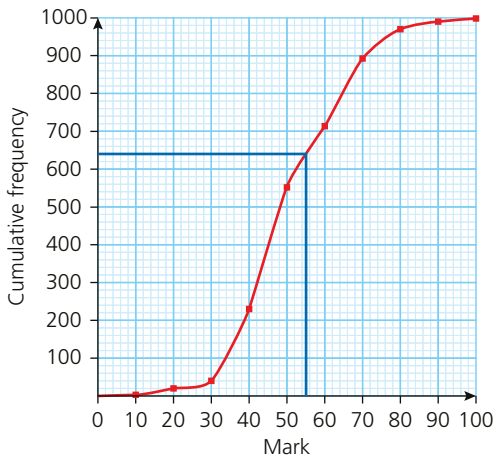
- Hybrid cars tends to have the lowest CO_2 emissions.
- Petrol cars tend to have the highest CO_2 emissions.
- Diesel cars have the lowest spread in CO_2 emissions.
- A petrol car with median CO_2 emissions is producing more CO_2 than the upper quartile diesel car.
- The data for diesel cars are positively skewed.

14 Interpret a cumulative frequency diagram

1000 students sit an examination which is marked out of 100. Their results are shown in the cumulative frequency diagram below.



How many students got over 55 marks?



The cumulative frequency associated with a mark of 55 is 640.

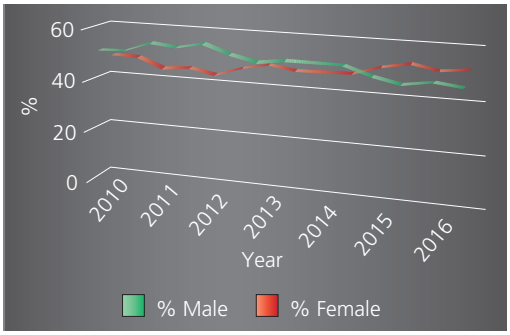
This means that 640 students got a mark of 55 or less.

1000 – 640 = 360 students got a mark over 55.

15 Select or critique data presentation techniques
The chart below shows the gender of mathematics trainee teachers in England.
State two ways to improve the presentation of the data.

Two distinct ways to improve the presentation of the data, e.g.

- Do not use a black background.
- Do not use a 3-D effect.
- Have a grid to allow reading values off.
- Just have one of the graphs – they add up to 100% so the other is superfluous.
- If you have both the graphs do not use green and red as this makes it hard for someone who is colour blind to tell the difference.



16 Calculate and interpret measures of central tendency: median, mode, mean
The age distribution of classroom teachers in state schools in England in 2016 is shown in the following table.

Age	Frequency (thousands)
Under 25	30.8
25–29	83.6
30–34	78.0
35–39	66.7
40–44	55.9
45–49	46.9
50–54	38.1
55–59	24.8
60 and over	11.7

Calculate an estimate of the mean age of classroom teachers. State any assumptions you make.

To estimate the mean, the midpoint of each group is needed.
Assumptions must be made about the starting point of the first group and the end point of the last group.

Reasonable assumptions are that no teacher is below 20 (you might have 21 or 22 here instead) and that no teacher is above 65 (you might have a slightly different age here).

When calculating midpoints, remember that age 25–29 means from the 25th birthday till just before the 30th birthday and so the midpoint is $\frac{25+30}{2} = 27.5$.

Age	Midpoint	Frequency (thousands)
20–25	22.5	30.8
25–29	27.5	83.6
30–34	32.5	78.0
35–39	37.5	66.7
40–44	42.5	55.9
45–49	47.5	46.9
50–54	52.5	38.1
55–59	57.5	24.8
60–65	62.5	11.7

Make sure your calculator is set up for using frequencies. Notice that the frequencies in the table are in thousands and you will need to enter them as 30 800, etc. Your calculator may let you enter frequencies which are not whole numbers, but if you do this you will get some of the wrong values in the list of statistics that your calculator produces.

	X	freq
1	22.5	30800
2	27.5	83600
3	32.5	78000
4	37.5	66700

66700

$\bar{x}$ =38.46334479
$\sum x$ =16789250
$\sum x^2$ =694498125
$\sigma^2 x$ =111.6321043
σx =10.56560951
$s^2 x$ =111.63236
sx =10.56562161

The mean is 38.5 years (to 1 d.p.).

17 Calculate and interpret variance and standard deviation
For the data in question 16, calculate an estimate of the standard deviation.

Using calculator statistical functions as for question 16, the standard deviation is 10.6 years (to 1 d.p.).

18 Identify outliers and clean data

i For the data in question 16, show that there are some ages which could be classified as high outliers.

ii Are these values the result of errors? Justify your answer.

i Using the rule, mean $\pm$ 2 $\times$ standard deviation, a high outlier is more than $38.5 + 2 \times 10.6 = 59.7$.
There are teachers in the table aged 60 and over so there are some high outliers.

ii There are 11.7 thousand teachers aged 60 and over. These could not all be errors.
OR
The mean and standard deviation used were estimates and 59.7 is close to 60. If the exact mean and standard deviation were used it is possible that mean $\pm$ 2 $\times$ standard deviation could be 60 or more.

19 Calculate probability for independent events

i For a particular type of chocolate bar, one bar in every 10 has a voucher for a free bar inside the pack. Jason buys 5 of these bars. What is the probability that he does not get a voucher for a free bar? Assume that vouchers occur independently of each other.

ii Is the assumption of independence reasonable?

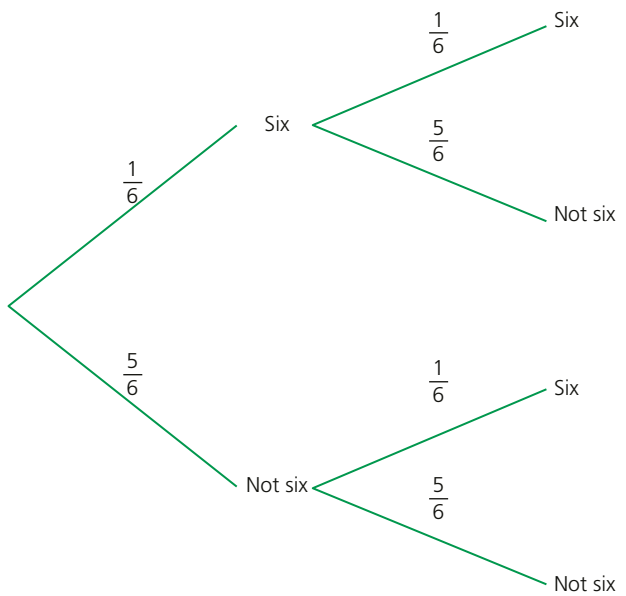
i For each bar, P(no voucher) $= 1 - \frac{1}{10} = \frac{9}{10}$.
The probability of no vouchers in 5 bars $= \left(\frac{9}{10}\right)^5 = 0.59049$

ii As long as the bars with vouchers are well mixed with the bars without vouchers, it is reasonable.

20 Calculate probability for mutually exclusive events

A player in a game throws two fair dice. What is the probability of getting exactly one six?

This could be done by using a tree diagram.

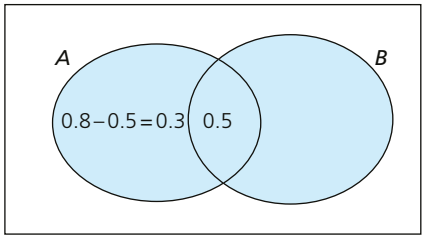


P(One Six) = P(Six, Not Six) + P(Not Six, Six) $= \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{5}{18}$

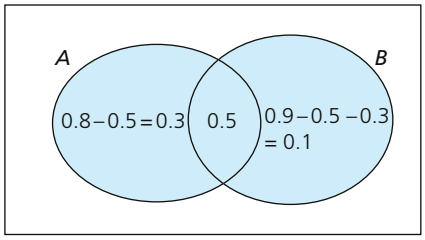
21 Use Venn diagrams to assist in calculation of probabilities

Events A and B are such that P(A) = 0.8, P($A \cap B$) = 0.5 and P($A \cup B$) = 0.9. Find P(B).

This could be done by using a Venn diagram.



Start by entering the probability for $A \cap B$ then use the probability for A to work out the probability of 0.3. Next use the probability for $A \cup B$.



$P(B) = 0.5 + 0.1 = 0.6$

22 Decide whether two events are independent
For the events A and B in question 21, determine whether they are independent.

$P(A) \times P(B) = 0.8 \times 0.6 = 0.48$

$P(A \cap B) = 0.5$

$P(A \cap B) \neq P(A) \times P(B)$ so the events are not independent.

23 Calculate conditional probability using the formula
For events S and T, $P(S) = 0.4$, $P(T) = 0.3$ and $P(S \cap T) = 0.2$.
Find $P(S \mid T)$.

$$P(S \mid T) = \frac{P(S \cap T)}{P(T)} = \frac{0.2}{0.3} = \frac{2}{3}$$

24 Calculate conditional probability using a tree diagram, Venn diagram or sample space diagram
All patients at a local health centre were asked whether they had made any of the following three changes during the past year.

- Eat healthier food.
- Increase exercise.
- Drink less alcohol.

The results of the survey are shown in the Venn diagram.

Given that a patient at the health centre has increased exercise, find the probability that the patient has drunk less alcohol.

$$\mathrm{P}(\text{drink less alcohol} \mid \text{increase exercise}) = \frac{\mathrm{P}(\text{drink less alcohol} \cap \text{increase exercise})}{\mathrm{P}(\text{increase exercise})} = \frac{3\% + 4\%}{8\% + 26\% + 4\% + 3\%}$$

$$= \frac{7\%}{41\%}$$

$$= \frac{7}{41} \approx 0.171$$

25 Recognise situations which give rise to a binomial distribution
Could the number of rainy days in a week be modelled by a binomial distribution? A rainy day is defined as a day when any rain falls. Justify your answer.

In a binomial situation, these criteria apply:

1. You are conducting an experiment or trial n times (a fixed number).
   This one is met – the weather is being observed on 7 days.
2. There are 2 outcomes, which you can think of as 'success' and 'failure'.
   This one is met – the day is either rainy or not.
3. The probability of 'success' is the same each time and independent of the outcome of previous trials (symbol p). The probability of 'failure' is $q = 1 - p$.
   This one is not met – if one day is rainy, the next day is more likely to be rainy.

A binomial model is not suitable.

26 Calculate probabilities for a binomial distribution
$X \sim \mathrm{B}(48, 0.3)$. Calculate $\mathrm{P}(X > 20)$.

The values of X which are included are 21, 22, 23, …, 48.

Calculator which gives $\mathrm{P}(X \leqslant x)$	Calculator which uses upper and lower limits
$\mathrm{P}(X > 20) = 1 - \mathrm{P}(X \leqslant 20)$ Binomialcdf TRIALS=n= 48 P(SUCCESS)= 0.3 X=20 $\mathrm{P}(X > 20) = 1 - \mathrm{P}(X \leqslant 20)$ $= 1 - 0.969812994$ $= 0.030187006$	Binomial C.D Data : Variable Lower : 21 Upper : 48 Numtrial : 48 p : 0.3

0.0302

27 Calculate expected value and expected frequencies for a binomial distribution
A biased coin has a probability 0.46 of landing heads. It is tossed 20 times. What is the expected number of heads?

Expected number $= np = 20 \times 0.46 = 9.2$

28 Understand terminology associated with hypothesis testing
Neil is conducting a statistical hypothesis test. The p-value is 0.03. Neil says this means there is only a 3% chance of the null hypothesis being true. Is Neil right? Justify your answer.

Neil is not right. The p-value is the probability of seeing values at least as extreme as those observed if the null hypothesis is true. It is a conditional probability:

P(observing something this extreme|null hypothesis true).
Neil says it is

P(null hypothesis true|observing something this extreme).
That is the other way round and will not be 3%.

29 Know when to apply 1-tail or 2-tail tests when setting up a binomial hypothesis test
A website claims that half of men don't like football. Sadiq thinks that in his college most men like football. He will take a random sample of 50 male students to carry out a hypothesis test. State suitable null and alternative hypotheses for this test.

You can state the hypotheses either in term of the proportion of men who **do** like football or the proportion of men who **do not** like football – it is particularly important to define what p stands for.

$H_0 : p = 0.5$

$H_1 : p > 0.5$ where p is the proportion of male students in the college who like football.

Sadiq is looking for evidence that most male students in the college like football so that is what the alternative hypothesis is.

30 Find a critical region for a binomial hypothesis test
Historical data suggests that 30% of adults in an area have high blood pressure. A researcher suspects that there are more adults with high blood pressure. A random sample of 60 adults from the area will have their blood pressure checked. What is the critical region for a suitable hypothesis test at the 5% level of significance?

The hypotheses for the test are:
$H_0 : p = 0.3$

$H_1 : p > 0.3$ where p is the proportion of adults with high blood pressure.

If the null hypothesis is true then $X \sim B(60, 0.3)$ where X is the number of adults with high blood pressure in a sample of 60.

1-tail test at the 5% level with high numbers of adults with high blood pressure providing evidence in favour of the alternative hypothesis.

Using cumulative binomial probabilities on a calculator:

$P(X \geqslant 24) = 0.0632$

$P(X \geqslant 25) = 0.0362$ so the critical region is $X \geqslant 25$

31 Conduct a hypothesis test using the binomial distribution and interpret the results in context
The researcher from question 30 finds that the number of adults with high blood pressure in the sample is in the critical region. What should she conclude?

There is enough evidence to reject the null hypothesis so she should conclude that there is sufficient evidence at the 5% level that more than 30% of adults have high blood pressure.

32 Calculate probabilities for a simple discrete random variable

A discrete random variable can take the values 1, 2, 3 or 4 with probabilities as shown in the table below.

x	1	2	3	4
$P(X = x)$	0.3	0.3	0.2	k

i Find the value of k.

ii Find $P(X > 1)$.

i The probabilities add up to 1 so

$0.3 + 0.3 + 0.2 + k = 1$

$k = 0.2$

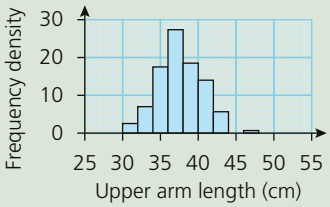
ii $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4)$

$= 0.3 + 0.2 + 0.2$

$= 0.7$

33 Use the Normal distribution as a model

The histogram shows the upper arm lengths, in centimetres, of a sample of American adults.



Does the shape of the histogram suggest that a Normal model is suitable? Justify your answer.

The Normal model appears to be suitable because the histogram is approximately symmetrical and bell-shaped.

34 Calculate and use probabilities from a Normal distribution

Upper arm lengths of adults are Normally distributed with mean 37.4 cm and standard deviation 2.8 cm. Find the probability that a randomly chosen adult has upper arm length greater than 45 cm.

$X \sim N(37.4, 2.8^2)$

$P(X > 45)$

```
Normal    C.D
Data      : Variable
Lower     : 45
Upper     : 10000
σ         : 2.8
μ         : 37.4
```

```
Normal  C.D

P       = 3.3209E−03
Z : Low = 2.71428571
Z : Up  = 3558.0713
```

So the probability is 0.0033

35 Know the sampling distribution of the sample mean from a Normal distribution
Upper arm lengths of adults are Normally distributed with mean 37.4 cm and standard deviation 2.8 cm. What is the distribution of means of random samples of 10 adult upper arm lengths?

$X \sim \mathrm{N}(37.4, 2.8^2)$

For samples of size 10, $\bar{X} \sim \mathrm{N}\left(37.4, \frac{2.8^2}{10}\right)$

36 Identify suitable null and alternative hypotheses for a Normal hypothesis test
Reaction times for adults are Normally distributed with mean 0.2 seconds and standard deviation 0.04 seconds. A researcher suspects that tired adults will have longer reaction times. She will find the reaction times of a random sample of 20 tired adults and conduct a suitable hypothesis test. What should be the null and alternative hypotheses for the test?

$\mathrm{H}_0 : \mu = 0.2$

$\mathrm{H}_1 : \mu > 0.2$ where μ is the mean reaction time in seconds for the population of tired adults.

The researcher is looking for evidence that the reaction time is longer so the alternative hypothesis is that μ is greater than 0.2.

37 Find a *p*-value for a Normal hypothesis test
The researcher conducting the hypothesis test in question 36 finds that the sample mean reaction time is 0.23 seconds. Find the *p*-value. Assume the standard deviation remains 0.04 seconds.

If the null hypothesis is true, $\bar{X} \sim \mathrm{N}\left(0.2, \frac{0.04^2}{20}\right)$ where $\bar{X}$ is the sample mean.

The *p*-value is the probability of observing a sample mean at least as extreme as 0.23 if the null hypothesis is true. This is a 1-tail test.

Need to calculate $\mathrm{P}(\bar{X} > 0.23)$.

Using cumulative Normal probability on a calculator:

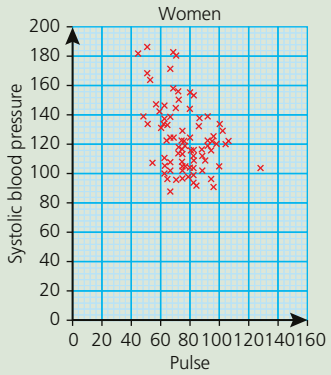
```
Normal    C.D
Data      : Variable
Lower     : 0.23
Upper     : 10000
σ         : 8.9442E−03
μ         : 0.2
```

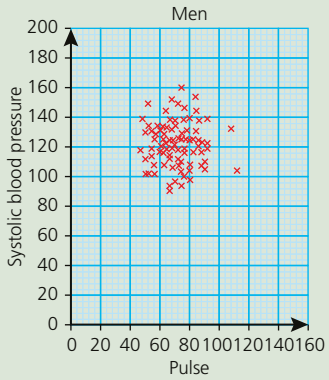
```
Normal  C.D
P       = 3.9812E−04
z : Low = 3.35410197
z : Up  = 1.118E+06
```

The *p*-value is 0.0004 (to 4 s.f.).

38 Interpret a scatter diagram and best fit model, describe correlation

Measurements of pulse rate and blood pressure are used to monitor people's health. Measuring pulse rate does not need special equipment so it can be done by anyone. A student is investigating whether measuring pulse rate also gives information about blood pressure. He takes a random sample of adults and draws the following scatter diagrams for systolic blood pressure, in mm Hg, against pulse rate, in beats per minute.



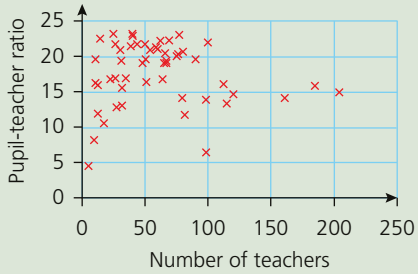


The correlation coefficient for women is –0.375 and for men is 0.005.
The student says that systolic blood pressure could be calculated from pulse rate for women but not for men. Is the student correct? Justify your answer.

There is no relationship between pulse rate and systolic blood pressure for men so the student is right to say that the systolic blood pressure could not be calculated from the pulse rate for men. There is negative correlation for women so a line of best fit could be drawn and it would be possible to calculate the systolic blood pressure from the pulse rate but the results would not be reliable – there is a lot of scatter in the scatter diagram.

39 Recognise outliers and different groups in scatter diagrams

The scatter diagram shows the number of teachers and the pupil–teacher ratio in all the schools in one area of the country. The correlation coefficient is –0.06.



Make two distinct comments about what the scatter diagram shows.

Two distinct correct comments, e.g.

- There is no correlation between number of teachers and pupil-teacher ratios.
- The pupil-teacher ratio for larger schools is fairly consistent at around 15.
- There is a lot of variability in pupil-teacher ratio in smaller schools.
- It is likely that more than one type of school is represented in the scatter diagram.

40 Use a given correlation coefficient to make an inference about correlation or association
In a triathlon event, athletes swim, cycle and run. A sports scientist suspects there is positive correlation between the cycling times and the running times so he conducts a hypothesis test using a random sample of 20 athletes. The correlation coefficient is 0.52; the 1-tail p-value is 0.0094 and the 2-tail p-value is 0.0188. What conclusion should the sports scientist draw from the test at the 1% level of significance?

This is a 1-tail test with the following hypotheses.

$H_0 : \rho = 0$

$H_1 : \rho > 0$ where ρ is the population correlation coefficient

0.0094<1% so the p-value is smaller than the significance level.

There is enough evidence to reject the null hypothesis.

There is sufficient evidence at the 1% level to suggest that there is positive correlation between cycling time and running time for triathletes.

Data collection (page 11)

A market researcher wants to know whether customers of a particular supermarket can taste the difference between the supermarket own brand digestive biscuits and a well-known brand.

i What is the population for this investigation?

ii Explain why it is not possible to take a simple random sample from the population.

The market researcher will visit one branch of the supermarket one Monday morning and ask 20 customers to taste both types of biscuit. She will choose 10 males and 10 females.

iii State the sampling method being used.

iv Suggest one possible improvement to the sampling method to get more reliable results from the investigation.

i All customers of the supermarket.

> The investigation is about customers of the supermarket so the population is all customers of the supermarket.

ii There is no list of all customers.

> Simple random sampling needs a sampling frame; this is often a list of the population and there is no list of all the customers of a supermarket.

iii Quota sample.

> The researcher will choose 10 males and 10 females; strata in the population are being used but the researcher chooses the members of the sample herself so this is a quota sample.

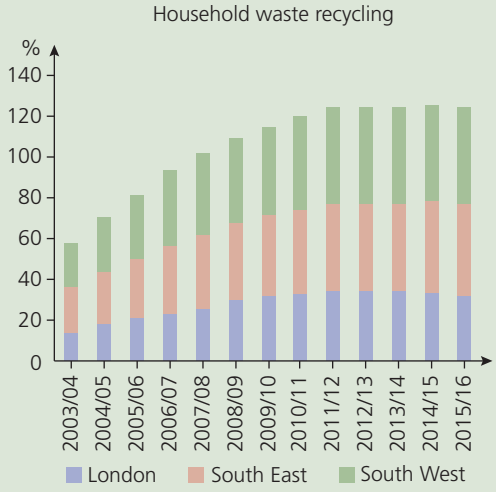
iv One possible improvement. Two possible examples are given below.

- Increase sample size.
- Sample at more than one time of day.

> - There are several possible improvements; you only need to give one improvement. Giving any more answers than you have been asked for wastes time. Examiners will have to choose which ones to mark according to strict exam board rules. If you change your mind about an answer, cross out the one you no longer want.
> - In addition to the two improvements suggested above, it might be possible to sample at random from a list of customers who had registered to have a loyalty card.

Interpreting graphs for one variable (page 18)

The two charts below have been drawn using a spreadsheet. They show the percentage of household waste recycled in three regions of England.



i One of the charts is not an appropriate way to display the data. State which one and why it is not appropriate.

ii Make two statements about the percentage of household waste recycled over the time period shown.

iii A newspaper reports that less than a third of households in London recycle their waste. Explain why this statement is unlikely to be true.

i The first chart is not appropriate because adding percentages for different areas is not meaningful.

> The later bars have percentages which add up to more than 100%; this is usually a sign that there is something wrong with the graph. A compound bar chart should only be used if the total height of each bar means something and in this case it does not.

ii The recycling rate in each area has gone up over the time period. In each area, the rate has stopped increasing.
Other answers include: recycling rates in London are lower than in the other two areas.

iii The chart shows that about 30% of household waste in London is recycled. This is less than a third but it does not refer to the percentage of households. If less than a third of households recycled their waste, they would need to recycle everything while the other households recycled nothing.

> **Common mistakes:**
> - It is easy to be confused about what a percentage is showing; look carefully at the information. These charts show percentage of household waste recycled but the statement is about households; these are not the same thing. Point this out in your answer.
> - Notice that the question asks why the statement is unlikely to be true so you need to do more than just point out that the percentage has been misinterpreted. Imagine that the statement is true; if less than one third of households recycled waste (and the rest of households did not) then to fit in with the 30% figure in the chart the households which recycled would have to recycle nearly all their waste but this is not possible so the statement cannot be true.

> **Hints:**
> - Look at the trend shown for each area. You can also compare the three areas – is there a consistent difference between them? You are asked to make two statements, so this is what you should do; your two statements should be different – be careful not to make one statement in two different ways.
> - There are more than two possible statements which could be made.

Averages (page 23)

The table below shows the ages of the residents of the City of London at the time of the 2011 census.

Age	0–15	16–19	20–29	30–44	45–59	60–74	75–89	90 and over	Total
Frequency	620	164	1501	2045	1547	1050	406	42	7375

i Alan is calculating an estimate of their mean age. He starts by saying that the midpoint of the first group is 7.5. Is this correct? Explain your answer.

ii Britney correctly calculated an estimate of the mean age as 42.3 years (to 1 d.p.). To do this, she had to choose an upper age for the last group. Show that this must have been at least 94.

iii The National Statistics website reports the mean age as 41.4. Why is this different from the mean Britney calculated?

i 7.5 is not correct; this is half way between 0 and 15 but the oldest person in the group could be nearly 16 so the midpoint is $\frac{0+16}{2} = 8$.

The ages in the 0 to 15 group can be as low as 0 but the oldest person in the group could be just one day short of their 16th birthday; you can think of the group as being 0 to 16.

ii

Age	0–15	16–19	20–29	30–44	45–59	60–74	75–89	90 and over	Total
Midpoint	8	18	25	37.5	52.5	67.5	82.5	x	
Frequency	620	164	1501	2045	1547	1050	406	42	7375

Start by working out the midpoints of the groups; write them down so that you can use them and so that the examiner can see your working.

For the first group, the midpoint is 8; you worked this out in part i.

For the second group, remember it is ages 16 up to 20 and so the midpoint is 18.

You cannot work out the midpoint for the last group yet. You might want to use a letter (like x) to stand for this value.

	List 1	List 2	List 3	List 4
SUB	midpt	freq		
5	52.5	1547		
6	67.5	1050		
7	82.5	406		
8				

You need to find the sum of the data values (estimated by the midpoints) for the first seven groups. You can do this by entering the data into your calculator.

1–Variable

$\bar{x}$	=41.9626346
Σx	=307712
Σx^2	=1.5718E+07
σx	=19.5602645
sx	=19.5615984
n	=7333

The sum of the data values is 307 712.

Notice that the number of data values is 7333; this is 42 less than the total 7375 and is the number you should have without the last group.

From calculator, $\sum xf$ for the first seven groups is 307 712.

Write down the total you have from your calculator, saying what the number represents.

$$\frac{307\,712 + 42x}{7375} = 42.3$$

You know that Britney calculated the estimated mean to be 42.3; this is the total of all the midpoints divided by the number of people so you can form an equation, using x to stand for the midpoint of the final group.

You could try to solve the equation to find x but the mean has been rounded and this may lead to inaccuracy. Your task is to show that Britney assumed that the last group went up to at least 94 so you need to show that assuming 94 will give the calculated mean but 93 will not.

Suppose Britney assumed the last group was 90 to 94. The oldest person in the group could have been just less than 95 so the midpoint would be 92.5.

$$\frac{307\,712 + 42 \times 92.5}{7375} = 42.2504... = 42.3 \text{ (1d.p.)}$$

If Britney assumed the last group was 90 to 93, the midpoint would be 92.

$$\frac{307\,712 + 42 \times 92}{7375} = 42.2475... = 42.2 \text{ (1d.p.)}$$

So assuming the last group went up to age 93 would not give the calculated mean hence she must have assumed the group went up to at least 94.

Hint: This shows that assuming the last group is 90 to 94 will work; it is important that you show the unrounded mean you would get because the 42.3 was given in the question.

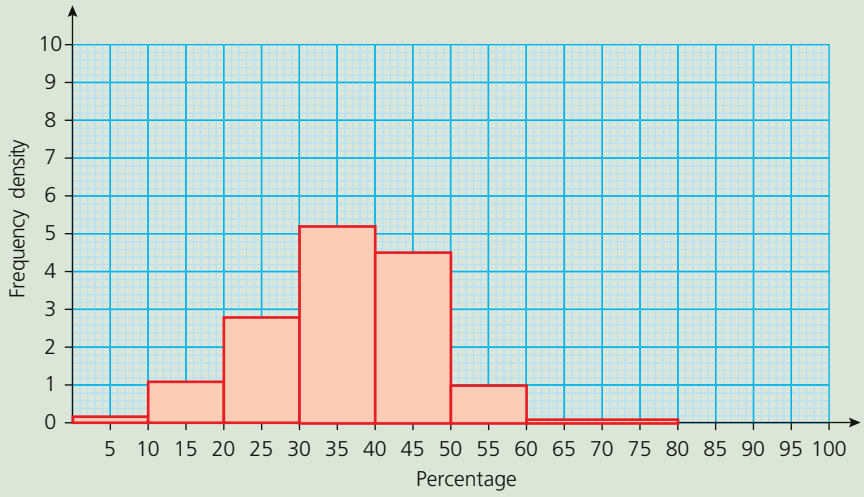
Now that you have completed the task of showing that assuming that the final group goes up to 94 does give the stated mean but assuming it goes up to 93 does not, you should state your conclusion clearly.

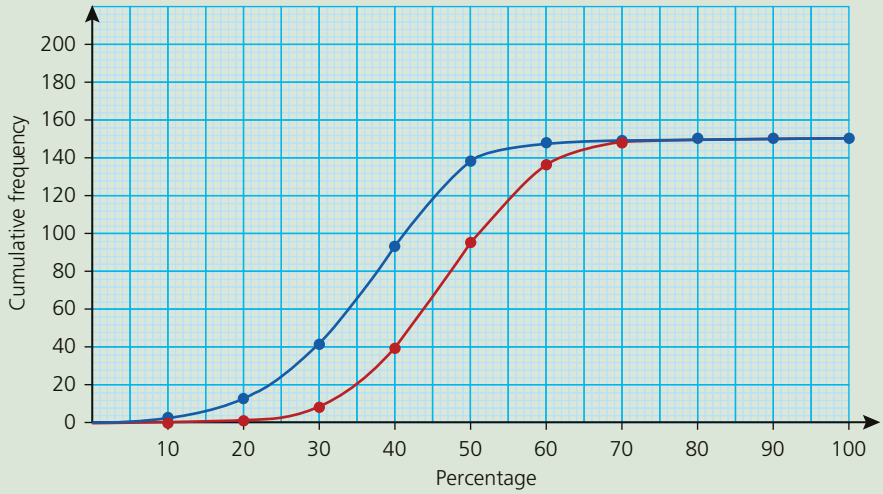
iii Britney's answer is only an estimate because the data were grouped. The exact ages are not known and so the midpoint of each group has been used to estimate them. The National Statistics' value could have used the exact ages.

It might occur to you that choosing a different upper limit for the last age group would give a different mean but it could not give 41.4; notice that your working without the last group gives a mean of 41.96 – no matter what upper limit you choose for the last group, the mean will be over 42 so the difference between Britney's working and the official statistic must be due to the estimation from working with grouped data.

Diagrams for grouped data (page 32)

150 students sat two exam papers. The histogram shows the percentages they gained in paper 1.





i One of the cumulative frequency graphs is for the same data as the histogram. Determine which one.

The other cumulative frequency graph is for the data from paper 2.

ii **A** Find the number of students who got between 50% and 60% on paper 1.

B Find the number of students who got between 50% and 60% on paper 2.

iii Compare the students' performance on the two papers.

i For the blue graph, in the 20 to 30 group there are $40 - 12 = 28$ people.

> The key differences between the graphs are that the red graph goes up more slowly at the start so it makes sense to look at a group where the numbers on both graphs are different.

For the red graph, there are $8 - 0 = 8$ people.
In the histogram, there are $10 \times 2.8 = 28$ people.

> The height of the 20 to 30 group is 2.8 and the width is 10. The frequency is the area of the bar so work out width times height.

The blue graph must be the one representing paper 1.

> You need to show that the blue graph is consistent with the data in the histogram; it is also helpful to show that the red graph is not; the difficulty of reading exactly from graphs could result in the numbers from the histogram and cumulative frequency graph not matching exactly.

ii A From the histogram, it is $1 \times 10 = 10$.

> You could use either the histogram or the cumulative frequency graph to calculate this – it is easier to read accurately from the histogram and you are sure that this is for Paper 1 so it is best to use that.
>
> The height of the 50 to 60 bar is 1; the width is 10. The area is length times width and this gives the frequency.

B $136 - 94 = 42$

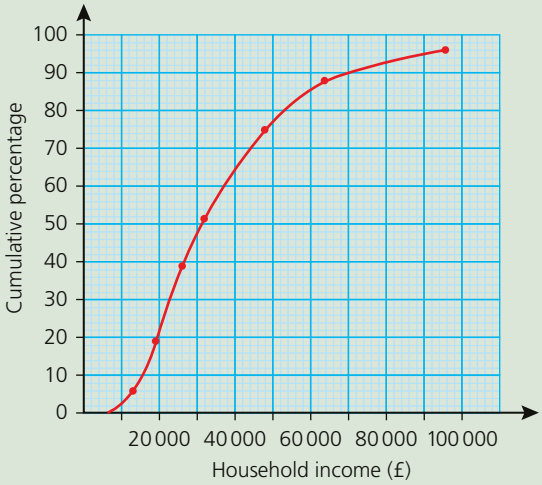
> The number of people with 60% or less minus the number with 50% or less will give the number between 50 and 60.
>
> The values 136 and 94 are difficult to read off exactly; values close to 42 coming from correct working would be acceptable here.

iii Overall, the students did better on paper 2 than on paper 1.

> The cumulative frequencies go up slower at first for the red graph so there are fewer students in the lower mark categories on this paper (and so more in the higher mark categories).

Median and quartiles (page 40)

The cumulative percentage graph shows the annual income of households in the UK in 2015–16.



i 3.8% of households had an income over £96 000; they do not appear on the graph. What additional information is needed for these households to be added to the graph?

ii A Explain why you should read off from 50 on the vertical axis to estimate the median income of **all** households from the graph.

B Estimate the median income of **all** households from the graph.

iii What household incomes might be considered to be outliers?

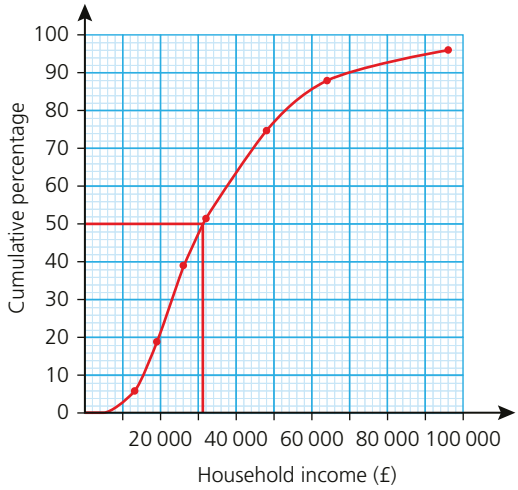
i The upper limit of the group's earnings.

> The households with income over £96 000 would be plotted at the upper end of the graph but the highest income in the group would be needed to plot the point.

ii A There are 100% of households altogether and half of this is 50%.

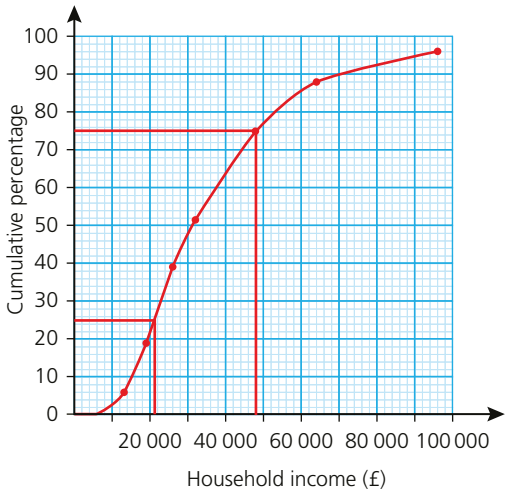
> Notice that not being able to plot the data for the highest earners does not prevent us calculating the median because the middle of the graph is needed for this.

B Reading off from a cumulative percentage of 50 gives a value between £30 000 and £32 000; any value in this range will be acceptable.



The median income is £31 000.

The method for outliers using quartiles will be the easiest one to use here so use the graph to find the quartiles and the interquartile range. The values for all households are needed so you should read off from 25 and 75 on the vertical axis.



iii Lower quartile = £22 000
Upper quartile = £48 000

It is hard to read accurately from the graph; values close to these are acceptable.

IQR = £48 000 − £22 000 = £26 000
1.5 × IQR = 1.5 × £26 000 = £39 000

It is not possible to be £39 000 below the lower quartile but you should still show this.

£22 000 − £39 000 is negative so no low outliers.
£48 000 + £39 000 = £87 000
Those with income over £87 000 would be considered outliers.

Standard deviation (page 45)

The table below shows the heights in cm for a sample of 58 women

Height, x cm	Frequency
$145 < x \leqslant 150$	5
$150 < x \leqslant 155$	5
$155 < x \leqslant 160$	19
$160 < x \leqslant 165$	15
$165 < x \leqslant 170$	11
$170 < x \leqslant 175$	3

i Find estimates of:
- the mean
- the standard deviation.

ii For this sample of 58 women, using the ungrouped data gives $\sum x = 9278$ and $\sum x^2 = 1\,486\,243.38$.

Find:
- the mean
- the standard deviation.

iii Comment on any differences between your answers to parts **i** and **ii**.

i

The estimates will both be given at once using calculator statistical functions; the midpoints and frequencies of the groups must be entered into a calculator. It is helpful to write down the midpoint for each group.

Height, x cm	Frequency	Midpoint (cm)
$145 < x \leqslant 150$	5	147.5
$150 < x \leqslant 155$	5	152.5
$155 < x \leqslant 160$	19	157.5
$160 < x \leqslant 165$	15	162.5
$165 < x \leqslant 170$	11	167.5
$170 < x \leqslant 175$	3	172.5

Now enter the midpoints and frequencies into a calculator. Make sure the calculator is set up to use the frequencies.

1–Variable
$\bar{x}$ =160.172413
Σx =9290
Σx^2 =1.4903E+06
σx =6.37989279
sx =6.43561344
n =58

The mean is 160.17 cm (2 d.p.).

The standard deviation is 6.38 cm (2 d.p.).

Write down the mean and standard deviation, rounding answers sensibly.

Check that the value of n is the number of data values, 58 in this case, and check that the mean is between the lowest and highest values of data.

Remember that the standard deviation you want is the smaller of the two possible values which the calculator gives.

ii $\bar{x} = \frac{9278}{58} = 159.965 \ldots$

Work out the mean first and round your answer sensibly.

The mean is 159.97 cm (2 d.p.).

You will need to work out S_{xx} before working out the standard deviation.

$$S_{xx} = \sum x^2 - \frac{\left(\sum x\right)^2}{n} = 1486\,243.38 - \frac{9278^2}{58}$$

$$S_{xx} = 2083.311 \ldots$$

$$\text{standard deviation} = \sqrt{\frac{S_{xx}}{n}} = \sqrt{\frac{2083.311 \ldots}{58}}$$

$$\text{standard deviation} = \sqrt{35.919 \ldots} = 5.993\,325\,9 \ldots$$

Hint: Do not round until you have got your final answer; then round your answer sensibly.

The standard deviation is 5.99 cm (2 d.p.).

iii The answers in part **i** are estimates because the data have been grouped and so they are less accurate than the values in part **ii**.

Remember that when you work with grouped data by using midpoints this loses information from the original data and so the mean and standard deviation are estimates and not as accurate as you could get from the raw data but they are usually good enough to work with. A different sample from the population would have given slightly different values of mean and standard deviation anyway.

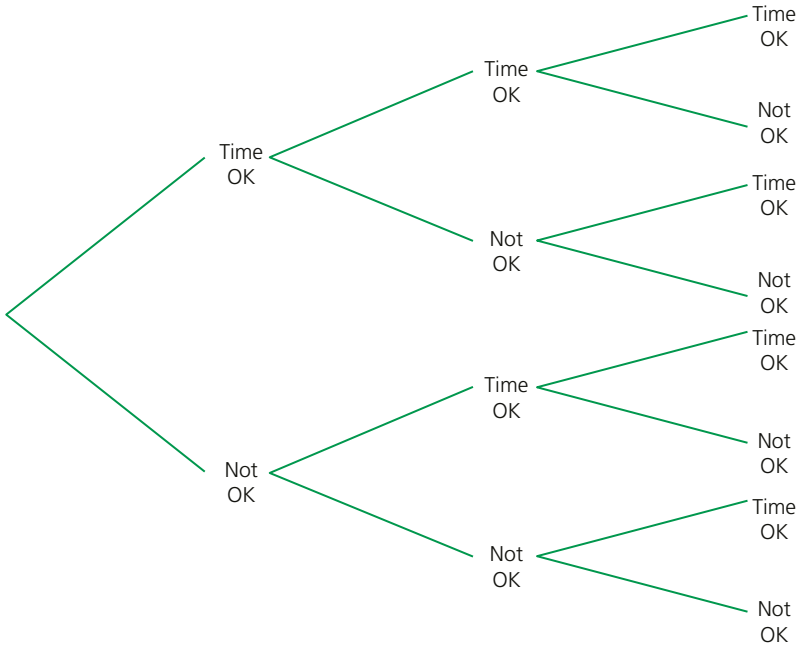
Working with probability (page 54)

To be in the finals of a swimming competition, swimmers need to meet the qualifying time in at least two out of three trial swims. Based on his previous performance, Duncan estimates his probability of meeting the qualifying time in the first trial as 0.8. If he meets the time in any trial, his probability of succeeding (i.e. meeting the qualifying time) in the next trial is 0.8, but if he does not meet the qualifying time in a trail, his probability of succeeding in the next trial is reduced to 0.6.

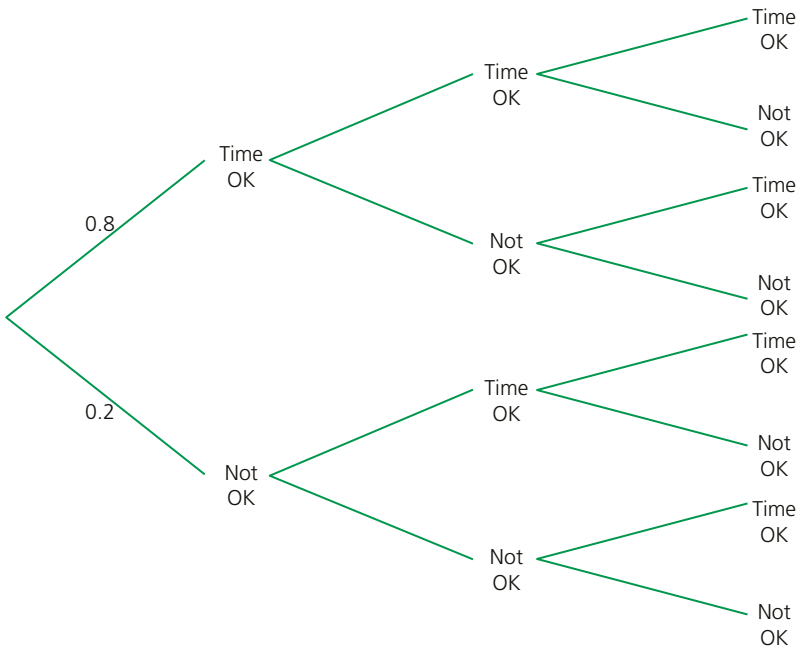
i Find the probability that Duncan will succeed in meeting the qualifying time in at least one trial.

ii Find the probability that Duncan will be in the finals of the competition.

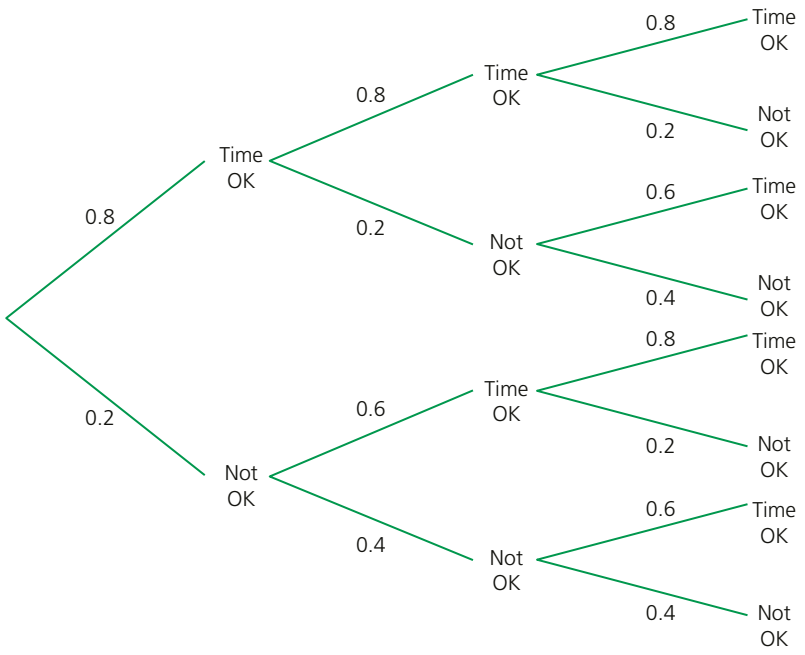
The changing probabilities in this question could be hard to keep track of without a tree diagram so that is the best way to start the question. There are three trial swims and, for each one, either Duncan meets the qualifying time or he does not. Start the tree diagram by drawing the branches and putting the outcomes on the ends of them.



At each trial you are interested in whether he meets the qualifying time or not so these are the two outcomes which should go on the branches.



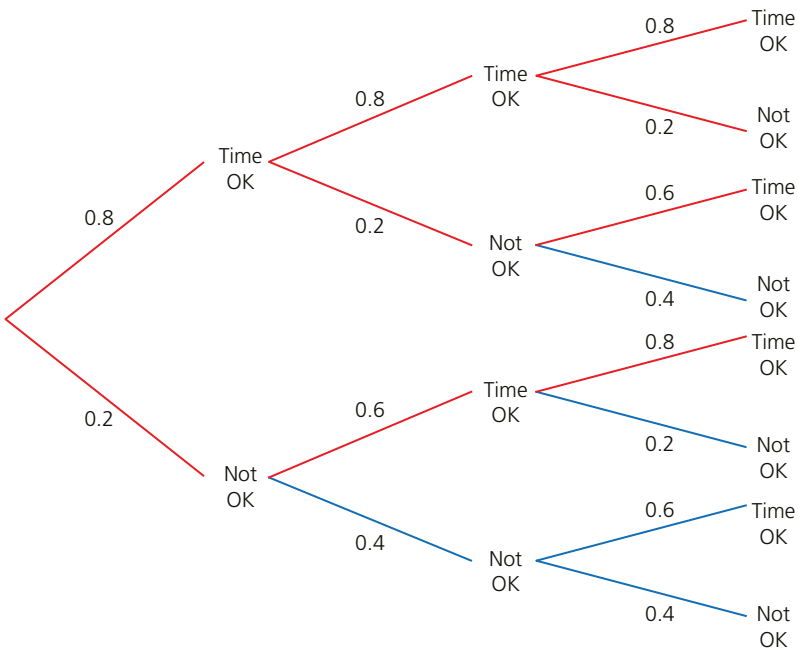
- The probability of meeting the time for the first trial is 0.8. This means that the probability of not meeting the time is 0.2.
- The probability that he meets the qualifying time is not always 0.8 so be careful not to fill in all the probabilities automatically.



- If he meets the time in any trial, his probability of succeeding (i.e. meeting the qualifying time) in the next trial is 0.8. You can use this information to fill in the probabilities for every pair of branches that follow the event "Time OK". The probability of not meeting the qualifying time is 0.2, as before.
- If he does not meet the qualifying time in a trial, his probability of succeeding in the next is reduced to 0.6. This information will allow you to fill in the probabilities for pairs of branches that follow the event "Not OK". The probability of not meeting the qualifying time for these pairs of branches is 1 – 0.6 = 0.4.

i P(meets the qualifying time in at least one trial) = 1 − P(never meets qualifying time).
P(never meets qualifying time) = $0.2 \times 0.4 \times 0.4 = 0.032$
P(meets the qualifying time in at least one trial) = $1 - 0.032 = 0.968$

ii In part **ii**, you are asked to find the probability that Duncan will be in the finals. This means that he has met the qualifying time at least twice. Find the routes in the tree diagram where this happens; they are shown in red below.



Hints:

- In part **i** you are asked to find the probability he meets the qualifying time in at least one trial. The easiest way to find this is to first find the probability that he never meets the qualifying time.
- The branches where he never meets the qualifying time are the lowest ones in the tree diagram.

$P(\text{OK, OK, OK}) = 0.8 \times 0.8 \times 0.8 = 0.512$
$P(\text{OK, OK, not OK}) = 0.8 \times 0.8 \times 0.2 = 0.128$
$P(\text{OK, not OK, OK}) = 0.8 \times 0.2 \times 0.6 = 0.096$
$P(\text{not OK, OK, OK}) = 0.2 \times 0.6 \times 0.8 = 0.096$

- Work out the probability of each set of three outcomes separately.
- These possibilities are mutually exclusive, only one of them could have happened, so the probability of Duncan being in the finals is found by adding them.

$$P(\text{in finals}) = 0.512 + 0.128 + 0.096 + 0.096$$
$$= 0.832$$

Conditional probability (page 57)

Bill's cat has eight kittens: three female and five male. Alex is going to have two of them.

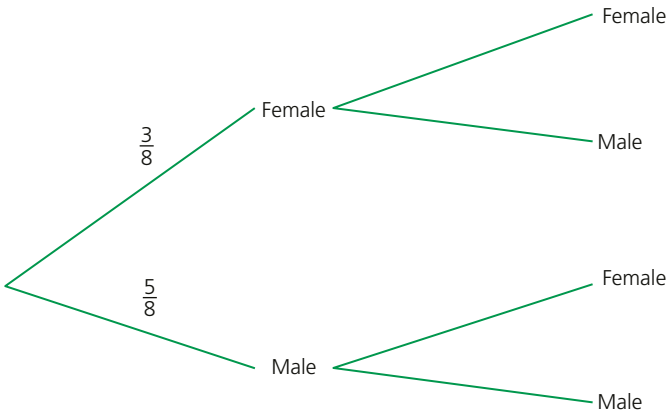
Two of them are selected at random for Alex.

i What is the probability that two females are selected?
ii What is the probability that one of each gender is selected?
iii What is the probability that two females are selected given that both kittens chosen are of the same gender?

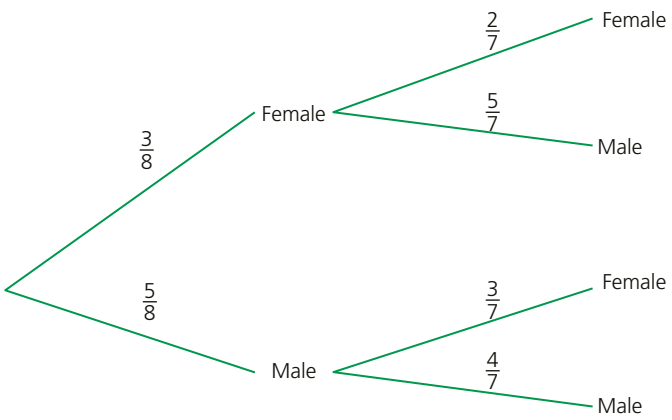
A good way to start is by drawing a tree diagram.

Conditional probability will feature in the tree diagram because once one kitten is chosen, there is one less to choose from at the next stage.

You want to know about the genders of the kittens so make the ends of the branches of the tree diagram refer to this.



For the first kitten, there are 8 to choose from: 3 female and 5 male. The probability of the first kitten chosen being female is 3 out of 8; it is 5 out of 8 for male. Put in the probabilities on the first pair of branches.



For the second kitten, there are 7 to choose from.

- If the first kitten chosen was female, there are 2 female and 5 male kittens left. So the probabilities on the branches after the first kitten being female are $\frac{2}{7}$ for female and $\frac{5}{7}$ for male. Put in the probabilities on the 1st pair of branches for the second kitten.
- If the first kitten chosen was male, there are 3 females and 4 males left to choose from. Fill in the remaining probabilities. They are $\frac{3}{7}$ for female and $\frac{4}{7}$ for male.

i $P(\text{female, female}) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

ii $P(\text{female, male}) = \frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$

$P(\text{male, female}) = \frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$

In part **ii**, you are asked to find the probability of each gender; it could either be female then male or male then female. These two possibilities are mutually exclusive; they cannot both happen together, so you add the probabilities to get the overall probability of one of each gender.

$P(\text{one of each}) = P(\text{female, male}) + P(\text{male, female})$

$P(\text{one of each}) = \frac{15}{56} + \frac{15}{56} = \frac{15}{28}$

iii $P(\text{both female} \mid \text{both same gender})$

In part **iii** you are asked to find the probability that two females are selected given that both kittens chosen are of the same gender. The words 'given that' tells you this involves conditional probability. You will find it easiest to use the formula correctly if you use words in it rather than trying to have letters stand for the events.

$$P(\text{both female} \mid \text{both same gender}) = \frac{P(\text{both female} \cap \text{both same gender})}{P(\text{both same gender})}$$

Using the formula $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$.

One advantage of using words in the formula is that you can more easily see that the numerator simplifies. If the kittens are both female **and** both the same gender then that simply means that they are both female.

$$P(\text{both female} \mid \text{both same gender}) = \frac{P(\text{both female})}{P(\text{both same gender})}$$

You have already worked out the probability that both are female in part **i**. You need to work out the probability that they are both the same gender. They could either be both female or both male.

$P(\text{both same gender}) = P(\text{both female}) + P(\text{both male})$

$P(\text{male, male}) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$

$P(\text{both same gender}) = \frac{3}{28} + \frac{5}{14} = \frac{13}{28}$

Now you are ready to finish part **iii**.

$$P(\text{both female} \mid \text{both same gender}) = \frac{P(\text{both female})}{P(\text{both same gender})}$$

$$\begin{aligned} P(\text{both female} \mid \text{both same gender}) &= \frac{3}{28} \div \frac{13}{28} \\ &= \frac{3}{28} \times \frac{28}{13} \\ &= \frac{3}{13} \end{aligned}$$

Introducing the binomial distribution (page 63)

10% of the population are dyslexic. A school has classes which contain 30 students.

i Use a binomial probability model to find the following probabilities.

A The probability that a class has one dyslexic student in it.

B The probability that a class has at least one dyslexic student in it.

ii State one assumption needed for the binomial model you used to be valid.

i $X \sim B(30, 0.1)$ where X is the number of dyslexic students in a class.

- This question tells you to use a binomial probability model; the parameters needed for the model are n and p.
- n is the number of trials; in this case a trial is finding out whether a student in a class is dyslexic so $n = 30$.
- p is the probability of success; in this case success is finding out that a student is dyslexic – that is what we are interested in here.
- The random variable in this case is the number of dyslexic students in a class.

A $P(X = r) = {}_nC_r p^r q^{n-r}$

$P(X = 1) = {}_{30}C_1 \times 0.1^1 \times 0.9^{29} = 0.1413$ (to 4 d.p.)

To find the probability of one student being dyslexic, you can either use the binomial probability formula or you can find the probability directly using your calculator binomial probability function. You should round your answer sensibly.

B $P(X \geqslant 1) = 1 - P(X = 0)$

Remember that the probability of at least one event is one minus the probability that the event does not happen.

$P(X = 0) = {}_{30}C_0 \times 0.1^0 \times 0.9^{30} = 0.0423...$

Don't round the probability of no dyslexic students; that is not your final answer but do use your calculator memory or the ANS key to avoid typing the number in to your calculator.

$P(X \geqslant 1) = 1 - 0.0423... = 0.9576$ (4 d.p.)

ii For the binomial model used to be valid, the probability of students at the school being dyslexic needs to be 0.1, the same as in the general population.

The conditions for a binomial distribution are as follows:

- You are conducting an experiment or trial n times (a fixed number); there is no problem with this one as you are seeing whether each of 30 students is dyslexic.
- There are two possible outcomes; in this case either the student is dyslexic or not.
- The probability of 'success' is the same each time; in this case success is a student being dyslexic – it may be that the probability is not 0.1 in each case; the school could have more (or less) than average dyslexic students in their area.

An alternative assumption you could put down is that students are randomly assigned to classes; it could be that the school puts more dyslexic students in some classes than in others so that they can receive specialist teaching.

Cumulative probabilities and expectation (page 69)

23% of people who live in Wales can speak Welsh.

A researcher will interview a random sample of 50 people who work for a particular employer.

i Give one reason why a binomial distribution with $n = 50$, $p = 0.23$ might not be a suitable model for the number of people in the sample who speak Welsh.

For the rest of this question you should assume that a binomial distribution with $n = 50$, $p = 0.23$ is a suitable model for the number of people in the sample who speak Welsh.

ii Find the probability that at least five people in the sample speak Welsh.

iii The researcher interviews the people one at a time. What is the probability that the first one he interviews who speaks Welsh is the 5th person?

i The percentage of people working for that employer who speak Welsh may be different to 23%. There are areas where Welsh is more widely spoken than in other areas. So, depending on where the employer is based, the probability may be greater or less than 0.23.

Conditions for a binomial distribution are as follows:

- You are conducting an experiment or trial n times (a fixed number); there is no problem with this one as you are seeing whether each of 50 people speak Welsh.
- There are two possible outcomes; in this case either the person speaks Welsh or not.
- The probability of 'success' is the same each time; in this case success is a person speaking Welsh – it may be that the probability is not 0.23 in each case; the employer could have more (or less) than the average percentage of Welsh speakers.

ii $X \sim B(50, 0.23)$ where X is the number of people who speak Welsh in the sample.

$P(X \geqslant 5)$

Possible values are 5, 6, 7, 8, …….50

Hint: At least 5 means 5 or more. You will use calculator functions to find the probability – it will help you to get the right answer if you write down the possible values of X.

Hint: If you are using a calculator which calculates $P(X \leqslant x)$ it will help to think about the values of X you cannot have; they are 0 to 4 inclusive. Now you can write the probability in a way that will allow you to use your calculator.

$P(X \geqslant 5) = 1 - P(X \leqslant 4)$

Hint: If you are using a calculator which will give the probability of a binomial random variable being in a range, the lower value is 5 and the upper value is 50 because the possible values are 5, 6, 7, 8, …….50.

Now enter the information into your calculator. Remember you want cumulative binomial probabilities. The number of trials is 50 with probability of success 0.23.

Binomial C.D X :4 N :50 p :0.23	Binomial C.D Data : Variable Lower : 5 Upper : 50 Numtrial : 50 p : 0.23 Save Res : None
Your calculator will tell you $P(X \leqslant 4)$. P= $5.237815857 \times 10^{-3}$	This will give you the probability you want directly and you just need to round it sensibly. Binomial C.D P=0.99476218
Hint: Now you need to subtract this probability from 1 to get the answer but you should write down the probability you have found first. $P(X \leqslant 4) = 0.005\ 23...$ $P(X \geqslant 5) = 1 - 0.005\ 23...$	

$P(X \geqslant 5) = 0.9948$ (4 d.p.)

iii

Common mistake: This question is about a binomial distribution but this does not mean that other techniques cannot be used. For the last part of the question you will be using the probability of 0.23 and the techniques you know for working out probabilities.

Hint: The first four people he interviews must not be Welsh speakers and then the fifth is a Welsh speaker. P(W', W', W', W', W) where W stands for a Welsh speaker.

$0.77^4 \times 0.23$

The probabilities are multiplied because each of the events is independent of the others.

0.080851…
0.0809 (to 4 d.p.)

Common mistake: Check that your calculator is doing the calculation that you want it to. The screenshot below shows a calculation with '× 0.23' within the index, which is wrong.

$0.77^{4 \times 0.23}$

0.7862695684

Introducing hypothesis testing using the binomial distribution (1-tail tests) (page 76)

A report from a few years ago says that 90% of children (aged 11–16) in an area have mobile phones. A researcher suspects that the percentage with mobile phones is greater now. She asks a random sample of 200 children aged 11–16 from the area and finds that 193 have mobile phones. A suitable statistical test will be carried out to see whether there is evidence that the proportion with mobile phones has increased.

i Write down suitable null and alternative hypotheses.
ii Carry out the test at the 5% level of significance, stating your conclusions clearly.
iii Would your conclusions be different at the 1% level of significance? Explain your reasoning.

This is a test of a population proportion so it will be a binomial hypothesis test. The hypotheses for a binomial hypothesis test should be stated in terms of p. Your first task is to decide what p stands for. This question is about the proportion of children in an area who have mobile phones, so that is what p stands for. Your answer to part **i** will consist of the two hypotheses and a statement of what p stands for.

i $H_0 : p = 0.9$

The null hypothesis is always of the form that p is equal to some particular value, 0.9 in this case. It expresses the belief that there has been no change.

$H_1 : p > 0.9$

The test will look for evidence that there has been an increase in the proportion of children with mobile phones; so the suspicion is that p is greater than 0.9. This is the alternative hypothesis

where p = the proportion of children aged 11-16 in the area who have mobile phones.

p will always refer to the whole population and not just to the sample. You do not need to test the proportion in the sample; you just work it out.

ii X = the number of children in a sample who have mobile phones

A sample of 200 children has been taken and the number of them who have mobile phones has been observed. This will be the random variable, X. Although you know how many had phones in this particular sample, it would be different for different random samples.

If H_0 is true, $X \sim B(200, 0.9)$.

- The size of the sample is 200 and, if the null hypothesis is true, $p = 0.9$. You should write down what the distribution of X is, assuming that H_0 is true. The distribution will be binomial because you are repeating the action of choosing someone at random 200 times and, each time, the chance of getting someone who has a mobile phone is 0.9, assuming the null hypothesis is true.
- You are looking for evidence that p is greater than 0.9 so large values of X will provide evidence for p being large. The question does not tell you whether you should work out a p-value or a critical region so you can choose which method to use. You will probably find it quicker to work out the p-value.

Need $P(X \geqslant 193)$

- To work out a p-value, notice that the observed value of X was 193; larger values would indicate that the alternative hypothesis could be true. You need to find the probability that X is greater than or equal to 193.
- You should use the cumulative binomial function on your calculator. Remember n is 200 and p is 0.9.

$P(X \geqslant 193) = 1 - P(X \leqslant 192)$

```
Binomialcdf
TRIALS=n= 200
P(SUCCESS)= 0.9
X=192
                    CALC
```

```
Binomial    C.D
Data        : Variable
Lower       : 193
Upper       : 200
Numtrial    : 200
p           : 0.9
Save Res    : None
```

```
Binomialcdf
VALUE= 0.999515003
STORE: No yztabcd
SOLVE AGAIN QUIT
```

```
Binomial  C.D
        p=4.85E−04
```

$P(X \geqslant 193) = 1 - 0.999\,515\,003$

$P(X \geqslant 193) = 0.000\,485$
$0.000\,485 < 0.05$

Compare the p-value with the significance level; you will need them both as decimals or percentages to do so. In this case, the p-value is very small and so there is evidence in favour of the alternative hypothesis.

Common mistake: Be careful not to get confused between the following two things which sound very similar.

- The value of p in the null hypothesis (0.9 in this case).
- The p-value; this is a technical term meaning the probability of getting values at least as extreme as the observed value, if the null hypothesis is true.

There is sufficient evidence, at the 5% level, to suggest that the proportion of children with mobile phones has increased.

iii For the last part, you are asked whether your conclusion would have been different at the 1% level of significance. Having a p-value allows you to do this quickly; the p-value is still smaller than the significance level so the conclusion does not change.

No because the p-value is 0.05% which is smaller than 1%.

2-tail tests (page 81)

A plate factory has historically had 5% of the plates produced being faulty. Following changes to production methods, the management wants to investigate whether the proportion of faulty plates has changed. A random sample of 40 plates contains 1 faulty plate.

i Write down suitable hypotheses.
ii What is the critical region for a suitable hypothesis test, at the 5% level?
iii Carry out the test, stating your conclusions clearly.
iv If the null hypothesis is true, what is the probability of the test statistic being in the critical region?

The null hypothesis will always be p = a particular value. The historical percentage of faulty plates is 5% so this will give you the null hypothesis; you will only believe there has been a change if the evidence is strong enough. It is best to write the hypotheses using decimals, rather than percentages.

i $H_0 : p = 0.05$

$H_1 : p \neq 0.05$

where p is the proportion of faulty plates produced.

The management are looking for evidence of a difference so the alternative hypothesis will be that p is not equal to 0.05.

p is a letter you have introduced into the question so you should say what it stands for.

Remember that the hypotheses should be written before gathering the data so do not let yourself be confused by the fact that they only found one faulty plate.

ii If H_0 is true, $X \sim B(40, 0.05)$ where X is the number of faulty plates in a sample.

- To find the critical region you need to assume that the null hypothesis is true and write down the distribution for the sample. This is a binomial situation. You are conducting 40 trials so $n = 40$. In each of these you select a plate at random and find out whether it is faulty. There are two possible outcomes. These are 'success' when the plate is faulty (this is what you are looking for) and non-success or 'failure' when the plate is not faulty. The probability of getting a faulty plate is constant, at 0.05, according to the null hypothesis, and independent of outcomes from other plates. The number of plates from the sample of 40 which are faulty is denoted by X.
- X is a letter you have introduced so you should say what it stands for.
- The alternative hypothesis is that p is not equal to 0.05 so the test is 2-tailed. Either large, or small, numbers of faulty plates in the sample will make you doubt the null hypothesis.
- For a 5% level of significance 2-tailed critical region, you need 2.5% for each tail.
- For the lower tail, look for a probability of 2.5% (or less).

$P(X \leqslant 0) = 0.1285 = 12.85\%$

$12.85\% > 2.5\%$ so the critical region for the lower tail is empty.

For the upper tail, use cumulative binomial probabilities to look for probabilities of the form $P(X \geqslant x)$ which are close to 2.5%. Write down each probability you calculate so that you can keep track of your working.

$P(X \geqslant 5) = 0.0480 = 4.8\%$
$P(X \geqslant 6) = 0.0139 = 1.39\%$
$1.39\% < 2.5\%$ but $4.8\% > 2.5\%$ so the critical region for the upper tail is $X \geqslant 6$.
The critical region is $X \geqslant 6$.

It is helpful to write down the overall critical region; this will help you to make a decision in the hypothesis test in part **iii**. In this case, the critical region for the lower tail was empty so the critical region for the upper tail is all there is.

iii $X = 1$. This is not in the critical region so do not reject H_0.

Since you have worked out the critical region in part **ii**; you will be able to get the answer to part **iii** very quickly if you use it.

The critical region tells you the values of X that would make you reject the null hypothesis. The observed value of X was 1. This is not in the critical region since $1 < 6$ so you do not reject the null hypothesis.

There is no evidence, at the 5% level, that the proportion of faulty plates has changed.

Remember to relate your decision to the original situation. The management were investigating whether the proportion of faulty plates was different from 5%. You can't say this for definite without checking all the plates but the sample does not provide evidence of any difference. You should really say that there is no evidence of difference at the 5% significance level. If a different level of significance was used, this could lead to a different critical region and so a different conclusion.

iv If the null hypothesis is true, the probability of the test statistic being in the critical region is 1.39%.

Discrete random variables (page 87)

A spreadsheet shows four random integers, each between 1 and 4 inclusive, in a row. X is the number of random digits needed (starting from the left) to get a total of 4 or more. For example, when the random digits are as shown below, $X = 4$ because $1 + 1 + 1 = 3$ and $1 + 1 + 1 + 2 = 5$.

1	1	1	2

The probability distribution for X is shown in the table below.

r	1	2	3	4
$P(X = r)$	$\frac{1}{4}$	$\frac{9}{16}$	p	$\frac{1}{64}$

i Find the value of p. **ii** Show that $P(X = 1) = \frac{1}{4}$. **iii** Show that $P(X = 2) = \frac{9}{16}$.

i $\frac{1}{4} + \frac{9}{16} + p + \frac{1}{64} = 1$

You have all the probabilities in the table in the question, apart from p, so you can use the fact that the probabilities for a discrete random variable have to add up to 1. Start by writing this down so that it is clear how you are doing the question.

$\frac{53}{64} + p = 1$

Write down the total of the known probabilities as a fraction. This will keep your working clear.

$p = 1 - \frac{53}{64}$

$p = \frac{11}{64}$

ii X is the number of random digits needed (starting from the left) to get a total of 4 or more.
If $X = 1$, the first random digit must be 4.

Each digit is an integer from 1 to 4. If the first digit is 4 or more then it must be 4.

Each digit is equally likely so the probability that the first digit is 4 is $\frac{1}{4}$.

iii If $X = 2$, the first random digit cannot be 4.

To find the probability that $X = 2$ start by thinking about what outcomes would lead to that. The first two digits add up to 4 or more.

(3, 1) (3, 2) (3, 3) (3, 4)	(2, 2) (2, 3) (2, 4)	(1, 3) (1, 4)
If the first digit is 3 then the second digit can be anything and the total will be 4 or more.	If the first digit is 2 then the second digit can be 2, 3 or 4 and the total will be 4 or more.	If the first digit is 1 then the second digit can be 3 or 4 and the total will be 4 or more.

There are 9 ways for the event $X = 2$ to happen; each of these is equally likely with probability $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.

So $P(X = 2) = \frac{9}{16}$.

Hints: Listing systematically all the ways that $X = 2$ can happen will allow you to show the given result easily.

The Normal distribution (page 98)

A random sample of 180 students is asked to complete a puzzle. The times taken are shown in the table below.

Time, x minutes	$0 < x \leqslant 2$	$2 < x \leqslant 3$	$3 < x \leqslant 4$	$4 < x \leqslant 5$	$5 < x \leqslant 8$
Frequency	2	52	82	27	17

i For these data, estimate
- the mean
- the standard deviation.

ii If times taken to complete the puzzle are Normally distributed, with mean and standard deviation the values you found in part **i**, what is the probability of a randomly chosen student taking between 4 and 5 minutes?

iii Do you think that the Normal distribution is a good model for these data? Give a reason for your answer.

i

The data are grouped so to find the mean and standard deviation, you will need to find the midpoint of each group. The answers you get will be approximations for the mean and standard deviation as you do not have a list of all the actual times. Start by working out the midpoint of each group.

Time, x minutes	Frequency	Midpoint, m
$0 < x \leqslant 2$	2	1
$2 < x \leqslant 3$	52	2.5
$3 < x \leqslant 4$	82	3.5
$4 < x \leqslant 5$	27	4.5
$5 < x \leqslant 8$	17	6.5

1 is half way between 0 and 2. A quick way to find the midpoint of a group is to average the highest and lowest value. For the group 2 to 3 the midpoint would be $(2 + 3)$ then divide the answer by 2. This gives 5 divided by 2, which is 2.5. Be careful with the last group, it is a wider group than the earlier ones so don't just follow the pattern in the table.

Hint: Now enter the midpoints and frequencies into your calculator.

SUB	List 1 midpt	List 2 freq	List 3	List 4
1	1	2		
2	2.5	52		
3	3.5	82		
4	4.5	27		

2

Common error: If your calculator uses lists, check that the variable and frequencies are set to the correct lists.

```
1Var  XList   : List1
1Var  Freq    : List2
```

1–Variable

$\bar{x}$ =3.61666666
Σx =651
Σx^2 =2596.5
σx =1.15962158
sx =1.16285624
n =180

Mean time is 3.617 minutes to 3 d.p.
Standard deviation is 1.160 minutes to 3 d.p.

Hint: Remember that the standard deviation you want is the smaller of the two possible values.

ii

In part **ii** you are asked to assume that the times are Normally distributed with the mean and standard deviation that you found in part **i**. If X stands for time taken, by a randomly selected person, to complete the puzzle, then X has a Normal distribution with mean 3.617 and standard deviation 1.160. You want the probability that X is between 4 and 5. Since time taken is a continuous variable, it will not change the answer for the probability whether you include the endpoints of 4 and 5 or leave them out.

X = time taken in minutes

$X \sim \mathrm{N}(3.617, 1.160^2)$

$\mathrm{P}(4 \leqslant X \leqslant 5)$

Hint: Use cumulative Normal probability on your calculator. Use as much accuracy as you can for mean and standard deviation.

```
Normal    C.D
Data      : Variable
Lower     : 4
Upper     : 5
σ         : 1.159621
μ         : 3.616666
Save Res  : None
```

```
Normal  C.D
p       =0.25403479
z : Low=0.3305676
z : Up  =1.19291789
```

The probability is 0.2540.

iii The model suggests that over a quarter of students take between 4 and 5 minutes.

From the data, the proportion taking between 4 and 5 minutes is $\frac{27}{180} = 0.15$ so the Normal model is not a good model for these data.

Hint: In this part of the question you are asked to comment on whether the Normal distribution is a good model. You could compare the theoretical probability you worked out in part **ii** with the experimental probability from the data.

Common error: The main thing in part **iii** is to give a good reason for your view of whether the Normal distribution is a good model; just saying that it isn't good will not gain you any credit.

Instead of using the probability from part **ii** you could say that the Normal distribution is symmetrical about the mean. In this case the mean is close to the middle of the 3 to 4 group but there are more data values below this group than above it, so the Normal distribution is not a good model for the data.

Hypothesis testing using the Normal distribution (page 107)

A national survey finds that the heights of men are Normally distributed with mean 176 cm and standard deviation 6.7 cm.

A historian is studying army records and he suspects that the soldiers he is studying were shorter, on average.

A random sample from the records gives the following heights of soldiers (all in cm):

165 158 169 176 175 179 172

i Write down suitable null and alternative hypotheses to test the historian's suspicion.

ii Assume that the standard deviation of the heights of soldiers is 6.7 cm. Carry out a suitable hypothesis test, at the 5% level of significance, to test the historian's suspicion. You should state your conclusions clearly.

i $H_0 : \mu = 176$

$H_1 : \mu < 176$

where μ is the mean height, in cm, of all the soldiers the historian is studying.

- The null hypothesis for a Normal test on the mean is always that μ is equal to a particular value so, in this case, it will be that $\mu = 176$. This says that the mean height of the soldiers the historian was studying is the same as that for men nowadays.
- The historian suspects that the soldiers were shorter so the alternative hypothesis is that $\mu < 176$.
- You need to say what μ stands for. In this question it stands for the mean height of all the soldiers that the historian is studying. μ is in cm.

ii Let X be the height, in cm, of a randomly chosen soldier.

Use a letter to stand for the height of a randomly chosen soldier. X will do.

If the null hypothesis is true, $\bar{X} \sim N\left(176, \frac{6.7^2}{7}\right)$.

- You will use the sample mean, $\bar{X}$, to come to a decision about the population mean.
- If the null hypothesis is true, the distribution that the sample mean comes from is Normal with mean μ and variance $\frac{\sigma^2}{n}$. In this case $\mu = 176$ and $\frac{\sigma^2}{n} = \frac{6.7^2}{7}$ because there were 7 soldiers in the random sample.

$\bar{x} = 170.571...$

Use statistical functions on a calculator to find the sample mean.

$\bar{x}$	=170.5714286
Σx	=1194
Σx^2	=203976
$\sigma^2 x$	=44.81632653
σx	=6.694499722
$s^2 x$	=52.28571429

- You are not told whether to use a critical region or a p-value; you will probably find it is quicker to use cumulative Normal probability to work out the p-value.
- The observed value of the sample mean was 170.57...; lower values would cast even more doubt on the null hypothesis so work out $P(\bar{X} < 170.57..)$
- Use a negative value that is large in size for the lower bound. For the standard deviation type in $\frac{6.7}{\sqrt{7}}$.

```
Normal      C.D
Data        : Variable
Lower       : –1E+05
Upper       : 170.571428
σ           : 2.53236196
μ           : 176
Save Res    : None
```

```
Normal  C.D
p        = 0.0160293
z : Low = –39558.326
z : Up   = –2.1436791
```

$P(X < 170.57...) = 0.016... < 0.05$

There is sufficient evidence at the 5% level to suggest that the mean height of the soldiers the historian is studying is below 176 cm.

Write down the probability you have calculated. This is a 1-tail test so this is the p-value; compare it to the significance level then state your conclusion clearly in context. Remember you can never be certain whether the conclusion from a hypothesis test represents the true situation so state your conclusion non-assertively.

Instead of working out a p-value, you could have worked out the test statistic $z = \frac{\bar{x} - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{170.571... - 176}{\sqrt{\frac{6.7^2}{7}}} = -2.14367...$ and compared it to the critical value which is –1.645. You will come to the same conclusion doing the test that way.

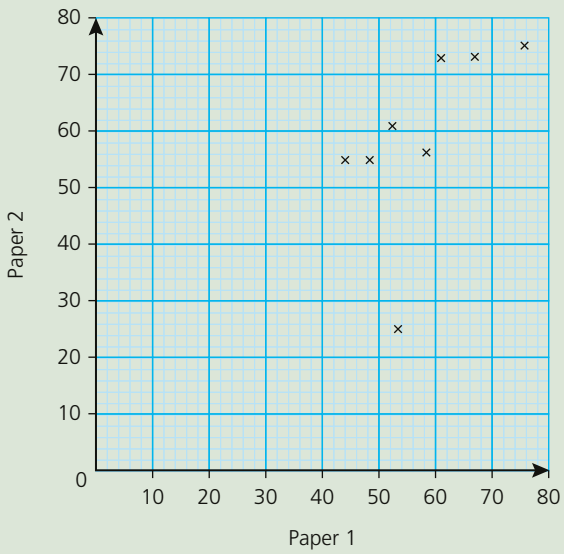
Bivariate data (page 117)

Eight history students are chosen at random to sit two examination papers. Their marks are shown in the scatter diagram.

i Describe the correlation in the scatter diagram.

ii One of the students did not notice the back page of questions on Paper 2.

- **A** Which point in the scatter diagram represents the marks of this student?
- **B** Should the data from this student be used when testing for correlation in the population? Justify your answer.



The examiners say that there should be a positive correlation between the marks on both papers.

iii The correlation coefficient for the data from the seven students who completed all questions is 0.8687. The p-value for a 1-tail test is 0.0056. Test, at the 5% level, whether the examiners' claim is true. You should state your hypotheses and conclusions clearly.

iv The examinations were tried out on a different random sample of students. Both the correlation coefficient and the p-value were lower. What can you say about the number of students in the second random sample? Give a reason for your answer.

i Positive correlation with an outlier.

- Students who got high marks on Paper 1 generally got high marks on Paper 2 as well so there is positive correlation; it would be possible to draw a line of best fit with a positive gradient.
- There is one point which does not seem to fit the pattern; this is an outlier. Without the outlier the correlation would be strong.

ii A The student who got 53 marks on Paper 1 and 25 marks on Paper 2.

Hint: This is the outlier in the scatter diagram. The reason for the lower performance on Paper 2 is failing to notice some of the questions. You could identify the point by referring to either the mark on Paper 1 or the mark on Paper 2 but if there was another point with the same mark on one of the papers then it would be safer to use both marks.

B No because he/she did not do all the second paper so the data are not the same kind as for the other students.

Hint: The main thing here is the reason for your answer. This student has not done all the questions on Paper 2 so his/her performance on that paper does not represent what he/she could have done so it is not like the other data values.

iii $H_0 : \rho = 0$

$H_1 : \rho > 0$

where ρ is the population correlation coefficient.

Hints:

- You are told to test whether the examiners' claim is true. They claim that there is positive correlation so the alternative hypothesis, H_1, is that $\rho > 0$. Remember that the null hypothesis, H_0, is always that $\rho = 0$.
- Remember to say that ρ stands for the population correlation coefficient.

$0.0056 < 5\%$ i.e. the p-value is smaller than the significance level.

Hints:

- You can conduct a hypothesis test by comparing the test statistic to the critical value or by comparing the p-value to the significance level. In this case, you are told the p-value and the significance level so compare these.
- You might want to change them both to decimals or percentages to help you compare them.
- Remember that smaller p-values provide stronger evidence in favour of the alternative hypothesis.
- Finish by stating your conclusion in terms of the original situation.

Reject H_0

There is enough evidence of positive correlation between the marks on both papers, at the 5% significance level.

iv The lower correlation coefficient means the correlation was weaker but the lower p-value shows that this provides stronger evidence of positive correlation in the population. This must mean that the second sample was larger, i.e. there were more than 7 students in the second sample.

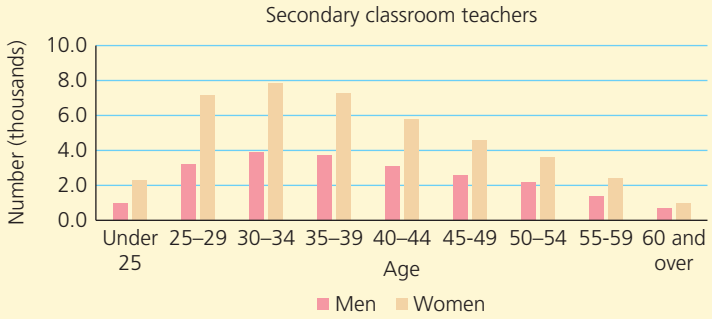
For small samples, it is possible to get a large correlation in the sample just by chance but for larger samples any correlation in the sample is more likely to represent the correlation in the population.

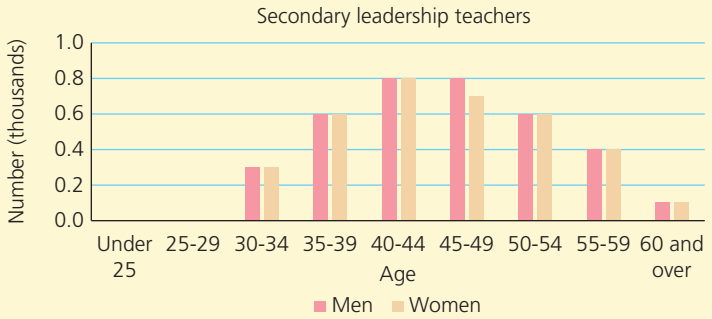
So weaker correlation in the sample but stronger evidence of correlation in the population means that the sample must be larger.

Review questions (Statistics) (pages 118–120)

1 Teachers in England are classified as leadership teachers if they are heads, deputy heads, assistant heads or advisory teachers. Other teachers are classified as classroom teachers.

The charts below show the number of teachers in state secondary schools in England in 2016.





i Wesley says that there are equal numbers of men and women in leadership positions so women teachers and men teachers are equally likely to go into leadership in secondary schools. Comment on Wesley's statement.

ii An advertising campaign is being planned to increase the number of teachers at secondary level. Suggest what kind of people it should be targeted at. Justify your answer.

Hint: There are two parts to Wesley's statement: a comparison of numbers of men and women in leadership and a comparison of how likely men and women are to go into leadership. You should comment on both parts.

i In each age group for leadership teachers (except age 45–49) there are roughly equal numbers of men and women but for age 45–49 there are more women so there are slightly more women in leadership than men.

Hint: You could try to read off the numbers in each age group and find the total number of women in leadership and the total number of men in leadership but it is not easy to read off accurately from the graphs and you only need a comparison of the totals so it is easier to look at the pair of bars for each age range and compare them.

In each age group for classroom teachers, there are about twice as many women as men but the numbers in leadership are the same so men are more likely to go into leadership than women.

Hint: You could try reading from the graphs to get total numbers of classroom teachers but it is not easy to read off and there are clearly more women classroom teachers than men so that is the thing to comment on.

ii Possible answers include the following:

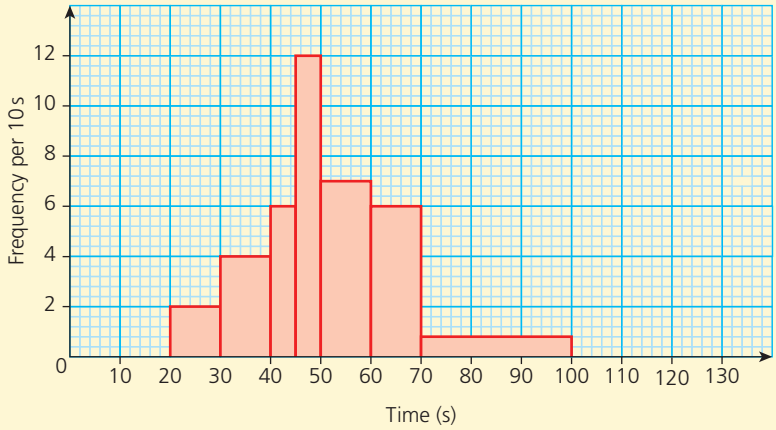
- Target men because there are fewer of them in teaching than women (and you would expect equal numbers).

- Target people aged 40 and over because there are fewer of them in teaching than for younger age groups.
- Target young women because teaching appeals to them.

You could target people who are under-represented in teaching (men or people aged 40 or over) or you could see which kinds of people teaching appeals to most and try to get more of them – the key thing is to justify your answer.

2 A sample of students are told to close their eyes; they are told to put up a hand when they think a minute has gone by. The times it takes them to put up their hands are summarised in the following table and histogram.

Time (x seconds)	Number of students
$20 \leqslant x < 30$	2
$30 \leqslant x < 40$	4
$40 \leqslant x < 45$	3
$45 \leqslant x < 50$	6
$50 \leqslant x < 60$	7
$60 \leqslant x < 70$	6
$70 \leqslant x \leqslant 100$	2



i Calculate an estimate of the median of the data.

ii Calculate an estimate of the mean of the data.

iii Joe suggests that a Normal distribution with a mean of 60 would be a good model for this situation. Comment on Joe's suggestion.

i The median is at the vertical line that cuts the area of the histogram in half.

Time (x seconds)	Area (Number of students)
$20 \leqslant x < 30$	2
$30 \leqslant x < 40$	4
$40 \leqslant x < 45$	3
$45 \leqslant x < 50$	6
$50 \leqslant x < 60$	7
$60 \leqslant x < 70$	6
$70 \leqslant x < 100$	2
TOTAL	30

You already have the frequencies in the original table and they are proportional to the areas of the bars so add the frequencies to find the total area. This is 30.

$30 \div 2 = 15$

Now find the area to the left of the median. The area each side of the median is 15.

$2 + 4 + 3 + 6 = 15$
The first four bars have a total area of 15.
The median is at the end of the fourth bar.
The median is 50 seconds.

Now start adding the area of the individual bars, one at a time, to see how many you need to get an area of 15. 2+ 4 is 6 and this is less than 15 so you have not got to the median yet. Adding the next bar gives an area of 9 and the area of the first 4 bars is 15. This means that the median is at the end of the fourth bar, so it is 50. Remember to include the units.

ii To calculate an estimate of the mean, you need the midpoint of each group.

Time (x seconds)	Frequency	Midpoint
$20 \leqslant x < 30$	2	25
$30 \leqslant x < 40$	4	35
$40 \leqslant x < 45$	3	42.5
$45 \leqslant x < 50$	6	47.5
$50 \leqslant x < 60$	7	55
$60 \leqslant x < 70$	6	65
$70 \leqslant x < 100$	2	85

To work out an estimate of the mean start by entering the numbers into your calculator so that you can use the statistical features.

	List 1	List 2	List 3	List 4
SUB	midpt	freq		
1	25	2		
2	35	4		
3	42.5	3		
4	47.5	6		

6

Hint: If your calculator uses lists, ensure it is set up to work with frequencies.

1Var XList : List1
1Var Freq : List2

1–Variable

$\bar{x}$ =51.5833333
Σx =1547.5
Σx^2 =86081.25
σx =14.4407313
sx =14.6875993
n =30

Mean ≈ 51.58 seconds (2 d.p.)

Common error:

- Find the mean in the list of statistics and round it sensibly, remembering to include units.
- Most calculators will also give the median in the list of statistics but using midpoints does not give a good estimate of the median for grouped data.

iii The histogram is fairly symmetrical so a Normal model might be suitable.

There are several aspects to consider when commenting on Joe's statement. Firstly, is a Normal distribution consistent with the shape of the distribution for the sample? Secondly, would a mean of 60 be appropriate?

The data suggest that a mean of 60 would not be suitable as both the mean and the median were about 50 – the fact that these values were about the same for the model confirms that a Normal model might be suitable.
However, the sample size is fairly small and the wide group at the upper end of the data makes it hard to judge.

You should also comment on whether the sample is large enough for deciding on a suitable model. You should not be too firm in your acceptance or rejection of the proposed model – it is not possible to be certain.

3 A random sample of word processed pages produced by a new secretary is being proof read. The random variable X denotes the number of errors per page.

The results are summarised as follows.

$$n = 50, \sum x = 53, \sum x^2 = 103$$

Calculate:

i the mean

ii the standard deviation.

i $\bar{x} = \dfrac{\sum x}{n} = \dfrac{53}{50} = 1.06$

The mean is 1.06.

To find the mean of the data you need to divide the total by the number of data values.

ii $S_{xx} = \sum x^2 - \frac{\left(\sum x\right)^2}{n} = 103 - \frac{53^2}{50} = 46.82$

To find the standard deviation, you will need to find the variance. First find S_{xx}.

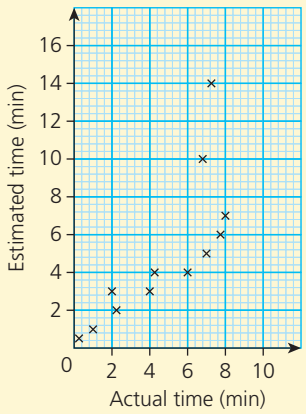
$$\text{variance} = \frac{S_{xx}}{n} = \frac{46.82}{50} = 0.9364...$$

The variance is S_{xx} divided by n. This is still not your final answer so do not round it. The standard deviation is the square root of the variance.

$$\text{standard deviation} = \sqrt{0.9364...} = 0.9676...$$

The standard deviation is 0.968 (to 3 d.p.).

4 When George instructs his computer to copy a folder, it gives an estimate of how long it will take. George wants to see how accurate the estimates are. He measures the actual times taken for a random sample of folders to copy. The results are shown in the scatter diagram.



i Describe the correlation in the scatter diagram.

ii The correlation coefficient is 0.7638. George suspects that removing the two points with the largest estimated times will increase the correlation coefficient. Is he right? Justify your answer.

i Positive correlation.

As the actual time increases, so does the estimated time and it is possible to draw a line of best fit with positive gradient so there is positive correlation.

ii Yes, he is right. The points would lie closer to a straight line with positive gradient if the two points with the largest estimated times were removed.

Hint: Remember correlation measures the strength of a linear relationship.

5 A and B are events with $P(A') = 0.3$, $P(B) = 0.5$, and $P(A \cup B) = 0.8$.

i Find $P(A)$.

ii Find $P(A \cap B)$.

iii Are A and B mutually exclusive? Explain how you know.

i $P(A') = 1 - P(A)$

so $P(A) = 1 - P(A')$

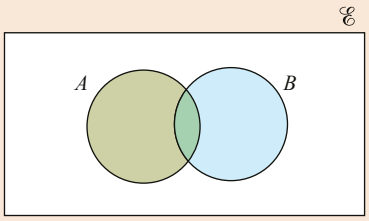
$P(A) = 1 - 0.3$

$= 0.7$

You are asked to find the probability of event A. The first piece of information you are given is that the probability of A dashed is 0.3; this is the probability of "not A", i.e. that A does not happen. Either A happens or it does not so the total probability of A and not A is 1.

ii $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

This formula comes from the Venn diagram because adding the probabilities of A and B gives you the probability of A union B but you have counted $A \cap B$, the region where they overlap, twice so you need to subtract one of them.



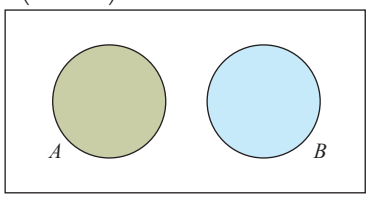
$0.8 = 0.7 + 0.5 - P(A \cap B)$

$0.8 = 1.2 - P(A \cap B)$

so $P(A \cap B) = 1.2 - 0.8$

$P(A \cap B) = 0.4$

iii For mutually exclusive events $P(A \cup B) = P(A) + P(B)$ and $P(A \cap B) = 0$.



A Venn diagram for a pair of mutually exclusive events is shown above.

A and B do not overlap so the probability of A intersection B is zero. In the example in this question, the probability of A intersection B is 0.4. This means that events A and B can both happen together and so they are not mutually exclusive.

Another way of doing this part is to use the result that for mutually exclusive events $P(A \cup B) = P(A) + P(B)$.

For these events $P(A \cup B) = 0.8$

but $P(A) + P(B) = 0.7 + 0.5 = 1.2$

These are not the same so A and B are not mutually exclusive.

6 A magazine conducts a survey of a random sample of its readers to find out if they would be interested in subscribing to an internet version of the magazine. The results are summarised in the table below which shows the number of people in each category.

	Age				Row totals
	under 25	26–40	41–60	over 60	
Would subscribe to internet version?	11	60	27	38	136
Would not subscribe to internet version?	45	113	56	50	264
Column totals	56	173	83	88	400

Assume that the sample is representative of all the readers.

i What is the probability that a randomly chosen reader would subscribe to the internet version?

ii What is the probability that a randomly chosen reader over 60 would subscribe to the internet version?

i $\frac{136}{400} = \frac{17}{50}$

Out of 400 readers, 136 would subscribe to the internet version. Give the probability as a fraction in its lowest terms or as a decimal.

ii $\frac{38}{88} = \frac{19}{44}$

Out of 88 readers over 60, 38 would subscribe to the internet version. This is a conditional probability.

7 A statistician is investigating the number of children that women have. She models the number of children per woman, X, by the probability distribution

$P(X = r) = k(5 - r)(4 + r)$ for $r = 0, 1, 2, 3, 4,$ where k is a constant.

i Find the value of k.

ii Another statistician says that some women have more than 4 children and that possible values of r should include 5 and 6. Would it be possible to use the same model, $P(X = r) = k(5 - r)(4 + r)$, but with a different value of k? Explain your answer.

i

Start by using the formula to find each value of probability in terms of k. The first possible value of r is zero and this gives a probability of $20k$. Continue for the other possible values of r, putting all the results in a table so that you can easily find and use them later in the question.

r	0	1	2	3	4
$P(X = r)$	$20k$	$20k$	$18k$	$14k$	$8k$

You know that the sum of all the probabilities for a discrete random variable has to be 1.

$$20k + 20k + 18k + 14k + 8k = 1$$
$$80k = 1$$
$$k = \frac{1}{80}$$
$$k = 0.0125$$

ii If the same model was used to include possible values 5 and 6, their probabilities would be:
$P(X = r) = k(5 - r)(4 + r)$
For $r = 5$, $P(X = 5) = k(0)(9) = 0$
For $r = 6$, $P(X = 6) = k(-1)(10) = -10k$
Probabilities for $r = 0, 1, 2, 3, 4$ are as before, in terms of k.

r	0	1	2	3	4	5	6
$P(X = r)$	$20k$	$20k$	$18k$	$14k$	$8k$	0	$-10k$

- To help you make the decision in part **ii**, see what happens if you try to use the same model, including possible values 5 and 6.
- For $r = 5$, the probability is zero. This means that it is impossible to have 5 children. This is clearly untrue but the probability might be so low that it is nearly zero and so rounds to zero.
- The probability for $r = 6$ is negative. It is not possible to have a negative probability, so this must be wrong.

This model gives a negative probability for $r = 6$ so it cannot be used to include possible values of 5 and 6 because a probability cannot be negative. If a negative value of k was used, this would make other probabilities negative.

8 An airline finds that, on average, 5% of people who book a seat fail to turn up for the flight.

i If 142 bookings are taken for a flight, how many people, on average, will fail to turn up?

142 bookings have been taken for a flight. You should assume that individual passengers are a random sample from the population and turn up for the flight independently of each other, with equal probability.

ii The aeroplane has 136 seats. Find the probability that more than 136 people turn up for the flight.

i 5% of 142 is $0.05 \times 142 = 7.1$

You are asked to find the average number of people who fail to turn up so you need to work out 5% of 142. This gives 7.1. Although you cannot have 7.1 people, it is fine to leave this unrounded as it is an average number and an average does not need to be a whole number.

ii X = the number of people who turn up.

You could do this part of the question either by considering the number of people who turn up or the number who fail to turn up. The question asks for the probability of more than 136 people turning up so you may find it easier to consider the number who turn up.

$X \sim B(142, 0.95)$

You know that 142 bookings are taken and 5% of people fail to turn up. Assuming that the people who book on a flight are a random sample from the population, the probability of each person failing to turn up will be 5% and so the probability of the person turning up is 95%. You are not asked about any assumptions you need to make in this part of the question but it is something that you should bear in mind when working with probability. The probability of each person turning up is the same and independent of what other people do so X will have a binomial distribution with $n = 142$ (as this is the number of people who booked) and $p = 95\%$.

$P(X > 136)$

You are asked to work out the probability that more than 136 people turn up. It will help you get the right answer if you decide what the possible values of X are.

Possible values are 137, 138, …, 142.

Hint: If you are using a calculator which calculates $P(X \leqslant x)$ it will help to think about the values of X you cannot have; they are 0 to 136 inclusive. Now you can write the probability in a way that will allow you to use your calculator.

$P(X > 136) = 1 - P(X \leqslant 136)$

Hint: If you are using a calculator which will give the probability of a binomial random variable being in a range, the lower value is 137 and the upper value is 142.

Now enter the information into your calculator. Remember you want cumulative binomial probabilities. The number of trials is 142 with probability of success 0.95. Each trial is a person booking the flight and success is the person turning up.

```
Binomialcdf
TRIALS=n= 142
P(SUCCESS)= 0.95
X=136 ■
```

```
Binomial      C.D
Data          : Variable
Lower         : 137
Upper         : 142
Numtrial      : 142
p             : 0.95
Save Res      : None
```

Your calculator will tell you $P(X \leqslant x)$.

```
Binomialcdf
VALUE= 0.718566379
STORE: ■ yztabcd
SOLVE AGAIN QUIT
```

This will give you the probability you want directly and you just need to round it sensibly.

```
Binomial  C.D
        p=0.28143362
```

Hint: Now you need to subtract this probability from 1 to get the answer but you should write down the probability you have found first.

$P(X \leqslant 136) = 0.718566...$

$P(X > 136) = 1 - 0.718566...$

$P(X > 136) = 0.2814$ (4 d.p.)

9 A survey shows that the systolic blood pressure of women in the UK is Normally distributed with mean 121 mm Hg and standard deviation 16.2 mm Hg.

i Hypertension is defined as a systolic blood pressure over 140 mm Hg. What proportion of women in the UK suffer from hypertension?

ii Ten women are chosen at random from the population. Find the probability that at least one of them has hypertension.

i It will help you organise your working if you start the question by summarising the information symbolically. The question deals with the distribution of systolic blood pressure for women. Use a letter to stand for the blood pressure of a randomly chosen woman. W is a suitable letter to use (P is best avoided as it might get confused with P for probability). Say what the letter stands for. The units of blood pressure are millimetres of mercury; stating the units at this stage means that you will not have to keep referring to them.

W = the systolic blood pressure of a randomly chosen woman in mm Hg.

$W \sim \mathrm{N}(121, 16.2^2)$

Hint: Remember the second number in the brackets is the variance.

You will need to use the standard deviation so do not actually work out 16.2 squared.

You are asked to work out the proportion of women with a blood pressure over 140 mm Hg. Start by working out the probability of this being the case for a woman chosen at random.

$\mathrm{P}(W > 140)$

Hints	What to do on your calculator
Select Normal CD (this gives cumulative probabilities).	1 : Normal PD 2 : Normal CD 3 : Inverse Normal 4 : Binomial PD
Enter the mean, standard deviation and lower and upper limits. There is no upper limit for the probability you are finding here so use a large number. Make sure you enter mean and standard deviation in the right places – on some calculators standard deviation is entered first.	Normal C.D Data : Variable Lower : 140 Upper : 10000 σ : 16.2 μ : 121 Save Res : None
On some calculators, you will get the standardised values of the upper and lower limits as well as the probability.	Normal C.D p = 0.12043008 z : Low = 1.17283951 z : Up = 609.814815

The proportion of women with hypertension is 12.0% (3 s.f.).

Hint: You were asked to find the proportion of women with hypertension. This will be the percentage that is equivalent to 0.1204. So it is 12.04%. It makes sense to round to 3 significant figures so your final answer to part (i) is 12.0%.

ii $\mathrm{P}(\text{at least one}) = 1 - \mathrm{P}(\text{none})$

$\mathrm{P}(W < 140) = 1 - \mathrm{P}(W > 140) = 1 - 0.12043... = 0.87957$

Hint: In part **ii** you are asked to find the probability that at least one of ten women chosen at random has hypertension. The probability of at least one occurrence is one minus the probability of none. For one woman, the probability she does not have hypertension is 0.87957.

$$\begin{aligned} P(\text{none}) &= 0.87957 \times 0.87957 \times 0.87957 \times \ldots \times 0.87957 \\ &= 0.87957^{10} \\ &= 0.277143\ldots \end{aligned}$$

Since this is a random sample from the population, you can assume that the probability of each one having hypertension is independent of the others. So the probability that among the 10 women no one has hypertension is 0.87957 times 0.87957 times 0.87957, and so on. This is 0.87957 to the power 10. Do not round this yet as it is not your final answer.

$$\begin{aligned} P(\text{at least one has hypertension}) &= 1 - 0.277143\ldots \\ &= 0.722856\ldots\ldots \end{aligned}$$

The probability that at least one of the women has hypertension is 0.723 (to 3 d.p.).

Hint: Round your final answer – this makes it easier to read and it has come from using a model, which is an approximation to reality.

10 A new fruit picker starts work. 8% of the baskets of strawberries she picks take her longer than 15 minutes. 5% of them take less than 10 minutes.

Assuming that the time the new picker takes to pick a basket of strawberries is Normally distributed, find the mean and standard deviation of her picking times.

Hint: You know that S is Normally distributed but you do not know the mean and standard deviation. You can use μ to stand for the mean and σ to stand for the standard deviation but you should say that this is what you are doing as these letters were not introduced in the question.

Hint: Write down the information you were given.

S is the time, in minutes, for the new picker to pick a basket of strawberries.

$S \sim N(\mu, \sigma^2)$

μ is the population mean.
σ is the population standard deviation.
$P(S > 15) = 0.08 \quad P(S < 10) = 0.05$

Hint: You need to standardise by subtracting the mean then dividing by the standard deviation. S minus μ over σ is the standard Normal variable, Z. Of course, you don't know the values of μ and σ – they are what you are trying to work out.

$$P\left(\frac{S-\mu}{\sigma} > \frac{15-\mu}{\sigma}\right) = 0.08 \qquad P\left(\frac{S-\mu}{\sigma} < \frac{10-\mu}{\sigma}\right) = 0.05$$

$$P\left(Z > \frac{15-\mu}{\sigma}\right) = 0.08 \qquad P\left(Z < \frac{10-\mu}{\sigma}\right) = 0.05$$

Hint: Now use inverse Normal probabilities on your calculator. If your calculator only does inverse probabilities for lower tails, you will need to get the lower tail that goes with the 0.08 upper tail.

$$P\left(Z < \frac{15-\mu}{\sigma}\right) = 1 - 0.08 = 0.92$$

```
invNormal
area=0.92
mean=mu=0
Sigma=1
```

Hint: Remember you are working with the standard Normal distribution so the mean is 0 and the standard deviation is 1.

```
invNormal
VALUE= 1.405071561
STORE: No xyztabcd
SOLVE AGAIN QUIT
```

$$\frac{15-\mu}{\sigma} = 1.40507...$$

$$15 = \mu + 1.40507\sigma$$

Hint: Now you have one equation involving μ and σ; use as accurate a value as you can for 1.40507...

Use the other probability statement to get another equation involving μ and σ.

$$\mathrm{P}\left(Z < \frac{10-\mu}{\sigma}\right) = 0.05$$

```
invNormal
area=0.05
mean=mu=0
Sigma=1
```

```
invNormal
VALUE= –1.644853626
STORE: ■ xyztabcd
SOLVE AGAIN QUIT
```

$$\frac{10-\mu}{\sigma} = -1.6448536...$$

$$10 = \mu - 1.6448536\sigma$$

Common error: Now you have two simultaneous equations to solve. If you are doing this in an examination check the instructions carefully to see whether you are expected to show a full method for solving them or whether you can use a calculator equation solver. The full method is shown below.

$$\begin{aligned} 15 &= \mu + 1.40507\sigma && (1) \\ 10 &= \mu - 1.6448536\sigma && (2) \quad \text{subtract} \\ \hline 5 &= \quad 3.049925\sigma \end{aligned}$$

$$\sigma = 5 \div 3.049925 = 1.63938...$$

$$\sigma = 1.6 \text{ (1 d.p.)}$$

Hint: Where possible use calculator memories to store the long decimals you are working with so that you do not have to keep typing them in. Round your answers but use unrounded answers in further working.

$15 - \mu = 1.40507 \times 1.63938....$

$15 - \mu = 2.30345...$

$\mu = 15 - 2.30345...$

$\mu = 12.6965...$

$\mu = 12.7$ (1 d.p.)

The mean is 12.7 minutes and the standard deviation is 1.6 minutes.

Finish by writing down the full answer; round answers as they are based on a model.

11 When a biscuit production line is working correctly it produces biscuits with a mean mass of 20 g and standard deviation 0.95 g. The masses of biscuits are Normally distributed. To check the process is working correctly, a random sample of 60 biscuits is taken and the mass, x g, of each biscuit is measured. The sample mean is 19.7 g.

i State suitable null and alternative hypotheses for a test to see whether the mean mass of biscuits produced has changed.

ii Carry out the test, at the 5% significance level, stating your conclusions carefully. You should assume that the standard deviation is unchanged.

i $H_0 : \mu = 20$

$H_1 : \mu \neq 20$

where μ is the mean mass, in grams, of the population of biscuits produced.

The null hypothesis will be that there is no change, i.e. that μ equals 20. You were asked to look for evidence of a change (rather than an increase or a decrease) so the alternative hypothesis is that μ is not equal to 20. Remember to say what μ stands for. It is the population mean but you need to be more specific than that. In this case, μ is the mean mass for the population of biscuits produced; it is measured in grams.

ii $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

If H_0 is true, $\bar{X} \sim N\left(20, \frac{0.95^2}{60}\right)$.

Using cumulative Normal probabilities:

Normal	C.D
Data	: Variable
Lower	: –10000
Upper	: 19.7
σ	: 0.12264447
μ	: 20

Normal	C.D
p	= 7.2207E–03
z : Low	= –81699.564
z : Up	= –2.4460947

Hint: The test will be based on the sample mean so start by writing down the distribution of the sample mean if the null hypothesis is true.

Hint: The sample mean was below 20 and lower values of the sample mean would cast even more doubt on the null hypothesis so calculate $P(\bar{X} < 19.7)$ – you will find it easiest to do this hypothesis test by using probability.

Hint: There is no lower value for the set of values you are finding the probability of so use a negative number which is large in size.

Hint: For the standard deviation, type in $\frac{0.95}{\sqrt{60}}$.

$P(\bar{X} < 19.7) = 0.00722...$

2-tail test, $0.00722 < 2.5\%$.
There is enough evidence to reject the null hypothesis at the 5% level.

The probability is less than half the significance level so there is enough evidence to reject the null hypothesis at the 5% significance level.

There is enough evidence, at the 5% significance level, that there has been a change in the mean mass of biscuits produced.

Finish by stating your conclusion in the context of the original problem. Be careful not to say definitely that the mass of biscuits has changed; you cannot be sure that this is the case. It would be possible to get a sample mean of 19.7 g even if the population mean was still 20 g but it would be unusual to get such a small value for the sample mean so it is reasonable to conclude that the population mean has changed.

You could do the test using a critical region instead of using probability. Using the standard Normal, the test statistic would be –2.446... which is less than the critical value of –1.96.

Hint: The test is 2-tail so the probability you get is not the p-value. The Normal distribution is symmetrical so you can double the probability to get the p-value and compare it to the significance level or you can compare the probability to half the significance level.

12 The following table shows the mean maximum temperature in April and the total April rainfall. The data are for a random sample of 12 years for the same place.

Temp (°C)	9.3	14.5	11.4	11.6	13.3	13.5	13.9	18.1	15.2	12.0	15.4	14.0
Rain (mm)	23.8	10.4	28.1	97.7	19.6	43.1	70.5	1.6	35.9	55.7	19.2	9.6

i A researcher thinks that warmer Aprils tend to have less rain. Write down suitable hypotheses for a test of this using a correlation coefficient.

ii The correlation coefficient is –0.4125. The critical value for a 1-tail test using the correlation coefficient is 0.4973 at the 5% level of significance. What is the conclusion for the hypothesis test?

i $H_0 : \rho = 0$
$H_1 : \rho < 0$

where ρ is the population correlation coefficient.

The null hypothesis is always that there is no correlation between the variables in the population. In this case the variables are temperature and rainfall. The researcher thinks that warmer Aprils tend to have less rain. If this is true, high temperature will go with low rainfall so the researcher is looking for evidence of negative correlation. The alternative hypothesis is that there is negative correlation between temperature and rainfall.

ii $0.4125 < 0.4973$
Do not reject the null hypothesis.
There is insufficient evidence, at the 5% level, that there is negative correlation between temperature and rainfall.

To decide whether or not to accept the null hypothesis, compare the size of the test statistic with the critical value. 0.4125 is smaller than 0.4973 so there is insufficient evidence to reject the null hypothesis. Remember to give your final conclusion in the original context of rainfall and temperature.

Target your revision (Mechanics) (pages 121–124)

1 Drawing a displacement–time graph

A boy runs 100 m from A to B at a constant velocity of $5\,\text{m s}^{-1}$. He waits for 5 s at B and then runs back to A at $4\,\text{m s}^{-1}$.

i Find the total time the boy takes from leaving A to returning there.

ii Draw a displacement–time graph for the boy.

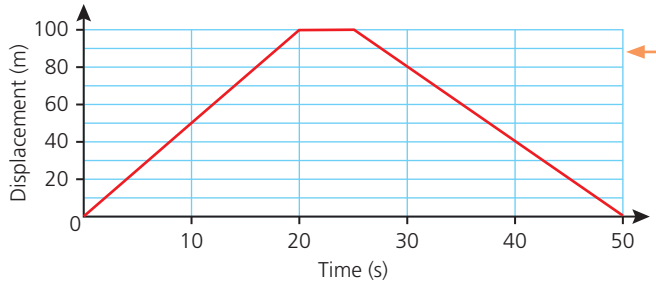
i Time A to B $= \dfrac{\text{displacement}}{\text{velocity}} = \dfrac{100}{5} = 20\,\text{s}.$

Use the fact that the velocity is constant.

Time B to A $= \dfrac{100}{4} = 25\,\text{s}.$

Total time $= 25 + 5 + 20 = 50\,\text{s}$

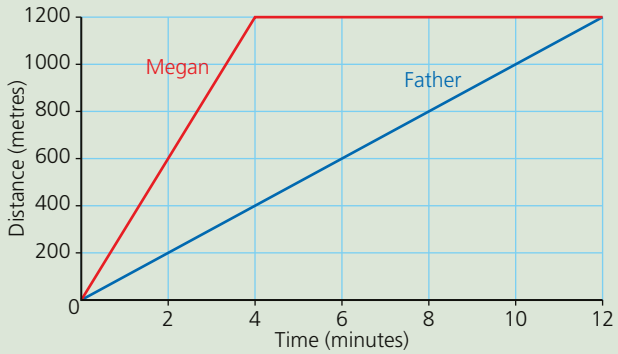
ii



Constant velocity means the graph consists of straight lines.

2 Interpreting the gradient of a position–time graph

Megan and her father leave home together and go to a football field. Megan runs on ahead and waits for her father when she gets there. Their journeys are shown on the distance–time graph in the figure below.



What is the difference in their speeds while they are travelling? Give your answer in m s^{-1} correct to 2 s.f.

Megan's speed is $\dfrac{\text{distance}}{\text{time}} = \dfrac{1200}{(4 \times 60)} = 5\,\text{m s}^{-1}$

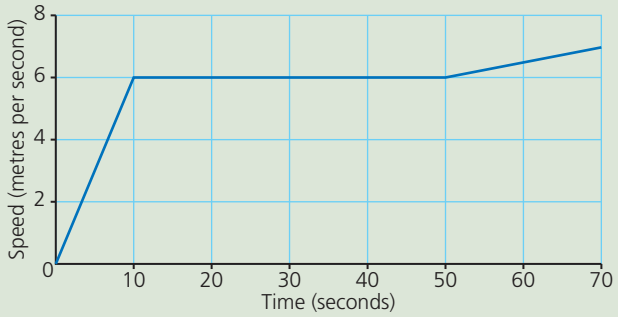
The distance–time graph is a straight line so the speed is constant.

Father's speed is $\dfrac{\text{distance}}{\text{time}} = \dfrac{1200}{(12 \times 60)} = \dfrac{5}{3}\,\text{m s}^{-1}$

Difference in speeds is $5 - \dfrac{5}{3} = \dfrac{10}{3}\,\text{m s}^{-1}$

$= 3.3\,\text{m s}^{-1}$ to 2 s.f.

3 Interpreting the gradient of a speed–time graph



The graph in the figure above shows the speed of a runner in a race.

i What was the magnitude of the runner's acceleration at each stage?

ii Calculate the distance travelled in total.

i In the first 10 seconds the magnitude of acceleration is

$\frac{\text{change in speed}}{\text{time}} = \frac{6}{10} = 0.6\ \text{m s}^{-2}$

In the middle section acceleration is $0\,\text{m s}^{-2}$.

In the third section the magnitude of the acceleration is $\frac{1}{20} = 0.05\,\text{m s}^{-2}$.

Magnitude of acceleration is needed as the graph gives information about speed and not velocity.

ii Total distance is represented by the area under the graph.

Divide the area into three sections.

Distance is $\left(\frac{1}{2} \times 10 \times 6\right) + (40 \times 6) + \left(\frac{1}{2} \times 20 \times (6 + 7)\right)$

Area of a triangle. Area of a rectangle. Area of a trapezium.

$= 30 + 240 + 130 = 400\,\text{m}$

4 Using constant acceleration formulae to find acceleration and distance travelled

A car is initially travelling at $12\,\text{m s}^{-1}$. The driver takes her foot off the accelerator and the car comes to rest in 7.5 s. Assuming the acceleration of the car is constant, calculate:

i the acceleration

ii the distance the car travels while coming to rest.

i Constant acceleration with $u = 12,\ v = 0,\ t = 7.5$

The car comes to rest.

Choose $v = u + at$

Choose the formula that does not involve s.

$0 = 12 + 7.5a$ giving $a = -\frac{12}{7.5} = -1.6\,\text{m s}^{-2}$

ii Choose $s = \frac{1}{2}(u + v)t$

You can use any of the *suvat* formulae here if your value for a is used. It may be better to choose the formula that does not involve a, which may be incorrect.

$s = \frac{1}{2}(12 + 0) \times 7.5 = 45\,\text{m}$

5 Finding the time taken and the final velocity for vertical motion under gravity

Atul drops an egg onto the floor from a shelf 1.2 m above the ground. Find:

i the time it takes to reach the ground

ii the velocity of the egg when it hits the ground.

i The object falls so the initial velocity is $0\,\text{m s}^{-1}$. Taking downwards as the positive direction:

$s = 1.5,\ u = 0,\ a = 9.8$

To find the time, use $s = ut + \frac{1}{2}at^2$ as the final velocity is not given.

$1.5 = 0 + \frac{1}{2} \times 9.8t^2$

$t = \sqrt{\frac{1.5}{4.9}} = \frac{\sqrt{15}}{7} = 0.553\,\text{s}$ (3 s.f.)

ii To find the velocity, use $v^2 = u^2 + 2as$.

$v^2 = 0^2 + 2 \times 9.8 \times 1.5 = 29.4$

$v = 5.42\,\text{m s}^{-1}$ (3 s.f.)

You can use any of the *suvat* formulae here if your value for t is used. It may be better to choose the formula that does not involve t, which may be incorrect.

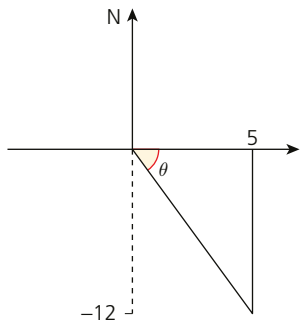
6 Understanding vector forms of displacement and velocity

In this question the x and y directions are East and North, respectively.

i The position of a model boat in metres is given by $\boldsymbol{r} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$. Find its distance and its bearing from the origin.

ii Initially the model boat is travelling at $4.5\,\text{m s}^{-1}$ on a bearing of 060°. Write down the initial velocity as a column vector. Give your answers correct to 3 significant figures where necessary.

i Distance $= |\mathbf{r}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13\,\text{m}$



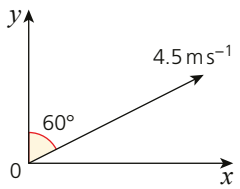
A diagram helps you to see the required angle.

$\tan\theta = \frac{12}{5}$ giving $\theta = 67.4°$

So the bearing of the boat is $90° + 67.4° = 157.4°$

In the triangle drawn, it is good enough to use 12 here to find the tangent of the angle.

ii



Resolve the velocity as you would any other vector.

$$\mathbf{v} = \begin{pmatrix} 4.5\sin 60° \\ 4.5\cos 60° \end{pmatrix} = \begin{pmatrix} 3.90 \\ 2.25 \end{pmatrix} \text{m s}^{-1}$$

7 Using the constant acceleration formulae in 2 dimensions

A particle accelerates from $\mathbf{u} = 9\mathbf{i} - 5\mathbf{j}$ to $\mathbf{v} = -\mathbf{i} - \mathbf{j}$ in 4 seconds.

i Find the acceleration of the particle.

ii Find the distance travelled in 4 seconds.

i Use $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ with $t = 4$, $\mathbf{u} = 9\mathbf{i} - 5\mathbf{j}$, $\mathbf{v} = -\mathbf{i} - \mathbf{j}$.

$(-\mathbf{i} - \mathbf{j}) = (9\mathbf{i} - 5\mathbf{j}) + 4\mathbf{a}$

$\mathbf{a} = \frac{1}{4}(-10\mathbf{i} + 4\mathbf{j}) = -2.5\mathbf{i} + \mathbf{j}$

Use the vector form of this equation as the acceleration is constant.

Leave your answer as a vector – only find its magnitude if asked to do so. Otherwise the examiner may think you do not know that acceleration is a vector quantity.

ii $\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t = \frac{1}{2}((9\mathbf{i} - 5\mathbf{j}) + (-\mathbf{i} - \mathbf{j}))4$

$\mathbf{s} = 2(8\mathbf{i} - 6\mathbf{j}) = 16\mathbf{i} - 12\mathbf{j}$

It is safer to use the formula that uses the information given in the question.

8 Using the position of a projectile at a given time

A ball is thrown from a point at ground level at $24\,\text{m s}^{-1}$ at an angle of 20° above the horizontal across level ground.

i Write expressions for the horizontal distance x and the height y of the ball above its initial position at time t seconds.

ii There is a fence 5 m away from the point of projection which is 2 m high. Does the ball go over the fence?

i Horizontal motion $u = 24\cos 20°$, $a = 0$

Vertical motion $u = 24\sin 20°$, $a = -9.8$

Use $s = ut + \frac{1}{2}at^2$ giving $x = (24\cos 20°)t = 22.6t$

and $y = (24\sin 20°)t - \frac{1}{2}gt^2 = (24\sin 20°)t - 4.9t^2 = 8.21t - 4.9t^2$

ii When the ball reaches the fence $x = 5$.

When information is given about the horizontal distance and the answer needs to consider the vertical height. Use the value of t to link the two directions.

$x = (24\cos 20°)t = 5$ gives $t = \dfrac{5}{24\cos 20°} = 0.2217...\text{s}$

When $t = 0.2217...$ $y = (24\sin 20°) \times 0.2217 - 4.9 \times 0.2217^2 = 1.579\,\text{m}$

This is less than 2 m, the height of the fence, so the ball does not go over the fence.

You can evaluate 24cos 20°. Make sure you use at least 3 significant figures.

An alternative method would be to find the times at which the ball is 2 m high and consider the vertical distances at these times.

Leave the answer on your calculator to avoid rounding errors.

Make sure you make the comparison with 2 m and state whether the ball goes over the fence or not.

9 Finding the greatest height and the range on level ground

A projectile is launched from ground level with an initial velocity $15\,\text{m s}^{-1}$ at an angle of 35° to the ground. The ground is a horizontal plane.

i Find the maximum height reached by the projectile.

ii Find the distance travelled horizontally by the time that the projectile hits the ground.

i Horizontal motion $u = 15\cos 35°$, $a = 0$

Vertical motion $u = 15\sin 35°$, $a = -9.8$

Maximum height when $v_y = 0$

Use $v^2 = u^2 + 2as$

$0 = (15\sin 35°)^2 - 2 \times 9.8 \times s$

$s = \dfrac{(15\sin 35°)^2}{19.6} = 3.78\,\text{m}$

Maximum height is 3.78 m (3 s.f.).

ii Time taken to reach the ground is found using $y = 0$.

Use $s = ut + \frac{1}{2}at^2$ giving $0 = (15\sin 35°)t - 4.9t^2$

$t(15\sin 35° - 4.9t) = 0$

So $t = 0$ or $t = \dfrac{15\sin 35°}{4.9} = 1.7558...$

Range is $x = 15\cos 35° \times 1.7558... = 21.6$ m (3 s.f.)

It is simplest to use the formula that does not involve t. Alternatively you can use a chain of equations to find the time to reach the top and use that value to find the height.

Take care with signs here.

You can use your calculator to solve this quadratic equation if you prefer. Make sure you give both roots as evidence of a complete method.

The ball is at ground level initially.

10 Finding the cartesian equation of the trajectory of the projectile

A ball is thrown horizontally at $7\,\text{m s}^{-1}$ from a point which is 1.5 m above the origin. Find the cartesian equation of the trajectory of the ball, using x and y for the horizontal and vertical displacements from the origin.

Horizontal motion $u = 7$, $a = 0$

Vertical motion $u = 0$, $a = -9.8$

There is no component of the initial velocity vertically.

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7t \\ 0 - 4.9t^2 + 1.5 \end{pmatrix}$$

This is found using $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{u}t^2$ and the initial position of the ball at $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix}$.

If you prefer you can consider the components separately and not use vector notation at all.

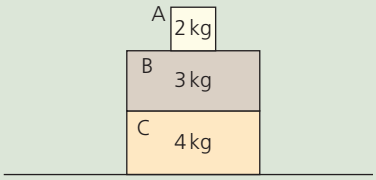
Rearrange the formula for the horizontal component $t = \frac{x}{7}$.

Substitute for t in the formula for the vertical component.

$y = -4.9t^2 + 1.5 = -4.9\left(\frac{x}{7}\right)^2 + 1.5$

Equation of the trajectory is $y = 1.5 - 0.1x^2$

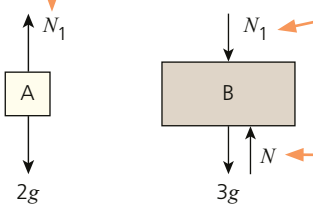
11 Using Newton's 3rd law



The diagram shows a pile of 3 blocks A, B and C in equilibrium on a rough horizontal table. Their masses are 2 kg, 3 kg and 4 kg. The block C exerts a force N on the block B. State the magnitude and direction of N.

The contact force N_1 between blocks A and B acts upwards on block A.

The contact force N_1 between blocks A and B acts downwards on block B.



N is the contact force from block B from block C and acts upwards on block B.

Each block has its own wieght which acts only on that block.

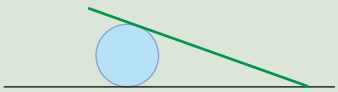
Block A is in equilibrium so $N_1 - 2g = 0$ giving $N_1 = 2g$

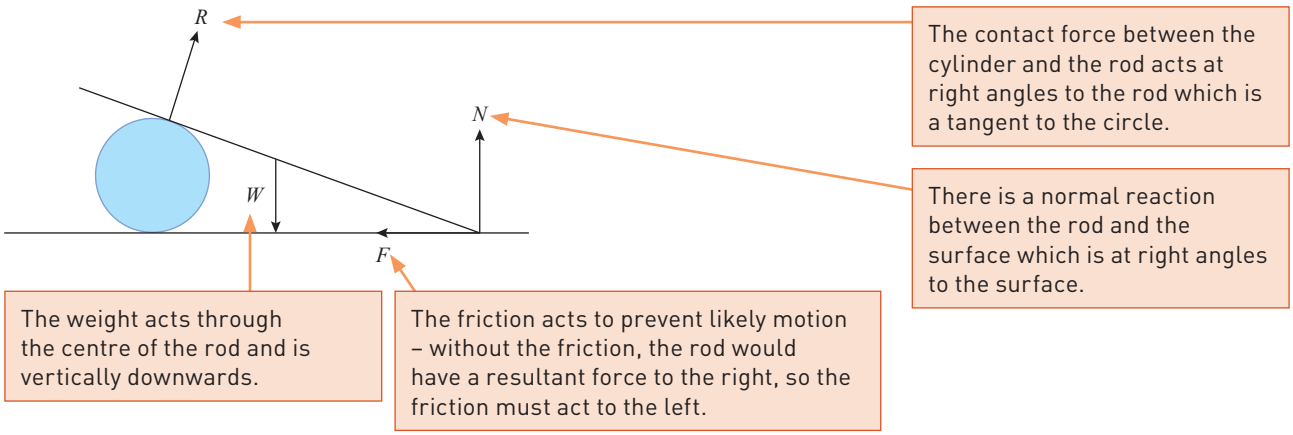
Block B is in equilibrium so $N - 3g - N_1 = 0$

So $N = 5g = 49$ newtons vertically upwards.

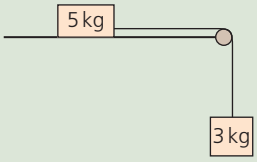
12 Identifying forces

A plank of wood rests on a fixed smooth cylinder with one end on rough horizontal ground as shown in the diagram. Show all the forces acting on the plank of wood.

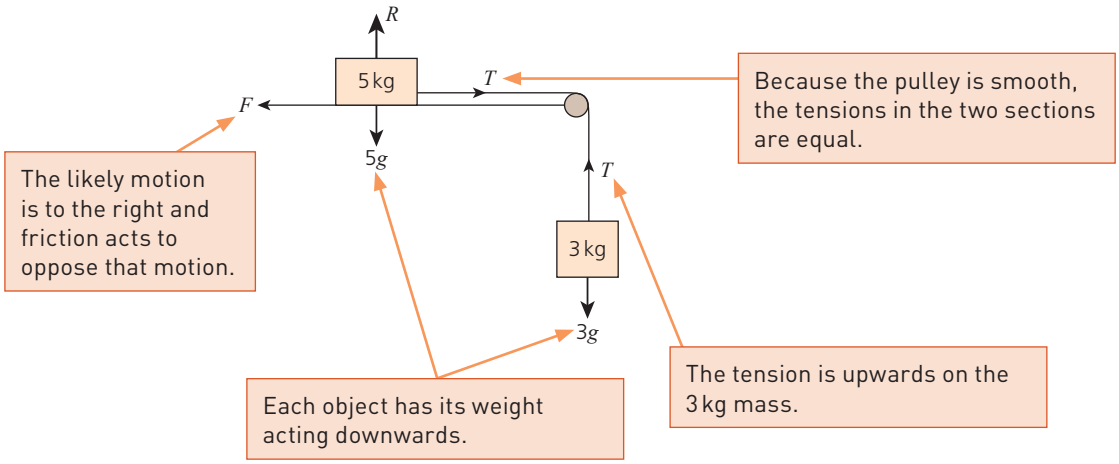




13 Using Newton's 1st law for forces in equilibrium



The diagram shows a block of mass 5 kg on a rough horizontal table. It is attached by a light string to a block of mass 3 kg. The string passes over a light smooth pulley. The system is in equilibrium at rest. Find the value of the frictional force acting on the larger block.



Vertically for the 3 kg mass $T - 3g = 0$ giving $T = 3g$.

Horizontally for the 5 kg block $T - F = 0$ giving $F = T = 3g = 29.4$ N.

14 Using Newton's 1st law for forces in two directions in equilibrium

In this question, **i** is a horizontal unit vector and **j** is the unit vector vertically upwards. A particle of mass 4 kg is in equilibrium under the action of three forces: its weight, **W**, and the tensions in two strings that are attached to it, $\mathbf{T}_1 = 5\mathbf{i} + y\mathbf{j}$ and $\mathbf{T}_2 = x\mathbf{i} + 12\mathbf{j}$.
Find the values of x and y.

The weight is $\mathbf{W} = -4g\mathbf{j} = -39.2\mathbf{j}$

The **j** vector is vertically upwards and the weight acts downwards.

The particle is in equilibrium so $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = 0$

The resultant force is zero.

Substitute for the forces: $(5\mathbf{i} + y\mathbf{j}) + (x\mathbf{i} + 12\mathbf{j}) - 29.4\mathbf{j} = 0$

In the **i** direction $5 + x = 0$ giving $x = -5$

In the **j** direction $y + 12 - 39.2 = 0$ giving $y = 27.2$

15 Using Newton's 2nd law to find acceleration

The mass of a helicopter at the time it is launched is 2 tonnes. The rotary blades produce an upwards force of D N.

i Find the greatest value of D for which lift-off will not be achieved.

ii The initial value of D is actually 23 000. Calculate the acceleration of the helicopter.

i Mass of helicopter is 2 tonnes = 2000 kg

Lift-off will not be achieved if $D <$ weight $= 2000 \times 9.8 = 19\,600$ N

When $D < 19\,600$ N, there will be a contact force R between the helicopter and the ground. Lift off is achieved when $R = 0$.

ii $D = 23\,000$

Newton's second law $D - 2000g = 2000a$

The upwards direction is taken to be positive in this equation.

$$a = \frac{23\,000 - 19\,600}{2000} = 1.7\,\text{m s}^{-2}$$

16 Using Newton's 2nd law to find a force

Iyana pushes a toy car of mass 0.7 kg along a straight horizontal track with a horizontal force for 0.8 s. The resistance to motion is 5 N. The track is straight and horizontal and the car accelerates from an initial speed of 0.4 m s^{-1} and travels 80 cm.

Calculate the pushing force that Iyana applies.

Calculate the acceleration using $u = 0.4,\ s = 0.8,\ t = 0.8$

Use $s = ut + \frac{1}{2}at^2$ giving $0.8 = 0.4 \times 0.8 + \frac{1}{2}a \times 0.8^2$

$a = \frac{0.48}{0.32} = 1.5\,\text{m s}^{-2}$

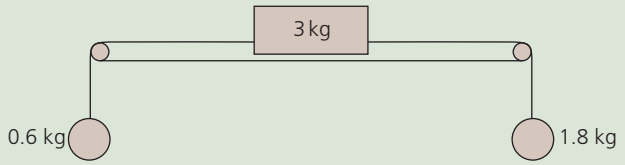
As the motion is horizontal, the weight of the car is not included in this equation.

To find the force use Newton's second law $P - 5 = ma = 0.7 \times 1.5$.

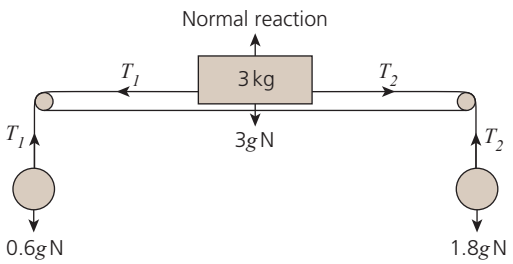
$P = 6.05$ N

17 Connected particles

A block of mass 3 kg is placed on a smooth horizontal table and is attached to objects of mass 0.6 kg and 1.8 kg on opposite sides as shown in the diagram. The strings are light and inextensible and pass over smooth light pulleys. The system is released from rest.



Write down the equations of motion for the block and each of the objects.



Horizontally to the right for the 3 kg block $\quad T_2 - T_1 = 3a$

Vertically down for the 1.8 kg object $\quad 1.8g - T_2 = 1.8a$

Vertically up for the 0.6 kg object $\quad T_1 - 0.6g = 0.6a$

18 Writing forces in vector form

In this question all quantities are in S.I. units and the x and y directions are East and North. Unit vectors in these directions are denoted by $\mathbf{i}$ and $\mathbf{j}$.
A particle of mass 3 kg experiences two forces, its weight $\mathbf{W}$ and the force $\mathbf{F}$ in the direction of $2\mathbf{i}+1.5\mathbf{j}$ with magnitude 10 N. Write these forces in vector form.

$\mathbf{W} = -3g\mathbf{j} = -29.4\mathbf{j}$

The **j** vector is vertically upwards and the weight acts downwards.

The force $\mathbf{F}$ is a multiple of $2\mathbf{i}+1.5\mathbf{j}$ and has magnitude 10.

This is the only way that vectors can be parallel to each other.

$|2\mathbf{i}+1.5\mathbf{j}| = \sqrt{2^2+1.5^2} = 2.5$

Find the magnitude of the direction vector and the choose the correct multiple to give a magnitude of 10.

So $\mathbf{F} = \frac{10}{2.5}(2\mathbf{i}+1.5\mathbf{j}) = 4(2\mathbf{i}+1.5\mathbf{j}) = 8\mathbf{i}+6\mathbf{j}$

19 Using the vector form of Newton's 2nd law to find acceleration

In this question the x and y directions are East and North, respectively.
A particle of mass 2 kg moves in a horizontal plane. The forces $\mathbf{F}_1 = -2\mathbf{i}-5\mathbf{j}$ and $\mathbf{F}_2 = 5\mathbf{i}+\mathbf{j}$ act on the particle. Find the acceleration of the particle.

The motion is in the horizontal plane so the weight of the object does not need to be included.

Newton's second law $\mathbf{F}_1 + \mathbf{F}_2 = m\mathbf{a}$

The resultant force is founding by adding the force vectors.

$(-2\mathbf{i}-5\mathbf{j})+(5\mathbf{i}+\mathbf{j}) = 2\mathbf{a}$

$\mathbf{a} = \frac{1}{2}(3\mathbf{i}-4\mathbf{j}) = 1.5\mathbf{i}-2\mathbf{j}$

20 Using Newton's law in vector form to find an unknown force

In this question, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors horizontally and vertically upwards respectively.
A particle of mass 0.5 kg experiences two forces, its weight and the force $\mathbf{F}$. The particle accelerates with $\mathbf{a} = 1.5\mathbf{i}-\mathbf{j}$ m s^{-2}. Find the force $\mathbf{F}$.

The weight is $\mathbf{W} = -0.5g\,\mathbf{j} = -4.9\mathbf{j}$

Newton's second law $\mathbf{W}+\mathbf{F} = m\mathbf{a} = 0.5\mathbf{a}$

Substitute for $\mathbf{W}$ and $\mathbf{a}$.

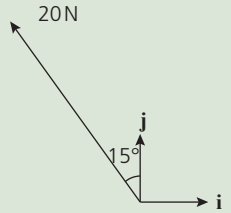
$-4.9\mathbf{j}+\mathbf{F} = 0.5\mathbf{a} = 0.5(1.5\mathbf{i}-\mathbf{j})$

$\mathbf{F} = 4.9\mathbf{j}+0.75\mathbf{i}-0.5\mathbf{j} = 0.75\mathbf{i}+4.4\mathbf{j}$

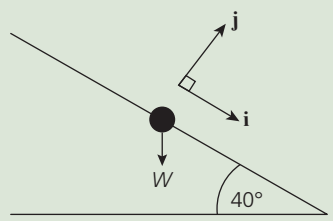
21 Resolving forces

In each case below, write the force in vector form, using the **i** and **j** vectors shown.

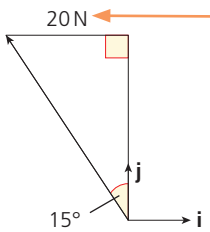
i



ii

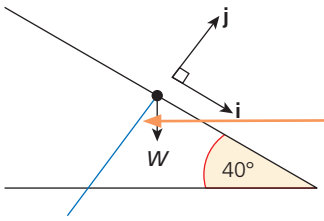


i Force given by $-20\sin 15°\mathbf{i} + 20\cos 15°\mathbf{j} = -5.18\mathbf{i} + 19.3\mathbf{j}$



Notice the **i** component is in the negative direction.

ii $W\sin 40°\mathbf{i} - W\cos 40°\mathbf{j}$



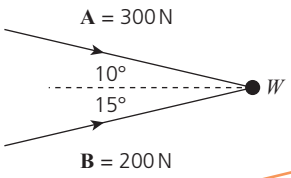
It is much easier to see which component is which if you add the line at right angles to the plane and mark the angle between it and the weight 40°.

22 Resolving forces to find acceleration

Annie and Bertie are pushing their car with mass 800 kg. Annie pushes with a force of 300 N at an angle of 10° to the direction of travel and Bertie with a force of 200 N at an angle of 15°.

i Show that the resultant force at right angles to the direction of travel is very small.

ii Find the acceleration of the car in the direction of motion.



Indicate clearly which direction you are taking to be positive.

i Resolve (↑) perpendicular to the direction of motion.

Resultant force is $-300\sin 10° + 200\sin 15° = -52.094 + 51.76 = 0.33$

0.33 N is very small compared to the forces involved.

A comment is needed here to get full credit.

ii Resolve (→) in the direction of motion.

$300\cos 10° + 200\cos 15° = ma$

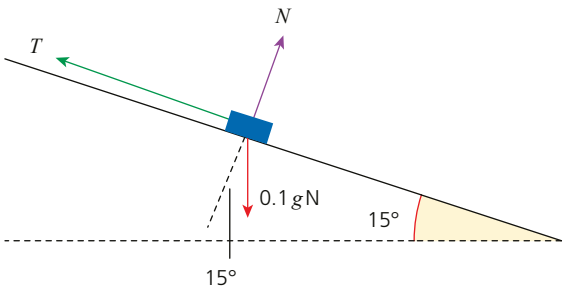
Newton's second law.

$488.627... = 800a$

$$a = \frac{488.627...}{800} = 0.611 \text{ m s}^{-2} \text{ (3 s.f.)}$$

23 Analysing motion on an inclined plane

A block of mass 0.1 kg is being pulled by a light string up a smooth plane. The plane makes an angle of 15° with the horizontal and the string is parallel to the slope. The block has an acceleration of 1.5 m s^{-2}. Find the tension in the string.



Resolve up the plane.

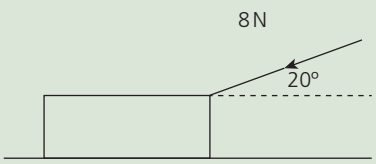
$T - 0.1g\sin 15° = 0.1a = 0.1 \times 1.5$

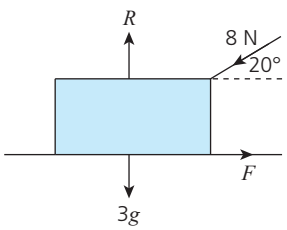
Remember to include the component of the weight down the plane.

$T = 0.1 \times 9.8\sin 15° + 0.15 = 0.404 \text{ N (3 s.f.)}$

24 Using the model for friction

A block of mass 3 kg rests on a rough horizontal surface. The coefficient of friction between the surface and the block is 0.2. A force of 8 N at 20° to the horizontal pushes the block. Calculate the acceleration of the block.





Draw a diagram showing all the forces acting on the block.

Resolve vertically to find the normal reaction between the block and the surface.

$R - 3g - 8\sin 20° = 0$

The block is not moving in this direction.

Common mistake: It is a common mistake to think that the normal reaction is always equal to the weight of the block. The vertical component of the 8 N force changes the normal reaction.

$R = 3 \times 9.8 + 8\sin 20° = 32.136...$

The block is moving so $F = \mu R = 0.2 \times 32.136... = 6.427...$

Newton's second law in the direction of motion

$$8\cos 20° - F = ma$$

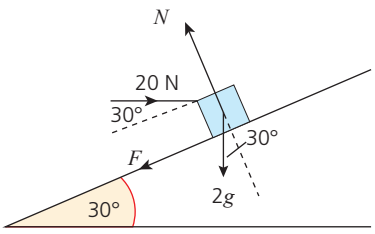
$$8\cos 20° - 6.427\ldots = 3a$$

$$a = 0.363\,\text{m s}^{-2}\ (3\text{ s.f.})$$

Right to left.

25 Finding the coefficient of friction

A block of mass 2 kg is placed on a rough plane inclined at 30° to the horizontal. A horizontal force of 20 N acts on the block which is on the point of sliding up the slope. Calculate the coefficient of friction between the block and the plane.



It is possible to solve this problem by resolving horizontally and vertically, but it will lead to simultaneous equations for F and N. It is usually easiest to resolve in the direction of the forces you are trying to find.

Resolve perpendicular to the plane.

$$N - 20\sin 30° - 2g\cos 30° = 0$$

$$N = 20\sin 30° + 2g\cos 30° = 26.974\ldots$$

Resolve parallel with the plane.

$$20\cos 30° - F - 2g\sin 30° = 0$$

Up the plane is the positive direction here.

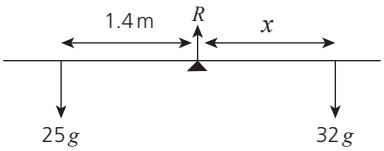
$$F = 20\cos 30° - 2\times 9.8\sin 30° = 7.5205\ldots$$

The block is on the point of sliding so $F = \mu N$.

So $\mu = \dfrac{7.5205\ldots}{26.974\ldots} = 0.279$ to 3 s.f.

26 Using moments in an equilibrium problem to find distance

Sally and Nasreen are balancing on a seesaw. Sally has a mass of 25 kg and Nasreen 32 kg. Sally sits 1.4 m from the middle of the seesaw. Calculate the distance from the middle that Nasreen should sit on the other side of the seesaw.



There is a reaction force at the pivot which we are not asked about but should be included in the force diagram.

Take moments about the centre.

The reaction at the pivot has no moment about the pivot and so it is not included in the equation.

$$25g\times 1.4 = (32g)x$$

$$x = \frac{25\times 1.4}{32} = 1.09\,\text{m}\ (3\text{ s.f.})$$

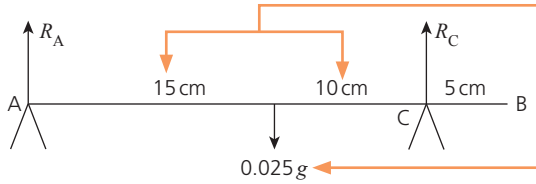
27 Using moments to solve equilibrium problems

A ruler AB is 30 cm long and has mass 25 g. It is held horizontally by two smooth pegs, one at A and the other at C which is 5 cm from B.

i Assuming that the ruler is uniform, find the contact forces at the pegs.

ii A downwards force is applied at B. Calculate the largest force that can be applied before the ruler tips.

i



Put the distances onto the diagram.

The weight needs to act at the centre of the ruler as it is assumed to be uniform.

Take moments about A $0.025g \times 0.15 = R_C \times 0.25$

Add the 15 cm and 10 cm to find the distance between A and C. You can leave all the lengths in cm if you prefer.

$$R_C = \frac{0.025 \times 9.8 \times 0.15}{0.25} = 0.147\,\text{N}$$

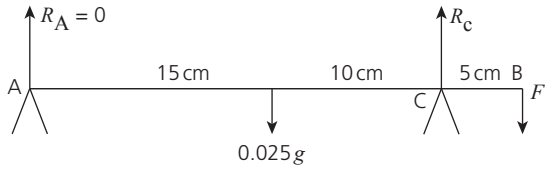
The vertical forces are in equilibrium so $R_A + R_C = 0.025g$

$$R_A = 0.025 \times 9.8 - 0.147 = 0.098\,\text{N}$$

You can check your working by taking moments about C. $R_A \times 0.25 = 0.025g \times 0.1$.

ii When the ruler is on the point of tipping, the reaction $R_A = 0$

As F increases, the reaction at A decreases and the reaction at C increases.



The reaction at C is not needed, so take moments about C.

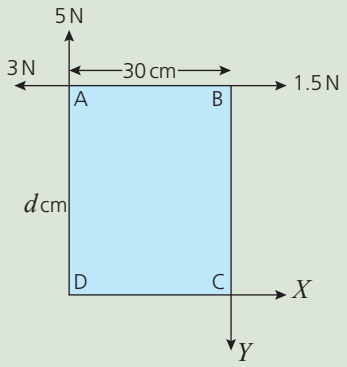
Take moments about C

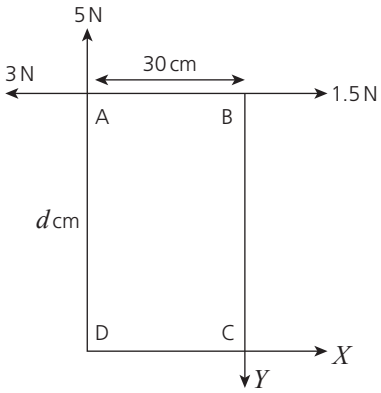
$F \times 0.05 = 0.025g \times 0.1$

$F = 0.05g = 0.49\,\text{N}$

28 Using moments for problems with a rectangular lamina

A rectangular lamina ABCD is in equilibrium under the action of the forces 3 N and 5 N at A, 1.5 N at B and X N and Y N at C as shown in the diagram. The length AB is 30 cm and AD is d cm. Calculate the values of X, Y and d.





Vertical forces in equilibrium $Y = 5\,\text{N}$

Horizontal forces in equilibrium $X + 1.5 - 3 = 0$ giving $X = 1.5\,\text{N}$

Notice the three given forces all have zero moment about A.

Take moments about A.

$Y \times 30 = 1.5d$

$$d = \frac{5 \times 30}{1.5} = 100\,\text{cm}$$

You can work in cm or m for lengths – you just need to be consistent.

29 Using differentiation to investigate motion

Abebe runs in a straight line and his displacement in metres from the origin O at time t seconds after the start of his run is given by $s = 15 - 1.47t + 0.01t^3$ for $0 \leqslant t \leqslant 20$.

i Determine whether the acceleration is constant.

ii Determine whether Abebe changes direction during his 20 s run.

i $s = 15 - 1.47t + 0.01t^3$

Differentiate to find velocity $v = \frac{ds}{dt} = -1.47 + 0.01 \times 3t^2 = -1.47 + 0.03t^2$

Differentiate again to find acceleration $a = \frac{dv}{dt} = 0.03 \times 2t = 0.06t$

This is a function of t so the acceleration is not constant.

You could also argue that if the acceleration is constant, the displacement is a quadratic function of t. This function is cubic so the acceleration is not constant.

ii Abebe changes direction when $v = 0$.

The velocity will change from positive to negative or vice versa and so must be zero as it changes.

$v = -1.47 + 0.03t^2 = 0$

$t = \pm\sqrt{\frac{1.47}{0.03}} = \pm 7\,\text{s}$

Reject the negative root.

So he does change direction after 7 s.

30 Motion with variable acceleration: integration

A particle travels in a straight line and at time t s its acceleration in m s^{-2} is given by $a = 1.5t + 2$. When $t = 0$, the particle is at the origin with a velocity of 4 m s^{-1} in the negative direction.

Find the distance of the particle from the origin when $t = 6$.

Integrate $a = 1.5t + 2$ to find an expression for velocity.

$v = \int v \, dt = \int (1.5t + 2) \, dt$

$v = 1.5 \times \frac{t^2}{2} + 2t + c = 0.75t^2 + 2t + c$

Common mistake: It is a common error to leave out the constant.

When $t = 0$, $v = -4$

The initial velocity is in the negative direction.

$-4 = 0.75 \times 0^2 + 2 \times 0 + c$ giving $c = -4$

So $v = 0.75t^2 + 2t - 4$

Integrate $v = 0.75t^2 + 2t - 4$ to find an expression for displacement.

$s = \int (0.75t^2 + 2t - 4) \, dt$

$= 0.75 \times \frac{t^3}{3} + 2 \times \frac{t^2}{2} - 4t + k$

This is a different constant to the one above, so a different letter should be used.

When $t = 0$, $s = 0$ giving $k = 0$

$s = 0.25t^3 + t^2 - 4t$

When $t = 6$, $s = 0.25 \times 6^3 + 6^2 - 4 \times 6 = 66\,\text{m}$

You could also obtain this answer by using the definite integral $\int_0^6 (0.75t^2 + 2t - 4) \, dt$. Your calculator may evaluate this integral which allows you to check that your answer is correct.

31 Using vectors and calculus to find acceleration

A child runs in a park and his displacement from an origin at time t seconds is given by $\mathbf{r} = \begin{pmatrix} 0.3t^2 \\ 0.2t^3 + 2t \end{pmatrix}$.

Find expressions for the child's velocity and acceleration. Determine whether the acceleration is constant.

Differentiate the expression for **s** to find an expression for **v**.

$$\mathbf{s} = \begin{pmatrix} 0.3t^2 \\ 0.2t^3 + 2t \end{pmatrix} \text{ so } \mathbf{v} = \frac{d\mathbf{s}}{dt} = \begin{pmatrix} 0.3 \times 2t \\ 0.2 \times 3t^2 + 2 \end{pmatrix} = \begin{pmatrix} 0.6t \\ 0.6t^2 + 2 \end{pmatrix}$$

Differentiate the expressions in each direction separately.

Differentiate **v** to find an expression for **a**.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \begin{pmatrix} 0.6 \\ 0.6 \times 2t \end{pmatrix} = \begin{pmatrix} 0.6 \\ 1.2t \end{pmatrix}$$

The expression for **a** is a function of t and so is not constant.

The x component does have constant acceleration but as the y component is a function of t, **a** is not constant.

32 Using vectors to find velocity and displacement

A particle moves with acceleration in m s^{-2} given by $\mathbf{a} = 0.12t\,\mathbf{i} - 0.1\mathbf{j}$. Initially the particle has a velocity of $\mathbf{v}_0 = -\mathbf{i} + 2\mathbf{j}$ and position $\mathbf{r}_0 = 5\mathbf{i} - \mathbf{j}$. Find expressions for the velocity and position of the particle at time t s.

Integrate $\mathbf{a} = 0.12t\mathbf{i} - 0.1\mathbf{j}$ to find an expression for **v**.

$$\mathbf{v} = \int \mathbf{a}\, dt = \int (0.12t\mathbf{i} - 0.1\mathbf{j})\, dt$$

$$\mathbf{v} = \left(0.12 \times \frac{t^2}{2}\right)\mathbf{i} - 0.1t\mathbf{j} + \mathbf{c} = 0.06t^2\mathbf{i} - 0.1t\mathbf{j} + \mathbf{c}$$

The expression needs a vector constant here. You could instead use a different constant in each direction if you prefer.

When $t = 0$, $\mathbf{v}_0 = -\mathbf{i} + 2\mathbf{j}$

$-\mathbf{i} + 2\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{c}$ so $\mathbf{c} = -\mathbf{i} + 2\mathbf{j}$

So $\mathbf{v} = 0.06t^2\mathbf{i} - 0.1t\mathbf{j} - \mathbf{i} + 2\mathbf{j}$

$$\mathbf{v} = (0.06t^2 - 1)\mathbf{i} + (2 - 0.1t)\mathbf{j}$$

It is usual to collect together all the terms in the **i** and **j** directions.

Integrate **v** to find an expression for position.

$$\mathbf{r} = \int \mathbf{v}\, dt = \int ((0.06t^2 - 1)\mathbf{i} + (2 - 0.1t)\mathbf{j})\, dt$$

$$\mathbf{r} = \left(0.06 \times \frac{t^3}{3} - t\right)\mathbf{i} + \left(2t - 0.1 \times \frac{t^2}{2}\right)\mathbf{j} + \mathbf{c}$$

$$\mathbf{r} = (0.02t^3 - t)\mathbf{i} + (2t - 0.05t^2)\mathbf{j} + \mathbf{c}$$

When $t = 0$, $\mathbf{r}_0 = 5\mathbf{i} - \mathbf{j}$

$\mathbf{r}_0 = 5\mathbf{i} - \mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{c}$ giving $\mathbf{c} = 5\mathbf{i} - \mathbf{j}$

$$\mathbf{r} = (0.02t^3 - t)\mathbf{i} + (2t - 0.05t^2)\mathbf{j} + 5\mathbf{i} - \mathbf{j}$$

$$\mathbf{r} = (0.02t^3 - t + 5)\mathbf{i} + (2t - 0.05t^2 - 1)\mathbf{j}$$

33 Finding the cartesian equation of the path

A particle moves so that its position at time t s is $\mathbf{r} = \begin{pmatrix} t - 2 \\ 0.1t^3 - 2t \end{pmatrix}$ m. Find the cartesian equation of the path.

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t - 2 \\ 0.1t^3 - 2t \end{pmatrix}$$

The key to this question is to remember that the position vector can be written $\begin{pmatrix} x \\ y \end{pmatrix}$ and then the vector equation can be rewritten as two separate equations.

This gives $x = t - 2$ and $y = 0.1t^3 - 2t$.

Rearrange $t = x + 2$.

Substitute for t in the expression for y.

$$y = 0.1(x + 2)^3 - 2(x + 2)$$

There is no need to rewrite this equation unless the question asks for a particular form.

Using graphs to analyse motion (page 133)

P and Q are two points 800 m apart on a straight road. A car passes the point P with a speed of 11 m s^{-1} and immediately accelerates uniformly, reaching a top speed of 25 m s^{-1} in 5 s. The driver continues at this speed for another 20 s before decelerating uniformly for T seconds at 1.25 m s^{-2} until he reaches Q. When the car passes Q its speed is V m s^{-1}.

i Sketch the speed-time graph for the journey between P and Q.

ii Find:

A the acceleration

B the distance travelled from P to reach the top speed.

iii Find the values of T and V.

iv **A** Find the average speed for the journey between P and Q.

B Draw the acceleration–time graph for the journey from P to Q.

i In the accelerating phase:

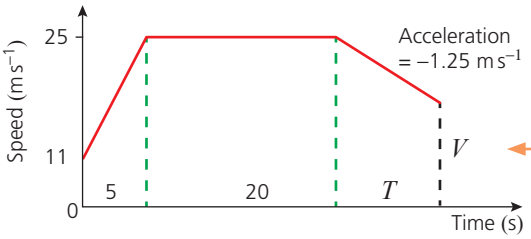
- the initial speed is 11 m s^{-1}
- the acceleration is uniform, so is a straight line on the graph
- the top speed is 25 m s^{-1}
- the car accelerates for 5 s
- the car then travels at 25 m s^{-1} for 20 s.

In the decelerating phase:

- the initial speed is 25 m s^{-1}
- the deceleration is 1.25 m s^{-2}
- the deceleration is uniform, so is a straight line on the graph
- the car decelerates for T s
- when the car passes Q its speed is V m s^{-1}.

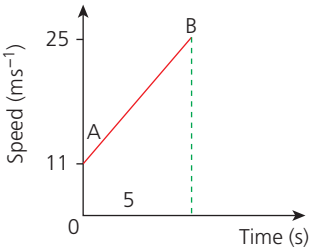
There is one further piece of information which you have been given, but it is not easy to put on the graph. The distance between P and Q is 800 m, so the total area under the graph is 800.

In order to sketch the speed-time graph it is a good idea to list the information you have been given.



Look at the accelerating phase on its own.

ii



A $\text{acceleration} = \dfrac{\text{change in speed}}{\text{time taken}}$

$\text{acceleration} = \dfrac{25-11}{5}$

$\text{acceleration} = 2.8\,\text{m s}^{-2}$

Acceleration is the gradient of the velocity–time graph.

B The distance is represented by the area under the graph.

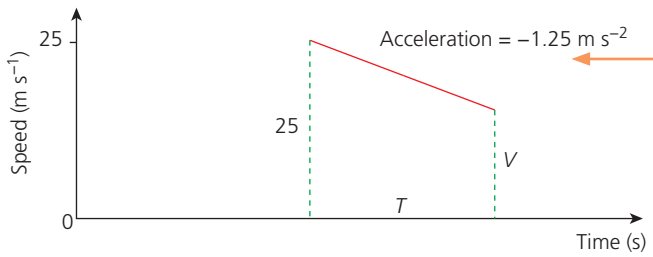
$\text{area} = \frac{1}{2}(11 + 25) \times 5$

$= 90$

The distance covered is 90 m.

Use the formula $A = \frac{1}{2}(a+b)h$ for the area of the trapezium.

iii acceleration $= \dfrac{\text{change in speed}}{\text{time taken}}$



This part deals with the decelerating phase, so you would find it helpful to begin with a sketch of the speed-time graph for the final Ts.

You are given that the deceleration is 1.8 m s^{-2} – this is the same as an acceleration of –1.8 m s^{-2}.

Look at the decelerating phase on its own. There are two unknowns, so a pair of equations must be made from the information in the question.

$-1.25 = \dfrac{V - 25}{T}$

$\therefore \quad V = 25 - 1.25T \qquad$ (A)

Acceleration is the gradient of the graph and can be used to form an equation linking T and V.

The car travels 90 m while accelerating.

$20 \times 25 = 500$ m at a steady speed.

$\therefore \quad (800 - 90 - 500) = 210$ m while decelerating

Distance can be used to form a second equation linking T and V.

So using the formula for the area of a trapezium:

$\frac{1}{2}(25 + V) \times T = 210 \qquad$ (B)

Using equations (A) and (B)

Substitute for V from equation (A) in equation (B)

$\frac{1}{2}(25 + (50 - 1.25T)) \times T = 210$

$50T - 1.25T^2 = 420$

This simplifies to give the quadratic equation:

$1.25T^2 - 50T + 420 = 0$

$\therefore \qquad T = 12$ or 28

Once you have collected the terms of one side of the equation, you can use your calculator to solve the quadratic equation.

If your calculator does not have this facility, you can solve the quadratic equation by dividing by 1.25 and factorising.

$(T - 12)(T - 28) = 0$

Alternatively, you can use the quadratic formula.

$T = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with

$a = 1.25, \quad b = -50, \quad c = 420$

So the car decelerates for 12 s before reaching Q.

When $T = 10$, $V = 25 - 1.25 \times 12 = 10$.

That is, the car decelerates for 12 s and has a speed of 10 m s^{-1} as it passes Q.

iv

In part **iv** you are asked two questions about the complete journey from P to Q.

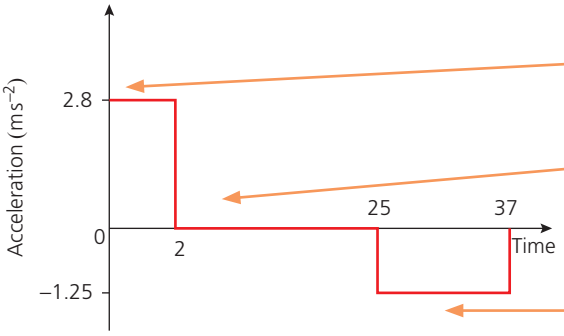
A average speed $= \dfrac{\text{total distance travelled}}{\text{total time taken}}$

total distance travelled = 800 m

total time taken = (5 + 20 + 12) = 37 s

$\therefore$ average speed $= \dfrac{800}{37} = 21.6 \text{ m s}^{-1}$

B The acceleration is constant in each of the three phases. It suddenly changes to the next value, giving a vertical line on the graph.



For the first 5 s the acceleration is 2.8 m s^{-2}.

For the next 20 s the acceleration is 0 m s^{-2} as the car is travelling at a steady speed.

For the final 12 s the acceleration is –1.25 m s^{-2}.

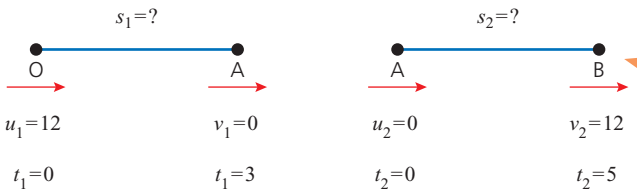
Using the constant acceleration formulae (page 137)

A motorcycle is travelling along a straight road with an initial speed of 12 m s^{-1} when it starts to slow down at a traffic light. Its deceleration is constant. The motorcycle stops after 3 s, waits for 10 s and then accelerates uniformly to its original speed of 12 m s^{-1} in 5 s.

i Calculate the distance travelled by the motorcycle during these 18 s.

ii How much less time would the motorcycle have taken to travel this distance if it had maintained the speed of 12 m s^{-1} throughout?

There are two stages to the motion and you can represent them on a diagram showing the distance travelled. In the first stage OA, it is slowing down. At A it is stationary for some time. Then from A to B it speeds up.



When you come to put in the values of u, v and t you come across a problem; they are different for the two parts of the motion, from O to A and from A to B. So a good idea is to call them u_1, v_1 and t_1, from O to A, and the required distance s_1.

Use $u_1 = 12$ at $t_1 = 0$, $v_1 = 0$ at $t_1 = 3$ and s_1 is to be found.

In the same way, for the motion from A to B,

$u_2 = 0$ at $t_2 = 0$, $v_2 = 12$ at $t_2 = 5$ and s_2 is to be found.

i $s_1 = \dfrac{1}{2}(u_1 + v_1)t_1$

Use this formula as it does not contain a which is neither given nor asked for.

$= \dfrac{1}{2}(12 + 0) \times 3$

$= 18$

Between C and D, $u = 0$, $v = 12$ and $t = 5$.

$s_2 = \dfrac{1}{2}(u_2 + v_2)t_2$

$= \dfrac{1}{2}(0 + 12) \times 5$

$= 30$

Total distance travelled = 18 + 30 m

= 48 m

It is easy to forget to add the two separate distances together.

ii Total distance (s) = 48 m total journey took 18 s.
Time at constant speed of 12 m s^{-1}.

$$\text{time} = \frac{\text{total distance travelled}}{\text{speed}}$$

$$t = \frac{48}{12}$$

$$= 4$$

Time at constant speed = 4 s

Time lost = 18 − 4 = 14 seconds.

You can use the equation $s = ut + \frac{1}{2}at^2$ using $s = 48$, $u = 12$ and $a = 0$. This gives the same equation

The extra time taken is the difference between the two times.

Vertical motion under gravity (page 141)

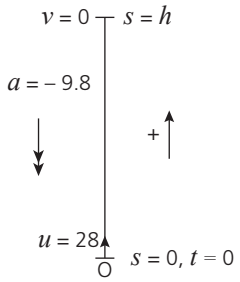
A particle, P, is projected vertically upwards at 28 m s^{-1} from a point O on the ground.

i Calculate the maximum height of P.

When P is at its highest point, a second particle, Q, is projected upwards from O at 25 m s^{-1}.

ii Show that P and Q collide 1.6 s later and determine the height above the ground that this takes place.

i Take the point of projection as the origin and upwards to be the positive direction.
At the highest point $v = 0$.



Put all the information onto a diagram. Be clear which you are taking as the positive direction.

$u = 28$, $a = -9.8$, $v = 0$ and $s = h$

$v^2 = u^2 + 2as$

$0 = 28^2 + 2 \times (-9.8) \times h$

$19.6h = 784$

$$h = \frac{784}{19.6}$$

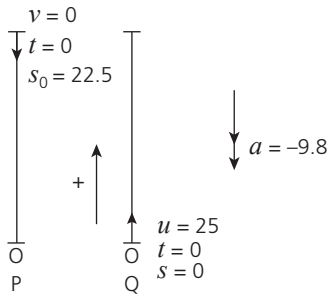
$h = 40$

The maximum height is 40 m.

Use the formula which does not use t.

Substitute all the values that you know. Rearrange to find h.

ii



Q: $u = 25$, $a = -9.8$.

- Reset the clock when the first particle begins to fall from the highest point.
- Keep the same origin and still take the positive direction to be upwards. You need a new diagram showing the initial situation for both particles.

For P: $t = 0$, $s_0 = 40$ and $u = 0$

For Q: $t = 0$, $s = 0$ and $u = 25$

For both particles $a = -9.8$.

$s = ut + \frac{1}{2}at^2$

$s = 25t - 4.9t^2$

P: $s = s_0 + ut + \frac{1}{2}at^2$

$s = 40 + 0 - 4.9t^2$

$s = 40 - 4.9t^2$

The particles collide when

$25t - 4.9t^2 = 40 - 4.9t^2$

$25t = 40$

$t = \frac{40}{25} = 1.6$

They collide after 1.6 s.

$t = 1.6$

$s = 40 - 4.9 \times 1.6^2$

$s = 40 - 4.9 \times 2.56$

$s = 40 - 12.544$

$s = 27.456$

They collide at 27.5 m (3 s.f.).

Find an expression for the height of Q above the origin at time t.

Remember to include the initial height of P above the origin.

It is often best to solve the simultaneous equations and check the answer against the given answer. It is important to show all the stages of working when the answer is given.

Another method is to use the given answer and find the height of each particle at time $t = 1.6$. You must make a very clear conclusion if you decide to use this method.

Substituting $t = 1.6$ in the other equation is useful as a check. $25 \times 1.6 - 4.9 \times 1.6^2$

Motion using calculus (page 147)

A particle moves along the x-axis with velocity v m s^{-1} at time t s given by $v = -5 + 6t - t^2$.

i Find an expression for the acceleration of the particle at time t.

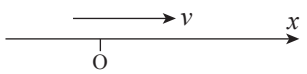
ii Find the times t_1 and t_2, where $t_1 < t_2$, at which the particle has zero velocity.

iii Find the distance travelled between the times t_1 and t_2.

iv At time t_1 the particle passes through the point A. Does the particle pass through A on any later occasion? At time t_2 the particle passes through the point B. Does the particle pass through B on any later occasion?

v Find the distance travelled from $t = 0$ to $t = 6$.

i Motion takes place in a straight line as shown in the diagram.



The question asks for displacement as well as distance, so the direction of motion will be important.

Acceleration is the derivative of velocity.

$v = -5 + 6t - t^2$

$a = \frac{dv}{dt} = 6 - 2t$

Differentiate the expression for velocity to get the acceleration.

ii When $v = 0$, $-5 + 6t - t^2 = 0$

- It is easy to make a mistake here and use the expression you have just found for acceleration when velocity is needed. Notice the question indicates that there are two values, which this quadratic equation will give.
- The values for t_1 and t_2 are the roots of this equation and the question states that t_1 is the smaller value.

Rewriting $t^2 - 6t + 5 = 0$

$t = 1$ or 5

So $t_1 = 1$ and $t_2 = 5$

Use your calculator to solve this equation if it has this facility. Otherwise factorise or use the quadratic formula. When the question asks for detailed reasoning, you should use an algebraic method to solve the equation.

The steps for solving the equation algebraically are

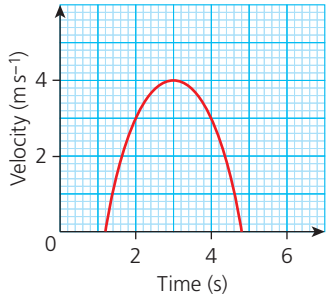
$-5 + 6t - t^2 = 0$

$\Rightarrow t^2 - 6t + 5 = 0$

$\Rightarrow (t - 1)(t - 5) = 0$

$\Rightarrow t = 1$ or $t = 5$

iii Between $t_1 = 1$ and $t_2 = 5$, the velocity is positive, so the distance travelled is the same as the displacement.



Integrate velocity between the limits:

The definite integral means that you do not need to find the constant of integration.

$$s = \int_1^5 \left(-5 + 6t - t^2\right) \mathrm{d}t$$

$$= \left[-5t + 6 \times \frac{t^2}{2} - \frac{t^3}{3}\right]_1^5$$

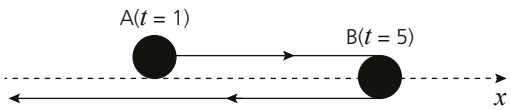
$$= \left[-5t + 3t^2 - \frac{1}{3}t^3\right]_1^5$$

$$= \left(-5 \times 5 + 3 \times 5^2 - \frac{5^3}{3}\right) - \left(-5 \times 1 + 3 \times 1^2 - \frac{1^3}{3}\right)$$

Take care with signs here. You may be able to use your calculator to check this answer.

$$= \left(\frac{25}{3}\right) - \left(-\frac{7}{3}\right) = \frac{32}{3} \text{ m}$$

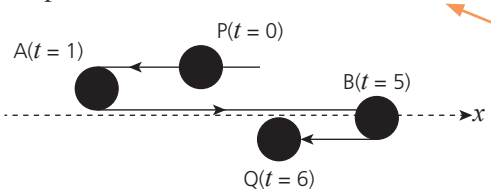
iv The particle is at point A at time $t = 1$ and at point B at time $t = 5$.



The particle travels in the positive direction from A to B and then in the negative direction at all later times. This means that the particle will pass again through A but not pass again through B.

You could also answer this question algebraically by equating the expression for s to $\frac{25}{3}$ and to $-\frac{7}{3}$ and solving the cubic equations. This is not too difficult if your calculator will solve the equations.

v Displacement in the interval $0 \leqslant t \leqslant 1$:



In the interval from $t = 0$ to $t = 6$, the particle changes direction twice. To find the total distance travelled, the displacement for the intervals $t = 0$ to $t = 1$ and $t = 5$ to $t = 6$ must be found and the distances added to the $\frac{32}{3}$ m travelled from A to B.

$$s = \int_0^1 \left(-5 + 6t - t^2\right) \mathrm{d}t = \left[-5t + 3t^2 - \frac{1}{3}t^3\right]_0^1$$

$$= \left(-5 \times 1 + 3 \times 1^2 - \frac{1^3}{3}\right) - \left(-5 \times 0 + 3 \times 0^2 - \frac{0^3}{3}\right) = -\frac{7}{3}$$

so the distance is $\frac{7}{3}$ m in the negative direction.

Displacement in the interval $5 \leqslant t \leqslant 6$:

$$s = \int_5^6 \left(-5 + 6t - t^2\right) \mathrm{d}t = \left[-5t + 3t^2 - \frac{1}{3}t^3\right]_5^6$$

$$= \left(-5 \times 6 + 3 \times 6^2 - \frac{6^3}{3}\right) - \left(-5 \times 5 + 3 \times 5^2 - \frac{5^3}{3}\right)$$

$$= 6 - \frac{25}{3} = -\frac{7}{3}$$

so the distance is $\frac{7}{3}$ m in the negative direction.

Total distance travelled = PA + AB + BQ $= \frac{7}{3} + \frac{32}{3} + \frac{7}{3} = 15\frac{1}{3}$ m.

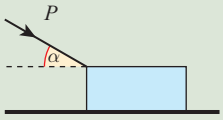
Note that evaluating $s = \int_0^6 \left(-5 + 6t - t^2\right) \mathrm{d}t$ gives you the displacement from P to Q which is 6 m.

Forces (page 153)

A courier is delivering a box of books of mass m kg.

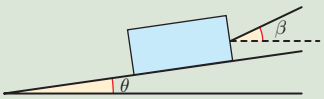
First he pushes it along rough level ground with a force P as shown in the diagram.

i Draw a diagram to show the forces acting on the box. Is the normal reaction with the ground equal to the weight of the box?



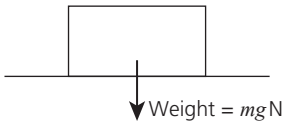
When the courier reaches a ramp, he decides to pull the box with a rope making an angle of β to the horizontal as in the diagram to the right. The ramp also is rough and makes an angle of θ with the horizontal.

ii Draw a diagram to show the forces now acting on the box.

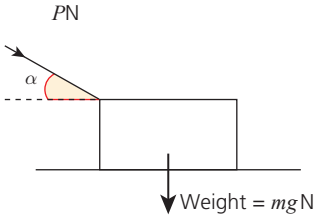


i The forces are added one at a time to this sequence of diagrams, but a single diagram with all the forces would be the usual answer to this question.

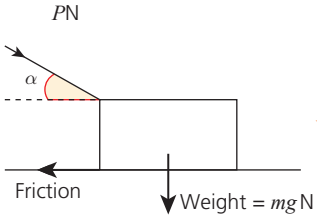
Begin with a drawing of the physical object.



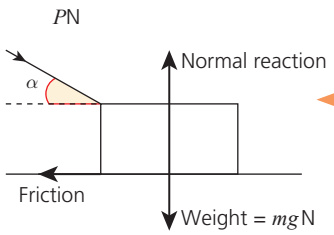
The weight must be vertically downwards. It is equal to the mass of the box multiplied by g, acceleration due to gravity.



Any forces mentioned in the question must be added to the diagram. Take care to indicate the direction of the force.



Under the action of the forces so far, the box would slide to the right, so the friction force must act in the opposite direction to prevent likely motion.

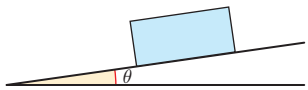


Wherever there is contact between two objects, there is a contact force which acts at 90° to the floor.

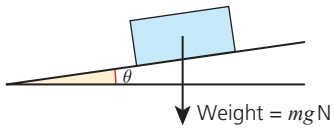
The normal reaction is not equal to the weight. Without the force P, the normal reaction would be equal to the weight, but the force P in part pushes the box into the floor, so the normal reaction is increased to balance this.

The study of components is covered in section 6 of this chapter.

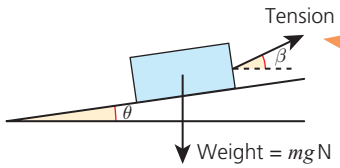
ii



Begin with a diagram showing the object in the question.



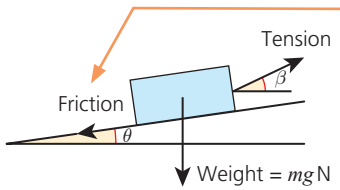
The weight acts vertically downwards through the middle of the box. It is an easy mistake to make to draw in the weight at 90° to the slope.

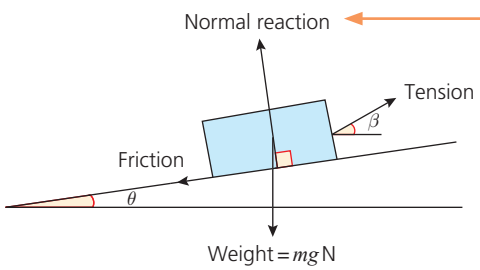


Draw in the force mentioned in the question.

The box is being pulled up the slope so the friction force opposes the motion.

Draw the frictional force along the surface pointing down the slope.

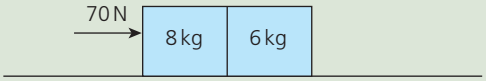




The normal reaction must be drawn at 90° to the slope – this is what the word 'normal' means in this context.

Newton's 1st law of motion (page 156)

The diagram shows two boxes of mass 8 kg and 6 kg standing together on a rough horizontal surface. Josh tries to push the boxes by applying a horizontal force of 70 N as shown in the diagram. There are frictional forces of $8k$ N and $6k$ N acting on the two boxes. The boxes do not move.



i Draw separate diagrams for each box showing all the forces acting on them.
ii Find the magnitude of the frictional forces acting on the 8 kg box.
iii Find the normal reaction between the 6 kg box and the surface.
iv Find the contact force between the two boxes.

i Draw in the weight of each box.

> Each weight is equal to the mass multiplied by g. It is measured in newtons.

Put in the normal reaction for each box.

> Notice these are not equal so need to be labelled differently.

The contact force is stopping the boxes crushing together.

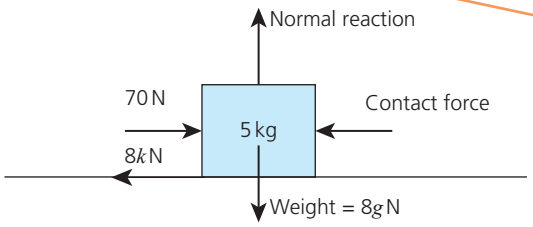
> It acts in the backward direction on the 8 kg box and forward direction on the 6 kg box.

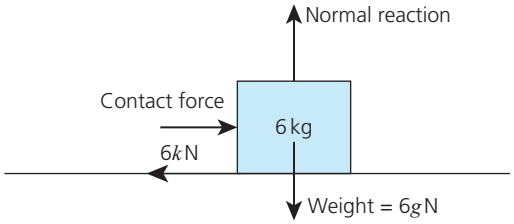
The 70 N force acts only on the 8 kg.

> Its effect is only experienced by the 6 kg box because the contact force changes.

The likely motion is in the direction of the 70 N force, so the friction forces are both in the backward direction.

> Use the values for these forces as given in the question.





> This question can also be done by considering each box separately and eliminating the contact force from the two equations.

ii Consider the two boxes together with combined friction force of $14k$ N.

$$70 = 14k$$

> The eternal force forward is balanced by the two friction forces added together.

$$k = \frac{70}{14} = 5$$

> Rearrange to find the value of k and use that value to find out the size of the friction force.

Friction force $= 8k = 40$ N

iii Consider the vertical forces on the 6 kg box.

> The vertical forces are in equilibrium so as there are only two, they are equal is size but opposite in direction.

$N_2 = 6g = 6 \times 9.8 = 58.8$ N

iv Consider the 8 kg box. The horizontal forces are in equilibrium.

> Either box can be used for this. You can check the answer by repeating this process for the other box.

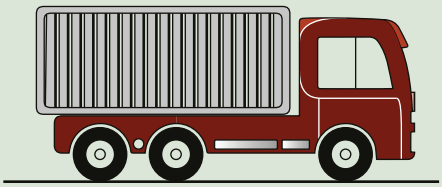
$$70 - 40 - C = 0$$

$$C = 30\,\text{N}$$

The resultant force is the total force in the positive direction. The friction and the contact force act in the negative direction and so are negative in the equation.

Applying Newton's 2nd law along a line (page 160)

A lorry of mass 40 tonnes is travelling along a straight, level road.



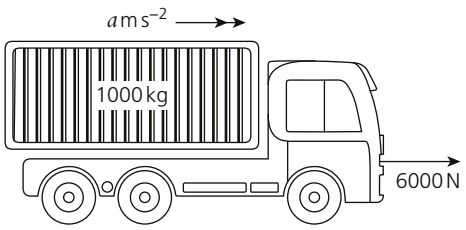
i Calculate the acceleration of the lorry when a resultant force of 60 000 N acts on it in the direction of its motion.

ii How long does it take the lorry to increase its speed from $5\,\text{m s}^{-1}$ to $12.5\,\text{m s}^{-1}$?

iii The lorry has an acceleration of $1.3\,\text{m s}^{-2}$ when there is a driving force of 80 000 N. Calculate the resistance to motion of the lorry.

i The mass of the lorry is 4 tonnes = 40 000 kg

Make sure the units are the ones you need – to obtain force in newtons you need mass in kg.



In this diagram, the forces in the vertical direction do not affect the motion, so the diagram shows only the resultant horizontal force.

$$F = ma$$

$$60\,000 = 40\,000a$$

$$a = \frac{60\,000}{40\,000}$$

$$a = 1.5$$

Use Newton's second law to link force and acceleration.

The acceleration is $1.5\,\text{m s}^{-2}$ and is constant.

Acceleration will be constant when the forces are constant. This means that the *suvat* equations are useful here.

ii $u = 5$, $v = 12.5$ and $a = 1.5$

$$v = u + at$$

Use this equation as the displacement is not given.

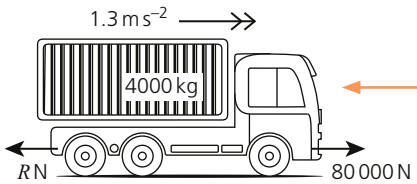
$$12.5 = 5 + 1.5t$$

$$7.5 = 1.5t$$

$$t = 7.5 \div 1.5 = 5$$

The time taken is 5 s.

iii



Draw a new diagram when the situation changes.

Resultant forward force: $80\,000 - R$

Acceleration: $a = 1.3$

Using $F = ma$,

$$80\,000 - R = 40\,000 \times 1.3$$
$$= 52\,000$$
$$80\,000 - 52\,000 = R$$
$$R = 28\,000$$

The resistance to motion is 28 000 newtons.

The resultant force is the difference between the forward force and the resistance which acts in the opposite direction.

It is very easy to make a mistake with these large numbers – take care not to type one extra zero or one too few zeros into the calculator.

Connected particles (page 163)

Blocks of mass 0.5 kg (A) and 0.3 kg (B) are attached to the ends of a light string which hangs vertically over a smooth light pulley. The system is released from rest.

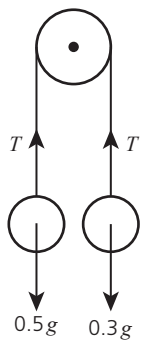
i Draw a diagram to show the forces and acceleration of the blocks.
ii Find the acceleration of the system.
iii Find the tension in the string.
iv Find the velocities of A and B after 2 s.

i Draw the objects first.
Put the weights of the blocks in next.

These act vertically downwards and are equal to the mass multiplied by g, acceleration due to gravity. This gives the weight in newtons.

The tension in the string acts upwards either side of the pulley. The pulley is smooth and the string light so the value of tension is the same on both sides.

Be sure that the value is the same before deciding to use the same letter. You will get fewer marks if you label them as T_1 and T_2.



ii The system will move so that the 0.5 kg mass falls and the 0.3 kg mass rises.
Use the direction of motion of each block as the positive direction for its equation of motion.
Newton's second law downwards for the 0.5 kg block: $0.5g - T = 0.5a$
Newton's second law upwards for the 0.3 kg block: $T - 0.3g = 0.3a$

Add the equations: $0.2g = 0.8a$

T is positive in one equation and negative in the other, so adding the equations will eliminate T from the simultaneous equations.

So $a = \dfrac{0.2g}{0.8} = \dfrac{g}{4} = 2.45\,\text{m s}^{-2}$

iii Substitute for a in one of the equations to find T.

$T = 0.3a + 0.3g = 0.3 \times 2.45 + 0.3 \times 9.8 = 3.675\,\text{N}$

The equation where T is positive is less likely to produce sign errors in your working. You can check your answer using the other equation.

iv The acceleration is constant so use the *suvat* equations with $u = 0,\ a = 2.45,\ t = 2$.
Use $v = u + at$ giving $v = 0 + 2.45 \times 2 = 4.9\,\text{m s}^{-1}$

Kinematics in 2 dimensions (page 171)

In this question the origin of position vectors is O and **i** and **j** are unit vectors east and north, respectively.

A toy car accelerates uniformly from $\mathbf{u} = -4\mathbf{i}\,\text{m s}^{-1}$ to $\mathbf{v} = 2\mathbf{i}+6\mathbf{j}\,\text{m s}^{-1}$ in 3 seconds.

i Find the acceleration of the car.

ii Determine what bearing the car is travelling on when $t = 1.5$ s.

iii Determine the times (if any) when the car is (instantaneously) at rest.

At time $t = 2$, the position vector of the car is $(-\mathbf{i}+2\mathbf{j})$m.

iv Find the position vector of the car when $t = 0$ and hence an expression for the position of the car at time t s.

v Determine the times (if any) when the car is southwest of O.

i Constant acceleration so use the vector form of the *suvat* equations.

$\mathbf{u} = -4\mathbf{i},\ \mathbf{v} = 2\mathbf{i}+6\mathbf{j},\ t = 3$

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

Use the equation that does not involve **s**.

$2\mathbf{i}+6\mathbf{j} = -4\mathbf{i}+3\mathbf{a}$

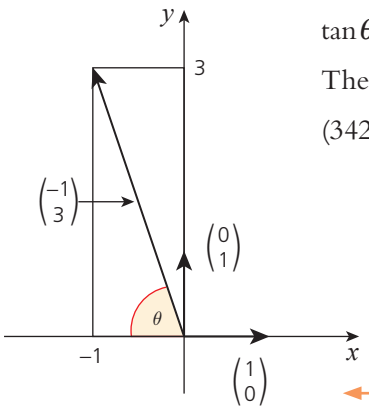
$$\mathbf{a} = \frac{1}{3}\left((2\mathbf{i}+6\mathbf{j})-(-4\mathbf{i})\right) = \frac{6\mathbf{i}+6\mathbf{j}}{3} = 2\mathbf{i}+2\mathbf{j}\ \text{m s}^{-2}$$

Leave your answer as a vector.

ii When $t = 1.5$ s,

$\mathbf{v} = \mathbf{u}+\mathbf{a}t = -4\mathbf{i}+(2\mathbf{i}+2\mathbf{j})\times 1.5 = -\mathbf{i}+3\mathbf{j}$

The direction of travel is given by the direction of the velocity vector at that time.



$\tan\theta = \frac{3}{1}$ giving $\theta = 71.6°$

The bearing is $270° + 71.6° = 341.6°$

(342° to the nearest degree)

A diagram makes it very clear which angle you are calculating and how to calculate the bearing from the angle.

iii The car is at rest if both components of **v** are zero at the same time.

$\mathbf{v} = \mathbf{u}+\mathbf{a}t = -4\mathbf{i}+(2\mathbf{i}+2\mathbf{j})t$

$\mathbf{v} = (-4+2t)\mathbf{i}+2t\mathbf{j}$

Do not write **i** = 0 and **j** = 0 as this makes no sense!

i component $-4+2t = 0$ giving $t = 2$

j component $2t = 0$ giving $t = 0$

They are not zero at the same time, so the car is never at rest.

Make sure you state your conclusion clearly from the evidence you have found.

iv Use the equation $\mathbf{s} = \mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$ to find the displacement of the car from its starting position.

$\mathbf{s} = \mathbf{u}t+\frac{1}{2}\mathbf{a}t^2 = (-4\mathbf{i})t+\frac{1}{2}(2\mathbf{i}+2\mathbf{j})t^2$

$\mathbf{s} = (t^2-4t)\mathbf{i}+t^2\mathbf{j}$

The displacement from the origin can be found by putting the arrow representing displacement from the origin to the starting position nose-to-tail with the subsequent displacement.

Be very clear about displacement and position – the displacement is the distance and direction that the car has moved from its starting position; position vectors are displacement from the origin.

The position vector of the car is $\mathbf{r} = \mathbf{r}_0 + \mathbf{s}$.

$\mathbf{r} = \mathbf{r}_0 + (t^2 - 4t)\mathbf{i} + t^2\mathbf{j}$

When $t = 2$, $\mathbf{r} = -\mathbf{i} + 2\mathbf{j}$

Use the information in the question to find a vector equation for the starting position.

$-\mathbf{i} + 2\mathbf{j} = \mathbf{r}_0 + (2^2 - 4 \times 2)\mathbf{i} + 2^2\mathbf{j}$

$-\mathbf{i} + 2\mathbf{j} = \mathbf{r}_0 - 4\mathbf{i} + 4\mathbf{j}$

giving $\mathbf{r}_0 = 3\mathbf{i} - 2\mathbf{j}$

The position vector at time t is

$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + (t^2 - 4t)\mathbf{i} + t^2\mathbf{j}$

$\mathbf{r} = (t^2 - 4t + 3)\mathbf{i} + (t^2 - 2)\mathbf{j}$

Collect the terms in the **i** and **j** directions together

v The car is southwest of the origin when the **i** component is equal to the **j** component and both are negative.

$t^2 - 4t + 3 = t^2 - 2$

Giving $4t = 5$ so $t = 1.25$ s.

To check it is southwest and not northeast:

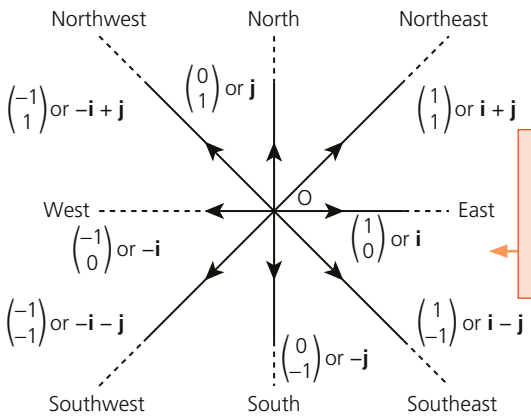
when $t = 1.25$,

$\mathbf{r} = (1.25^2 - 4 \times 1.25 + 3)\mathbf{i} + (1.25^2 - 2)\mathbf{j}$

$\mathbf{r} = -\frac{7}{16}\mathbf{i} - \frac{7}{16}\mathbf{j}$

So the car is southwest of the origin when $t = 1.25$ s.

The equation here cannot distinguish between northeast and southwest, so a check is needed.



The diagram gives the vectors representing the main compass points.

Vector form of Newton's 2nd law (page 175)

In this question **i** is a unit vector horizontally and **j** is the unit vector vertically upwards.

Forces $\mathbf{F}_1 = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{F}_2 = -5\mathbf{i} - a\mathbf{j}$ act on a particle of mass 0.7 kg.

i Write the weight of the particle in vector form.

ii Find the value of a and b if the particle is in equilibrium.

iii The values of a and b are 3 and 11, respectively. Find the acceleration of the particle.

iv Initially the particle is at rest. Find the displacement of the particle from its initial position after 5 s.

i Weight acts downwards in the negative **j** direction.

So $\mathbf{W} = -0.7g\mathbf{j} = -6.86\mathbf{j}$

The exam question may be structured like this to help you to remember to include the weight in the equations of motion in later parts of the question.

ii If the particle is in equilibrium,

resultant force is zero.

A vector is zero when each component is zero.

$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{W} = \mathbf{0}$

$(a\mathbf{i} + b\mathbf{j}) + (-5\mathbf{i} - a\mathbf{j}) - 6.86\mathbf{j} = \mathbf{0}$

In the **i** direction, $a - 5 = 0$ giving $a = 5$

Look at the **i** direction first as there is only one unknown in this direction.

In the **j** direction, $b - a - 6.86 = 0$

So $b = 5 + 6.86 = 11.86$

Substitute the value for a already found.

iii Given $a = 3,\ b = 11$

Resultant force $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{W} = 3\mathbf{i} + 11\mathbf{j} - 5\mathbf{i} - 3\mathbf{j} - 6.86\mathbf{j} = -2\mathbf{i} + 1.14\mathbf{j}$

Newton's second law with mass 0.7 kg gives $-2\mathbf{i} + 1.14\mathbf{j} = m\mathbf{a} = 0.7\mathbf{a}$

$\mathbf{a} = -\frac{20}{7}\mathbf{i} + \frac{57}{35}\mathbf{j} = -2.86\mathbf{i} + 1.63\mathbf{j}$ (to 3 s.f.)

iv Use the vector form of the *suvat* equations as the acceleration is constant.

$\mathbf{u} = \mathbf{0},\ \mathbf{a} = -2.86\mathbf{i} + 1.63\mathbf{j},\ t = 5$

Use $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

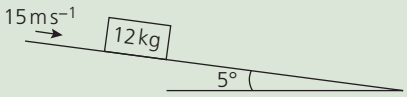
$$\mathbf{s} = \mathbf{0} + \frac{1}{2}(-2.86\mathbf{i} + 1.63\mathbf{j}) \times 5^2$$

$$\mathbf{s} = -35.75\mathbf{i} + 20.375\mathbf{j}$$

Resolving forces (page 183)

A block of mass 12 kg slides down a rough slope which is inclined at 5° to the horizontal.

The block starts with a speed of 15 m s^{-1} at the top of the slope and it is assumed that there is a constant resistance to motion of 15 N.



i Calculate the acceleration of the block.

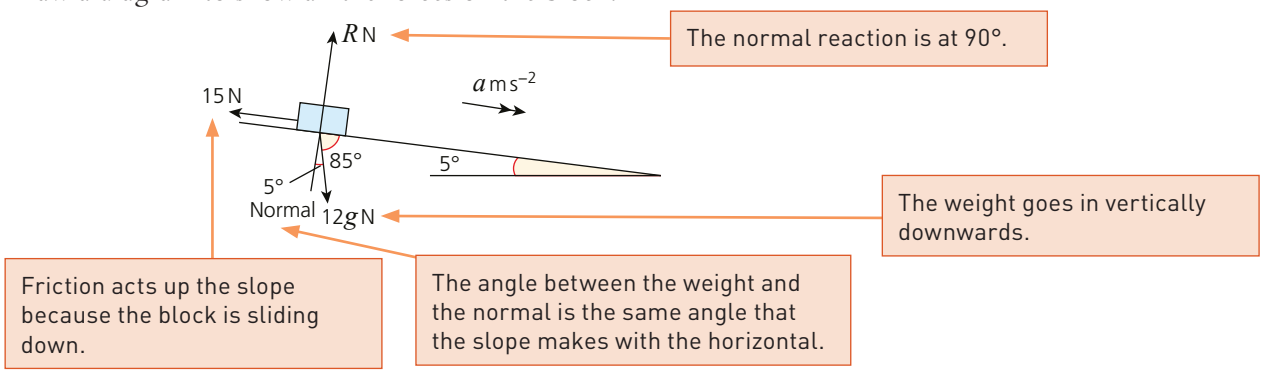
ii For how long does the block slide down the slope? How far does it travel?

Measurements show that the block actually comes to rest in 3.5 s.

There is an error caused by assuming the resistance is 15 N.

iii Calculate the true value of the resistance.

i Draw a diagram to show all the forces on the block.



Using Newton's second law down the slope:

$12g \sin 5° - 15 = 12a$ ← Take care with the signs here.

$10.25 - 15 = 12a$

$-4.75 = 12a$

$a = -0.396$ ← In this case the acceleration is negative because the block is slowing down.

ii Use *suvat* equations with $u = 1.5$ and $a = -0.396$.
The block comes to rest when $v = 0$.

Using $v = u + at$

$$0 = 1.5 - 0.396t$$
$$t = 1.5 \div 0.396$$
$$t = 3.7890...$$

Choose the equation that does not involve s.

The block slides for 3.8 s.

To find s use $v^2 = u^2 + 2as$

$$0 = 1.5^2 + 2 \times (-0.396)s$$
$$0.792s = 2.25$$
$$s = 2.84$$

It is best not to use the time if you can avoid it in case you have made a mistake.

The block slides 2.84 m.

Check using your value for t:

Use $s = \frac{1}{2}(u+v)t$

$$s = \frac{1}{2}(1.5 + 0) \times 3.7880...$$
$$= 2.84$$

It is good practice to develop checking strategies. This checks two answers in one step!

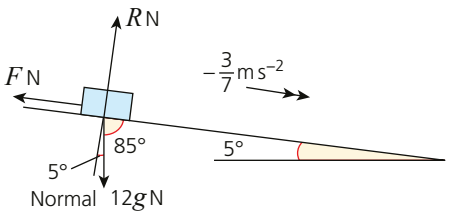
iii To find the acceleration using $u = 1.5$, $v = 0$, $t = 3.5$:

Use $v = u + at$

$$0 = 1.5 + 3.5a$$
$$-1.5 = 3.5a$$
$$a = -1.5 \div 3.5$$
$$= -\frac{3}{7} \ (= -0.4286)$$

When the information is about times and distances and the question about forces, find the acceleration to link the two parts.

Working with the fraction avoids errors due to rounding.



The new diagram is very similar to the first – it is only the values of the force and the acceleration that have changed.

When the friction force is F newtons, resolving down the plane gives:

$$12g\sin 5^\circ - F = 12 \times \left(-\frac{3}{7}\right)$$
$$10.249... - F = -5.142...$$
$$10.249... + 5.142... = F$$
$$F = 15.392$$

The resistance is 15.4 N.

As expected, the resistance is more than 15N. This also serves as a useful check.

Calculus and vectors (page 187)

In this question **i** and **j** are unit vectors East and North.

A model boat moves in a horizontal plane.

At time t, the velocity of the boat is given by $\mathbf{v} = bt^2\mathbf{i} + 4t\mathbf{j}$

where b is a positive constant.

At $t = 1$ second, the magnitude of the acceleration of the boat is $5\,\text{m}\,\text{s}^{-2}$.

i Show that $b = 1.5$.

At $t = 0$, the boat is at the origin O.

ii Work out the distance of the boat from the origin after the first three seconds of its motion and the bearing of the boat from the origin.

i Differentiate $\mathbf{v} = bt^2\mathbf{i} + 4\mathbf{j}$ to find an expression for acceleration.

This will be a vector.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2bt\mathbf{i} + 4\mathbf{j}$$

Differentiate each direction separately.

When $t = 1$, $\mathbf{a} = 2b\mathbf{i} + 4\mathbf{j}$

The magnitude of acceleration is $\sqrt{(2b)^2 + 4^2}$

The magnitude is a scalar and it is this that needs to be equated to the given value 5.

Given that $\sqrt{(2b)^2 + 4^2} = 5$,

$4b^2 + 16 = 25$

$4b^2 = 9$ giving $b = \pm\frac{3}{2}$

It is usually best to give both roots of the equation and to select the one you need. Give a reason why you reject the other.

But b is positive so $b = 1.5$.

Give the answer as a decimal as this way of writing it is given in the question.

ii Integrate $\mathbf{v} = bt^2\mathbf{i} + 4\mathbf{j}$ to find an expression for displacement.

The particle begins at the origin, so the displacement and the position vector are the same.

$$\mathbf{r} = 1.5 \times \frac{t^3}{3}\mathbf{i} + 4t\mathbf{j} + \mathbf{c}$$

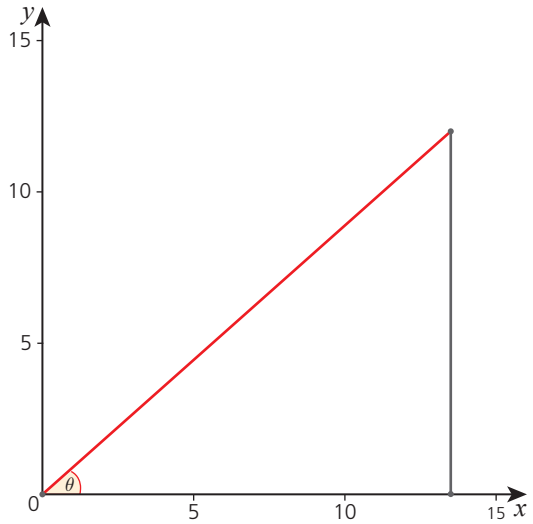
When $t = 0$, $\mathbf{r} = \mathbf{0}$ so $\mathbf{c} = \mathbf{0}$.

So the position vector at time t is $\mathbf{r} = \frac{t^3}{2}\mathbf{i} + 4t\mathbf{j}$.

When $t = 3$, $\mathbf{r} = \frac{3^3}{2}\mathbf{i} + 4 \times 3\mathbf{j} = 13.5\mathbf{i} + 12\mathbf{j}$

This is not the finished answer here – distance is the magnitude of the displacement vector.

Distance $= \sqrt{13.5^2 + 12^2} = \sqrt{326.25} = 18.1\,\text{m}$



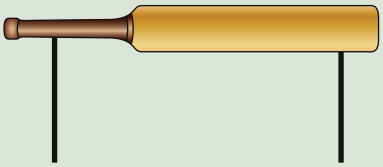
A diagram is the best way of seeing which angle is needed for the bearing – remember to give the angle clockwise from North with 3 figures before the decimal point.

$\tan\theta = \frac{12}{13.5}$ giving $\theta = 41.6°$

So the bearing is $90° - 41.6° = 048.4°$.

Moments (page 193)

A cricket bat from an historic match has length 96 cm and mass 1.4 kg. It is displayed in a museum. It lies horizontally at rest in equilibrium on two vertical supports, which are placed 18 cm and 84 cm from the end of the handle of the bat.



In an initial model, the centre of mass, G, of the bat is taken to be at its middle.

i Find the magnitude of the reaction force at each of the supports according to this model.

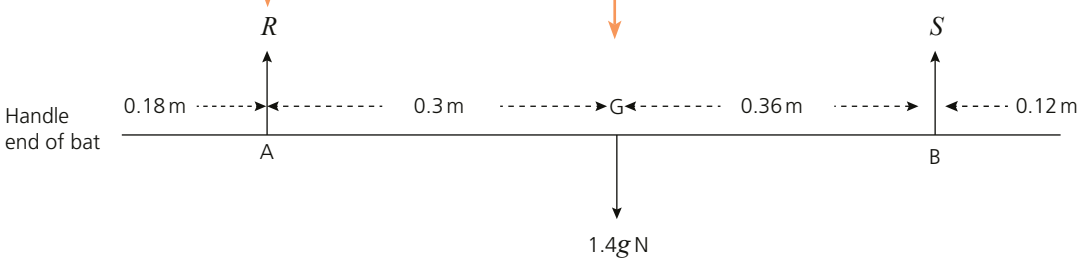
The initial model is not accurate. The supports have been carefully positioned so that the reaction force on each of them is exactly the same.

ii Find the position of the centre of mass of the bat.

i Draw a diagram showing all the forces and distances.

The reaction forces are not the same so use different letters

G is the centre of mass and is 0.48 m from either end. The weight acts here.



Taking moments about the point A:

Taking anti-clockwise as the positive direction.

The length AB is 0.3 + 0.36 = 0.66 m

The length AG is 0.3 m

$S \times \text{AB} - 1.4g \times \text{AG} = 0$

$S \times 0.66 - 1.4g \times 0.3 = 0$

$$S = \frac{0.3 \times 1.4g}{0.66}$$

$$= 0.636363g$$

$$= 6.236363$$

The reaction force at B is 6.24 N to 3 s.f.

Vertical equilibrium:

Take upwards to be the positive direction.

$R + S - 1.4g = 0$

$R + S = 13.72$

$R = 13.72 - 6.236363$

$= 7.483\ldots$

The reaction force at A is 7.48 N to 3 s.f.

Be very clear where the distance x m is on the diagram.

ii Centre of mass H is x m from A.

Vertical equilibrium:

The reaction forces are equal so use the same letter.

$T + T - 1.4g = 0$

$2T = 13.72$

$T = 6.86$

The reaction force at A, and so also at B, is 6.86 N to 3 s.f.

Taking moments about the point A:

$T \times \text{AB} - 1.4\text{g} \times \text{AH} = 0$

$6.86 \times 0.66 - 1.4\text{g} \times x = 0$

$$x = \frac{6.86 \times 0.66}{1.4g}$$

$$= 0.33 \text{ m}$$

You can check your answer by taking moments about B.

The centre of mass of the bat is 0.33 + 0.18 = 0.51 m from the 'handle end' of the bat.

Projectiles in flight (page 198)

A cricket ball is hit from ground level with a velocity of 24.5 ms^{-1} at an angle θ to the horizontal where $\cos\theta = 0.8$ and $\sin\theta = 0.6$.

i Show that, after t seconds, the position of the ball is given by $x = 19.6t$, $y = 14.7t - 4.9t^2$.

ii Find the greatest height reached by the ball.

iii Find the distance travelled before the ball bounces for the first time.

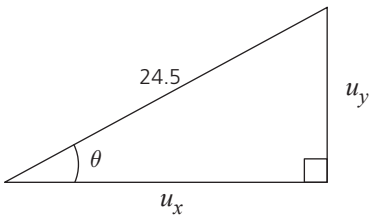
iv Show that the equation of the trajectory of the ball is

$$y = \frac{3}{4}x - \frac{5}{392}x^2.$$

i $u_x = 24.5 \cos\theta = 24.5 \times 0.8 = 19.6$
$u_y = 24.5 \sin\theta = 24.5 \times 0.6 = 14.7$

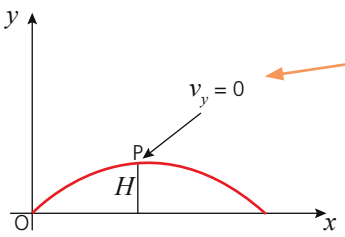
There is no need to find the size of the angle theta as the values of the sine and cosine are given and can be used directly.

In the horizontal direction:
$u_x = 19.6,\ a_x = 0$
Using $s = ut + \frac{1}{2}at^2$
$x = 19.6t + 0t^2$
$= 19.6t$



In the vertical direction:
$u_y = 14.7,\ a_y = -9.8$
Using $s = ut + \frac{1}{2}at^2$
$y = 14.7t + \frac{1}{2} \times (-9.8)t^2$
$= 14.7t - 4.9t^2$
So $x = 19.6t$ and $y = 14.7t - 4.9t^2$

ii The greatest height is the point P on the diagram.
In the vertical direction:
Using $v = u + at$
You know that $u_y = 14.7$ and $a_y = -9.8$
So, $v_y = 14.7 - 9.8t$
When $v_y = 0$, $14.7 - 9.8t = 0$
Therefore $9.8t = 14.7$
and $t = 1.5$ seconds

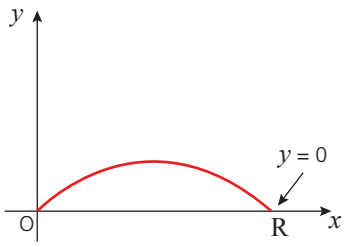


The ball reaches its greatest height when the vertical component of velocity is zero.

So the ball is at its greatest height after 1.5 seconds.
Substituting $t = 1.5$ into $y = 14.7t - 4.9t^2$
$H = 14.7 \times 1.5 - 4.9(1.5)^2 = 11.025$
So the greatest height is 11.0 m (3 s.f.).

This problem can also be done in one step using the formula $v^2 = u^2 + 2as$.

iii The distance travelled is the range of the ball, which is the distance OR on the diagram.
At R, $y = 0$
You know that $y = 14.7t - 4.9t^2$
So, $0 = 14.7t - 4.9t^2$
Rearranging gives $4.9t^2 - 14.7t = 0$
Factorising gives $t(4.9t - 14.7) = 0$



Therefore, $t = 0$ seconds or
$t = \dfrac{14.7}{4.9} = 3$ seconds

Notice this is double the time taken to reach the top found in part (ii).

This means that the ball is in the air for 3 seconds until its first bounce.
You know from part **i** that $x = 19.6t$.
Substituting in $t = 3$ gives OR $= 19.6 \times 3 = 58.8$.
So, the ball travels 58.8 m before its first bounce.

iv From part **i**
$x = 19.6t$ (1)
$y = 14.7t - 4.9t^2$ (2)
Making t the subject of equation (1) gives $t = \dfrac{x}{19.6}$

Substituting for t in equation (2) gives $y = \dfrac{14.7}{19.6}x - 4.9\left(\dfrac{x}{19.6}\right)^2 x^2$

$$y = \frac{14.7}{19.6}x - \frac{4.9}{(19.6)^2}x^2$$

$$\Rightarrow \quad y = \frac{3}{4}x - \frac{5}{392}x^2 \text{ as required}$$

Further projectiles (page 201)

Jack throws a small stone from a point that is 2 m above ground level towards a target that is 15 m away and 7.2 m above ground level. The initial velocity of the stone has horizontal and vertical components of 9 m s^{-1} and 12 m s^{-1}, respectively. (Take $g = 9.8$ m s^{-2}.)

i Calculate the speed of projection, u m s^{-1}, and the angle of projection, θ, of the stone.

ii Show that, t seconds after the stone has been projected, its height above ground level, y m, is given by the expression $y = 2 + 12t - 4.9t^2$.
Find the corresponding expression for the horizontal distance, x m.
Show that the stone misses the target.

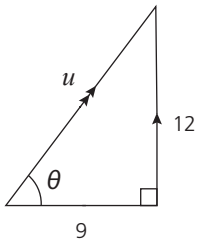
Jack tries again to hit the target, this time altering the angle of projection to 45°, but leaving the speed of projection, u m s^{-1} unchanged.

iii What are the x- and y-coordinates of the position of the particle after t seconds?
Show that the equation of the trajectory of the stone is $y = 2 + x - \frac{49}{1125}x^2$. Verify that the stone now hits the target.

iv At what angle to the horizontal is the stone travelling when it hits the target?

i The components of the initial velocity are given.

> More often these are quantities that you have to find.



The speed of projection is $= \sqrt{(9^2 + 12^2)} = 15$ m s^{-1}

> Use Pythagoras to combine the components together to find the magnitude of the initial velocity.

Angle of projection: $\tan\theta = \frac{12}{9}$, so $\theta = 53.1°$

> Draw a diagram to show which angle you are calculating.

ii In the vertical direction, the stone starts 2 m above the ground.

> It is an easy mistake to leave out the initial position.

Use $s = ut + \frac{1}{2}at^2 + s_0$ with $u_y = 12$, $a = -9.8$, $s_0 = 2$

$y = 12t + \frac{1}{2}(-9.8)t^2 + 2$ so $y = 2 + 12t - 4.9t^2$

> It is important to show this stage as the answer is given in the question.

In the horizontal direction, speed is constant so $x = u_x \times t = 9t$

Target is when $x = 15$, $15 = 9t$, so $t = \frac{5}{3}$

> Working with fractions avoids errors due to rounding.

Substituting $t = \frac{5}{3}$ in the equation for y gives

the height, $y = 2 + 12 \times \frac{5}{3} - 4.9 \times \left(\frac{5}{3}\right)^2$

$= 8.3888\ldots = 8.39$ m (to 3 s.f.)

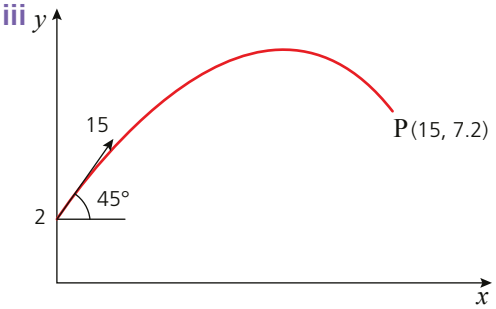
> Only round your answer at the end.

So the stone passes through the point (15, 8.39).

But the coordinates of the target are (15, 7.2), so the stone misses the target.

> Make your conclusion clear.

iii



$u_x = 15 \cos 45°$ and $u_y = 15 \sin 45°$

so, $u_x = \frac{15}{2}\sqrt{2}$ and $u_y = \frac{15}{2}\sqrt{2}$

> Leaving surds in your answers here helps avoid errors due to rounding.

Also $x_0 = 0$ and $y_0 = 2$

$a_x = 0$ and $a_y = -9.8$

At time t, displacement $s = s_0 + ut + \frac{1}{2}at^2$

therefore $x = \frac{15}{2}\sqrt{2}t$ (1)

and $y = 2 + \frac{15}{2}\sqrt{2}t - 4.9t^2$ (2)

Make t the subject of Equation (1)

$t = \frac{\sqrt{2}}{15}x$

Substituting for t in Equation (2)

$$y = 2 + \frac{15}{2}\sqrt{2} \times \frac{\sqrt{2}}{15}x - 4.9\left(\frac{\sqrt{2}}{15}x\right)^2$$

$$= 2 + x - \frac{49 \times 2}{10 \times 225}x^2$$

$$= 2 + x - \frac{49}{1125}x^2$$

The equation of the trajectory provides an alternative method for part **iv**.
Differentiate the equation of the trajectory.

$\frac{dy}{dx} = 1 - \frac{49}{1125} \times 2t = 1 - \frac{98}{1125}t$

When $x = 15$, $\frac{dy}{dx} = 1 - \frac{98}{1125} \times 15 = 1 - \frac{1470}{1125} = -\frac{23}{75}$

The angle this tangent makes with the x-axis is $\arctan\left(-\frac{23}{75}\right) = -17.0°$ so the angle is 17.0 below the horizontal.

This is the equation of the trajectory of the stone.

When $x = 15$, $y = 2 + 15 - \frac{49}{1125} \times 15^2 = 7.2$

Therefore the stone passes through the point (15,7.2), so hits the target.

Make a very clear conclusion.

iv In the vertical direction, $u_y = 15 \sin 45° = \frac{15}{2}\sqrt{2}$, $a_y = -9.8$

$v_y = \frac{15}{2}\sqrt{2} - 9.8t$

There is zero acceleration horizontally.

In the horizontal direction, $u_x = 15 \cos 45° = \frac{15}{2}\sqrt{2}$ giving $v_x = \frac{15}{2}\sqrt{2}$

To find the time at which the stone hits the target

use $x = \frac{15}{2}\sqrt{2}t$

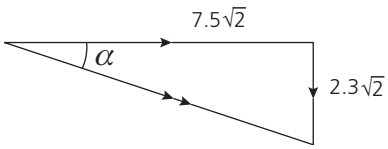
This was found in part **iii**.

When $x = 15$, $15 = \frac{15}{2}\sqrt{2}t$, so $t = \sqrt{2}$ s

Substituting $t = \sqrt{2}$ in the expressions for v_x and v_y:

$v_x = \frac{15}{2}\sqrt{2} = 7.5\sqrt{2}$

$v_y = \frac{15}{2}\sqrt{2} - 9.8\sqrt{2} = 7.5\sqrt{2} - 9.8\sqrt{2} = -2.3\sqrt{2}$



Show the components of velocity on a diagram.

$\tan\alpha = \frac{2.3\sqrt{2}}{7.5\sqrt{2}} = \frac{23}{75}$

$\Rightarrow \alpha = 17.0°$

The stone is moving in a downwards direction when it hits the target at an angle of 17.0° below the horizontal.

Friction (page 205)

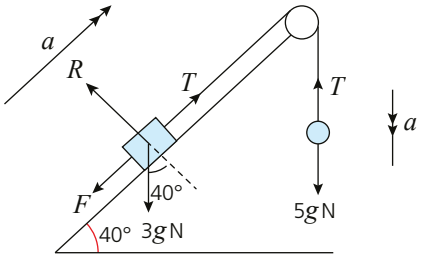
A 3 kg block is placed on a rough plane inclined at 40° to the horizontal. The coefficient of friction between the block and the plane is 0.4. A light inextensible string which is parallel to the slope is attached to the block. The string passes over a smooth pulley and is attached to a particle of mass 5 kg which hangs freely. The system is released from rest and the block is pulled up the slope.

i Calculate the acceleration of the system.

When the block is travelling at 1.5 m s^{-1}, the string snaps.

ii Calculate the distance that the block travels before coming to rest.

i Draw a diagram showing all the forces.



To find the size of the frictional force find the normal reaction.

Resolve perpendicular to the slope:

$R - 3g\cos 40° = 0$

The block has no acceleration in this direction. Notice the normal reaction is not equal to the weight.

$R = 3g\cos 40°$

The block is sliding so $F = \mu R = 0.4 \times 3g\cos 40°$

$F = 9.00868\ldots$ N

Resolve up the slope for the 3 kg mass:

$T - F - 3g\sin 40° = 3a$

Vertically downwards for the 5 kg mass:

$5g - T = 5a$

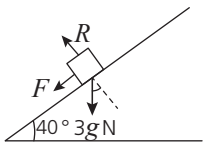
Add the equations to eliminate T.

$5g - 9.00868 - 3g\sin 40° = 8a$

$a = 2.64\text{ ms}^{-2}$

ii Without the string, there is a new acceleration.
The value of the frictional force is unchanged.

Draw a new diagram when the situation changes.



It is often easiest to use the direction of motion as the positive direction even when both forces act in the negative direction.

Resolve up the slope:

$-F - 3g\sin 40° = 3a$

This is Newton's second law in the direction of motion. Both forces act in the negative direction.

$a = -\frac{1}{3}(9.00868 + 3 \times 9.8\sin 40°) = -9.30221\ldots$

Acceleration is constant so use *suvat* equations.

Use $u = 1.5,\ v = 0,\ a = -9.30221\ldots$

Use $v^2 = u^2 + 2as$

Use the equation that does not involve t.

$0 = 1.5^2 + 2 \times -9.30221\ldots \times s$

$s = \dfrac{1.5^2}{18.60442} = 0.1209\text{ m}$

It travels 12.1 cm before coming to rest.

Review questions (Mechanics) (pages 206–208)

1 A boy starts at the origin and jogs to A which is 60 m due North in 8 s. He stays at A for 18 s and then jogs 30 m due South to B at the same speed as before. Find:

i the displacement after both parts of the journey

ii the total distance travelled

iii the total time taken

iv the average speed.

i Displacement = 60 – 30 = 30 m North

Travelling due South is in the negative direction and displacement takes direction into account."

ii Distance travelled = 60 + 30 = 90 m

For distance, the two parts of the journey are added together as the direction is not important for distance.

iii Time to travel 30 m is half the time to travel 60 m = 4 s.

The two parts of the journey have the same speed and the distance is half. Notice the velocity would be negative for this part of the journey.

Total time = 8 + 18 + 4 = 30 s

iv Average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{90}{30} = 3\ \text{m s}^{-1}$

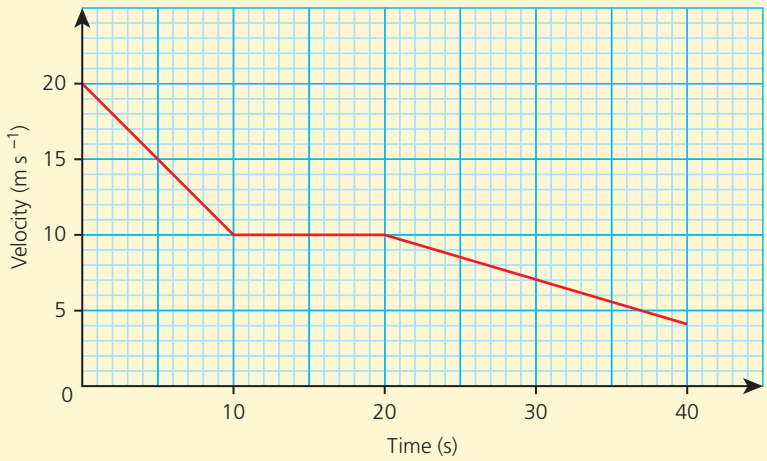
Make sure you use distance and not displacement to calculate speed. Using the displacement would give average velocity.

2 The motion of a particle is illustrated by the velocity–time graph. The positive direction is due North and the particle begins 200 m north of the origin.

i Describe the motion of the particle.

ii State the acceleration for each phase of the motion.

iii Find the position of the particle after 40 s.



i The particle is initially travelling north with a velocity of $20\,\text{m s}^{-1}$. It decelerates at a constant rate to $10\,\text{m s}^{-1}$ after 10 s. It then travels at constant speed for the next 10 s. Then it decelerates, again at a constant rate, reaching $4\,\text{m s}^{-1}$ after another 20 s.

ii Acceleration in the first phase is $\frac{10-20}{10} = -1\,\text{m s}^{-2}$

This is the gradient of the velocity–time graph.

Acceleration in the second phase is $0\,\text{m s}^{-2}$

Velocity is constant.

Acceleration in the third phase is $\frac{4-10}{20} = -0.3\,\text{m s}^{-2}$

Notice the gradient of the graph and hence the acceleration is negative here.

iii Displacement is given by the area under the graph.

This is the change in the position while the particle is moving. The initial position will be taken into account later.

$\frac{1}{2}(20+10)\times 10 + (10\times 10) + \frac{1}{2}(10+4)\times 20$

$= 150 + 100 + 140 = 390\,\text{m}$

Divide the area into three sections. Use the formula for the area of a trapezium $A = \frac{1}{2}(a+b)h$ and the area of a rectangle.

The starting position is 200 m north of the origin so the final position is 590 m north of the origin.

3 A space probe travels in a straight line. It accelerates uniformly from an initial velocity of $24\,\text{m s}^{-1}$ in the negative direction to a velocity of $120\,\text{m s}^{-1}$ in the positive direction in 120 s. It then continues to move along the same straight line with the same acceleration.

Find how long it takes the space probe to reach a point 50 km from its starting position. Give your answer in minutes and seconds to the nearest second.

Constant acceleration with $u = -24,\ v = 120,\ t = 120$

Choose $v = u + at$

Use the formula that does not involve s.

$120 = -24 + 120a$ giving $a = \frac{144}{120} = 1.2$

For the whole journey $u = -24,\ s = 50\,000,\ a = 1.2$

Take care not to misread the question here – it is not a further 50 km. The initial velocity is –24 and not $120\,\text{m s}^{-1}$.

$s = ut + \frac{1}{2}at^2$

Choose the formula that does not involve v. The final velocity for this longer journey is not $120\,\text{m s}^{-1}$.

$50\,000 = -24t + \frac{1}{2}\times 1.2t^2$

$0.6t^2 - 24t - 50\,000 = 0$

$t = 309.36...\ \text{s}$

309 ÷ 60 gives 5.156... but notice this is not 5 minutes and 15 seconds.

So journey takes 5 minutes 9 seconds.

4 Anastasia throws a ball vertically upwards from a point 1.2 m above the ground in her garden with a velocity of $15\,\text{m s}^{-1}$. The garden is surrounded by a hedge that is 3 m high.

For how long is the ball visible from outside the garden?

It is an arbitrary choice where the origin is, but if there is a choice, it is best to make it very clear what you have chosen. If the point of projection is taken to be the origin, the height of the hedge would be 1.8 m.

Take the ground to be the origin so $h_0 = 1.2\,\text{m}$.

Vertical motion, use constant acceleration with $u = 15,\ s = 3,\ a = -9.8$

$s = ut + \frac{1}{2}at^2 + h_0$

$3 = 15t - 4.9t^2 + 1.2$

$4.9t^2 - 15t + 1.8 = 0$

$t = 2.936,\ 0.125$

So the ball is visible for $2.936 - 0.125 = 2.81\,\text{s}$

You may be able to solve this equation with your calculator, or use the quadratic formula $t = \frac{15 \pm \sqrt{15^2 - 4 \times 4.9 \times 1.8}}{2 \times 4.9}$.

The ball is visible for times between these values.

5 In this question all quantities are in SI units. The acceleration of a particle is given by the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors East and North, respectively. At time $t = 0$, the particle has a velocity $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j}$ and position vector $\mathbf{r}_0 = -14\mathbf{i} + 8\mathbf{j}$.

i Show that the particle reaches the origin when $t = 2\,\text{s}$.

ii Calculate the speed of the particle when $t = 2\,\text{s}$.

i Constant acceleration so displacement $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = (6\mathbf{i} - 2\mathbf{j})t + \frac{1}{2}(\mathbf{i} - 2\mathbf{j})t^2$

The position of the particle is given by $\mathbf{r} = \mathbf{s} + \mathbf{r}_0$

When $t = 2$, $\mathbf{r} = \mathbf{s} + (-14\mathbf{i} + 8\mathbf{j})$

$\mathbf{r} = (6\mathbf{i} - 2\mathbf{j}) \times 2 + \frac{1}{2}(\mathbf{i} - 2\mathbf{j}) \times 2^2 + (-14\mathbf{i} + 8\mathbf{j})$

$\mathbf{r} = 12\mathbf{i} - 4\mathbf{j} + 2\mathbf{i} - 4\mathbf{j} + (-14\mathbf{i} + 8\mathbf{j}) = \mathbf{0}$

So the particle returns to the origin.

In this book, $\mathbf{s}$ is used for the displacement vector and $\mathbf{r}$ for the position vector.

$\mathbf{r}_0$ is the position of the particle when $t = 0$.

Notice the vector has both components equal to zero at the same time.

ii When $t = 2$, $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{v} = (6\mathbf{i} - 2\mathbf{j}) + (\mathbf{i} - 2\mathbf{j}) \times 2$

$\mathbf{v} = 8\mathbf{i} - 6\mathbf{j}$

$v = |\mathbf{v}| = \sqrt{8^2 + 6^2} = \sqrt{100}$

So speed is $10\,\text{m s}^{-1}$.

Speed is the magnitude of the velocity vector, so find the velocity vector first.

6 Jenny stands on the end of a pier 4 m above the sea. She throws a pebble at $12\,\text{m s}^{-1}$ at 35° above the horizontal out to sea.

i Calculate the maximum height of the pebble above the point of projection.

ii Calculate the horizontal distance travelled by the pebble when it hits the sea.

iii Find the direction of motion of the particle as it hits the sea.

At the highest point, the component of velocity in the vertical direction will be zero. The pebble will still be travelling in the horizontal direction.

i The pebble reaches maximum height when $v_y = 0$

In the vertical direction, $u_y = 12\sin 35°,\ a = -9.8,\ v_y = 0$

Use $v^2 = u^2 + 2as$

$0^2 = (12\sin 35°)^2 - 2 \times 9.8s$

$s = \frac{(12\sin 35°)^2}{19.6} = 2.417...$

So highest point is 2.42 m above the point of projection.

Resolve the initial velocity into its horizontal and vertical components.

Choose the formula that does not involve t.

You can leave all the calculations until the last line of working. If you prefer to work out 12 sin35°, make sure you work to more than 3 significant figures to avoid errors due to rounding.

ii The information is about the y-direction and the answer is about the x-direction. Use a value of t to link the two directions.

The pebble hits the sea when $y = -4$.

The sea is below the height of the pier and so the displacement must be negative.

Use $s = ut + \frac{1}{2}at^2$ in the vertical direction.

Choose the formula that does not involve v.

$-4 = (12\sin 35°)t - 4.9t^2$

$4.9t^2 - (12\sin 35°)t - 4 = 0$

$t = 1.8467, -0.442$

If you use your calculator to solve the equation, it is best to show both roots of the quadratic equation as evidence of a method. You could also use the quadratic formula $t = \dfrac{12\sin 35° \pm \sqrt{(12\sin 35°)^2 - 4 \times 4.9 \times (-4)}}{2 \times 4.9}$

Give a reason for choosing one root and not the other.

Use positive $t = 1.8467$ to find the horizontal distance travelled.

$x = (12\cos 35°)t = 12\cos 35° \times 1.8467$

So horizontal distance travelled is 18.2 m.

iii The direction of motion is given by the velocity vector.

It is not necessary to use vector notation if you are clear about which direction you are considering in each line of working.

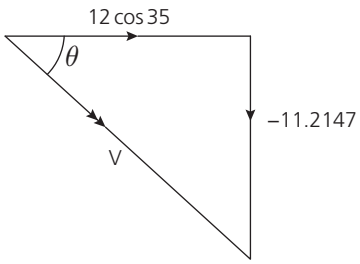
Horizontally, $v_x = 12\cos 35°$

When $t = 1.8467$,

$v_y = 12\sin 35° - 9.8t = 12\sin 35° - 9.8 \times 1.8467 = -11.2147...$

The direction of travel makes an angle θ to the horizontal.

An angle on its own is not quite the same as a direction. A diagram makes it very clear which direction is meant.



You could use the negative value here and get a negative value for θ. A negative angle would be interpreted as being in the clockwise direction from the x-axis.

$\tan\theta = \dfrac{11.2147}{12\cos 35°}$ giving $\theta = 48.8°$

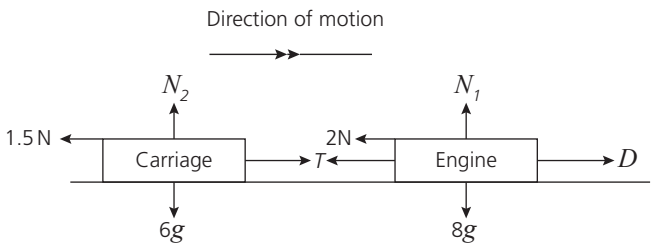
The direction of travel makes an angle of 48.8° to the sea.

7 A toy train consists of an engine of mass 8 kg and a carriage of mass 6 kg. It is travelling on level ground along a straight track at constant speed. The resistances to motion on the engine and the carriage are 2 N and 1.5 N, respectively. The normal reactions between the engine and the track and the carriage and the track are N_1 and N_2. The engine exerts a driving force of D N. The tensions in the coupling is T N.

i Draw a diagram showing all the forces acting on the engine and on the carriage.

ii Find the values of D, N_1, N_2 and T.

i



ii Horizontal motion for the whole train:

$D - 2 - 1.5 = 0$ giving $D = 3.5\,\text{N}$

This means that the tension in the coupling is an internal force and need not be considered in this equation.

Vertical forces on the engine, $N_1 = 8g = 78.4\,\text{N}$

Check carefully which is which!

Vertical forces on the carriage, $N_2 = 6g = 58.8\,\text{N}$

Horizontal motion for the carriage:

The carriage has fewer forces acting on it and so the equation is simpler. Notice the driving force does not act directly on the carriage.

$T - 1.5 = 0$ giving $T = 1.5\,\text{N}$

To check your answer, consider the engine $D - T - 2 = 0$ giving $3.5 - 1.5 - 2 = 0$ which is correct.

8 A woman of mass 56 kg is in a lift which is ascending. Find the contact force between the woman and the lift floor in each case.

i The lift is accelerating upwards at $2\,\text{m s}^{-2}$.

ii The lift is travelling at constant speed.

iii The lift is slowing down with a deceleration of $3\,\text{m s}^{-2}$.

i Upwards acceleration of $2\,\text{m s}^{-1}$

$R - 56g = 56a = 112$

$R = 56g + 112 = 660.8\,\text{N}$

ii No acceleration

$R - 56g = 0$

$R = 56g = 548.8\,\text{N}$

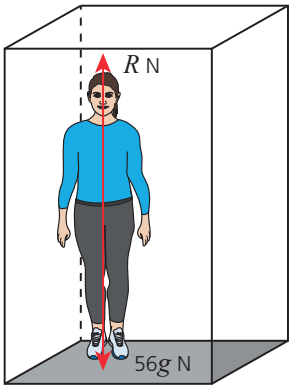
Notice the left-hand side of this equation is exactly the same as in part **i**. It is only the value of R that has changed.

iii Upwards acceleration of $-3\,\text{m s}^{-1}$

$R - 56g = 56a = 56 \times (-3) = -168$

$R = 56g - 168 = 492.8\,\text{N}$

Deceleration while travelling upwards means that the acceleration is in negative.



9 A car of mass 1500 kg pulls a caravan of mass 1200 kg. It accelerates from rest to $12\,\text{m s}^{-1}$ in 15 seconds.

i Calculate the acceleration of the car and caravan assuming that it is constant.

In an initial model, it is assumed that there is no resistance to motion.

ii Find the driving force the car must provide according to this model.

In a refined model it is assumed that the resistance is not negligible and that the driving force is 2500 N.

iii Calculate the total resistance to motion according to this model.

In this model it is also assumed that the resistance on the caravan is four times the resistance on the car.

iv Calculate the tension in the tow-bar.

i Constant acceleration with $u = 0,\ v = 12,\ t = 15$.

Use $v = u + at$

$12 = 15a$ giving $a = 0.8\,\text{m s}^{-2}$

Choose the formula that does not involve s.

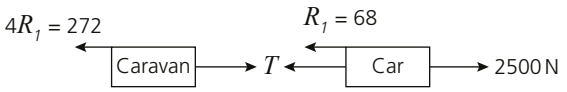
ii Newton's second law for the car and caravan:
$D = ma = (1500 + 1200) \times 0.8$
$D = 2160\,\text{N}$

It is also possible to consider each separately and eliminate the tension in the towbar from the equations.

iii Driving force = 2500 N
Newton's second law:
$2500 - R = 2700 \times 0.8$
$R = 340\,\text{N}$

iv $R_1 + R_2 = 340$ and $R_2 = 4R_1$
So $R_1 + 4R_1 = 340$ giving $R_1 = 68$ and $R_2 = 4 \times 68 = 272$

To find the tension in the tow-bar it is necessary to find the separate resistances on each part of the system.



Newton's second law for the caravan:
$T - 272 = 1200 \times 0.8$
So $T = 1232\,\text{N}$

You can check this by looking at forces on the car $2500 - T - 68 = 1500 \times 0.8$ which also gives $T = 1232\,\text{N}$.

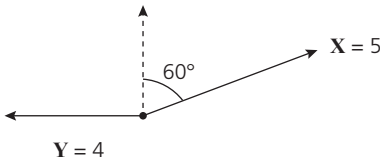
10 In this question, the x and y directions are East and North, respectively.
Forces **X** and **Y** act on a particle of mass 2 kg. **X** has a magnitude of 5 N and acts along a bearing of 060°. **Y** has a magnitude of 4 N and acts due West.

i Find the resultant of **X** and **Y** in vector form.

ii Find the acceleration of the particle.

iii The initial velocity of the particle is $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ m s^{-1}. Find the velocity after 5 seconds.

i



Resolve **X** in the x-and y-directions: $\begin{pmatrix} 5\sin 60° \\ 5\cos 60° \end{pmatrix}$

The y-component is the cos60° component as the angle between the force and the direction is 60°. You could write the x-component as cos30°. This approach might help if you are unsure which component is which.

Y acts in the negative x-direction: $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$

The resultant force is $\begin{pmatrix} 5\sin 60° \\ 5\cos 60° \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.330127... \\ 2.5 \end{pmatrix}$ N

The resultant force is found by adding the force vectors.

The resultant force is $\begin{pmatrix} 0.330 \\ 2.5 \end{pmatrix}$ correct to 3 significant figures.

ii Newton's second law gives:

Write Newton's second law in vector notation.

$\begin{pmatrix} 0.330127... \\ 2.5 \end{pmatrix} = m\mathbf{a} = 2\mathbf{a}$, $\mathbf{a} = \frac{1}{2}\begin{pmatrix} 0.330127... \\ 2.5 \end{pmatrix} = \begin{pmatrix} 0.165 \\ 1.25 \end{pmatrix}$ m s^{-2}

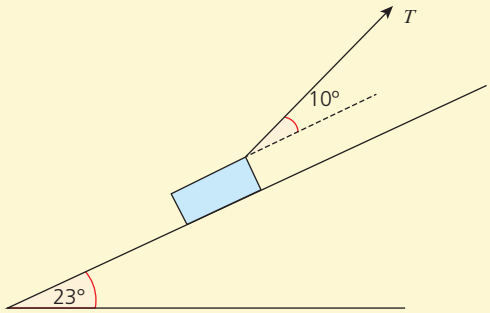
Use all the significant figures in your answer to part **i** to part get the acceleration correct to 3 significant figures.

iii Constant acceleration $\mathbf{a} = \begin{pmatrix} 0.165 \\ 1.25 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$, $t = 5$
Use $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

This *suvat* equation can also be written in vector notation.

$\mathbf{v} = \begin{pmatrix} 5 \\ -6 \end{pmatrix} + 5\begin{pmatrix} 0.165 \\ 1.25 \end{pmatrix} = \begin{pmatrix} 5.83 \\ 0.25 \end{pmatrix}$ ms^{-1} (to 3 s.f.)

11 A block of mass 8 kg rests on a smooth plane inclined at 23° to the horizontal. It is held in place by a string which makes an angle of 10° to the plane as shown in the diagram.



i Find the tension in the string.

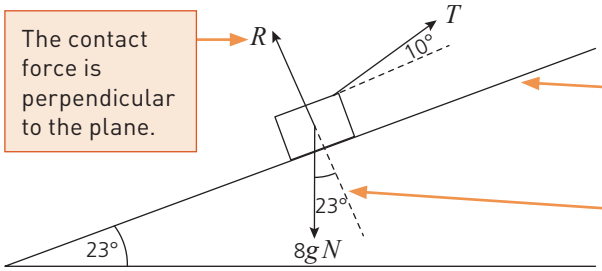
ii Find the normal reaction force between the block and the plane.

The tension is increased to 40 N.

iii Find the acceleration of the block up the plane.

i Resolve up the plane:

The contact force is perpendicular to the plane.



The plane is smooth so there is no frictional force.

The angle between the weight and the normal to the plane is the same as the angle between the plane and the horizontal. This helps to get the correct component of the weight along and perpendicular to the plane.

$T\cos 10° = 8g\sin 23°$

$T = 31.1\text{ N}$

ii Resolve perpendicular to the plane:

$R + T\sin 10° = 8g\cos 23°$

$R = 8g\cos 23° - 31.1\sin 10°$

$R = 66.8\text{ N}$

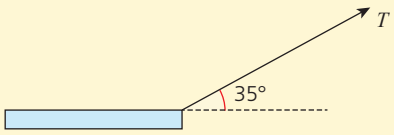
Notice that the normal reaction is not equal to the weight nor the component of the weight.

iii Resolve up the plane:

$40\cos 10° - 8g\sin 23° = 8a$

$a = \dfrac{8.758...}{8} = 1.09\text{ m s}^{-2}$

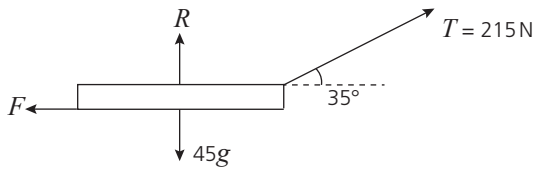
12 Sven uses a rope to pull a sledge of mass 45 kg across a rough horizontal surface. The rope makes an angle of 35° with the ground. The acceleration of the sledge is 2.5 m s^{-2} and the tension in the rope is 215 N.



i Calculate the magnitude of the frictional force acting on the sledge.

ii Calculate the coefficient of friction between the sledge and the surface, giving your answer correct to 3 significant figures.

i



Resolve horizontally using Newton's second law.
$T\cos 35° - F = 45a = 45 \times 2.5$
$F = 215\cos 35° - 45 \times 2.5 = 63.6$ N

ii Resolve vertically:
$R + 215\sin 35° = 45g$
$R = 317.68...$ N
As the sledge is moving $F = \mu R$
$\mu = \frac{F}{R} = \frac{63.6}{317.68....} = 0.200$ (3 s.f.)

Both the frictional force and the normal reaction force are needed to find the coefficient of friction.

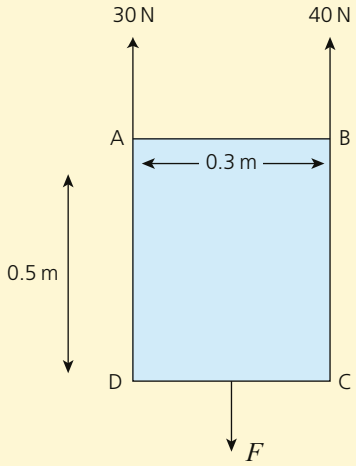
An accuracy mark would be lost in an exam for writing 0.2 as the answer when the question clearly states that 3 significant figures are required.

13 The diagram shows a rectangular lamina ABCD where the distances AB and BC are 0.3 m and 0.5 m, respectively. Forces of 30 N and 40 N act at A and B at right angles to the edge AB. A force F is applied to the edge CD to keep the lamina in equilibrium.

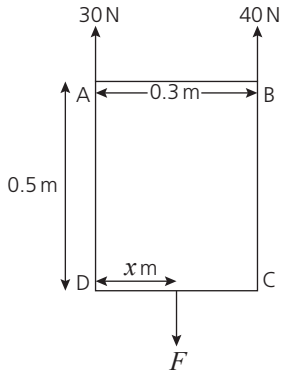
i Find the magnitude of the force F and the distance from D of the point of application.

The force F is now applied to M which is the midpoint of the edge CD. Additional forces at right angles to the lines AD and BC are applied at A and C to maintain equilibrium.

ii Find these forces, indicating clearly the direction in which they act.



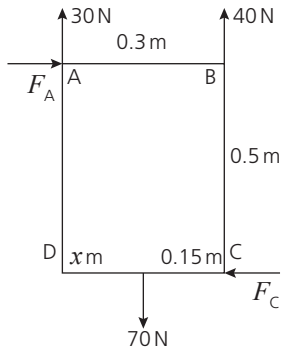
i



Vertical forces $30 + 40 - F = 0$ giving $F = 70$ N
Taking moments about A:
$40 \times 0.3 - 70x = 0$
So $x = 0.171$ m

The 30 N force has no moment about A as the line of action of the force passes through A.

ii



It is not a problem if you are not sure which direction to draw the forces. If you draw them in the opposite direction, their values will be negative which would need to be explained in your final line of working.

Horizontal forces in equilibrium so $F_A = F_C$
Take moments about A.

You can take moments about any point. The equation is simplest if you take moments about A as two of the forces have no moment about A.

$40\times0.3-70\times0.15-F_C\times0.5=0$

$F_C=2(12-10.5)=3\,\text{N}$ and $F_A=3\,\text{N}$

F_A acts in the direction AB and F_C acts in the direction CD as shown in the diagram.

The question reminds you that it is important to be clear about the direction in which a force acts.

14 A boy starts running the 100 m race from rest at the origin. His velocity $v\,\text{m s}^{-1}$ at time t seconds is given by

$v=4.2t-0.6t^2$ for $0\leqslant t\leqslant 3.5$

$v=7.35$ from $t=3.5$ until the end of the race.

i Find an expression for his acceleration at time t s.

Show that his acceleration is zero when $t=3.5$, and interpret this result.

ii Find the distance he has run after:

A 3.5 s

B 9.5 s.

iii Calculate the time he takes to run 100 m.

i Differentiate to find acceleration.

$a=\dfrac{dv}{dt}=4.2-1.2t$ for $0\leqslant t\leqslant 3.5$

$a=0$ for $t>3.5$

At $t=3.5$, $a=\dfrac{dv}{dt}=4.2-1.2\times3.5=0$

He has reached his maximum velocity.

This means that $t=3.5$ must give a maximum or minimum point. The context makes it clear that it is a maximum value.

ii A displacement s is given by $\int v\ dt=\int_0^{3.5}\left(4.2t-0.6t^2\right)dt$

$=\left[4.2\times\dfrac{t^2}{2}-0.6\times\dfrac{t^3}{3}\right]_0^{3.5}=\left(2.1\times3.5^2-0.2\times3.5^3\right)-(0)$

Distance is 17.15 m.

The boy does not change direction in this time interval so distance and displacement have the same value.

B For $3.5\leqslant t\leqslant 9.5$ speed is constant

so distance is $7.35\times6=44.1$ m

Speed = distance × time

Total distance in the first 9.5 s is $44.1+17.15=61.25$ m.

A common error would be to forget to include the distance in the first 3.5 s.

iii Extra distance still to run is $100-61.25=38.75$ m

Extra time needed $=\dfrac{\text{distance}}{\text{speed}}=\dfrac{38.75}{7.35}=5.272...$

Total time is $9.5+5.27=14.77$ s.

15 The velocity in m s^{-1} of a particle at time t seconds is given by $\mathbf{v}=\begin{pmatrix}5\\3t^2-1\end{pmatrix}$. The initial position of the particle is $\mathbf{r}=\begin{pmatrix}10\\-4\end{pmatrix}$ m from the origin.

i Find an expression for the acceleration of the particle at time t and determine whether the acceleration is constant.

ii Find the position of the particle after 3 seconds.

iii Find the cartesian equation of the path of the particle.

i $\mathbf{v}=\begin{pmatrix}5\\3t^2-1\end{pmatrix}$

Differentiate the velocity vector to find the acceleration vector. The expression for each component can be differentiated separately.

$\mathbf{a}=\dfrac{d\mathbf{v}}{dt}=\begin{pmatrix}0\\6t\end{pmatrix}$

This is a function of t and so is not constant

ii The position vector $\mathbf{r} = \int (\mathbf{v})\, dt$

$$\mathbf{r} = \int \begin{pmatrix} 5 \\ 3t^2 - 1 \end{pmatrix} dt = \begin{pmatrix} 5t \\ t^3 - t \end{pmatrix} + \mathbf{c}$$

This can be written as a single constant vector or two separate constants in each direction.

When $t = 0$, $\mathbf{r} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} 10 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \times 0 \\ 0 \end{pmatrix} + \mathbf{c}$$

$$\mathbf{r} = \begin{pmatrix} 5t + 10 \\ t^3 - t - 4 \end{pmatrix}$$

When $t = 3$, $\begin{pmatrix} 5 \times 3 + 10 \\ 3^3 - 3 - 4 \end{pmatrix} = \begin{pmatrix} 25 \\ 20 \end{pmatrix}$

iii $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5t + 10 \\ t^3 - t - 4 \end{pmatrix}$

You can use either expression to rearrange. Choose the direction which is simplest.

The expression for x rearranges to give $t = \dfrac{x - 10}{5} = 0.2x - 2$

Substitute for t in the expression $y = t^3 - t - 4$

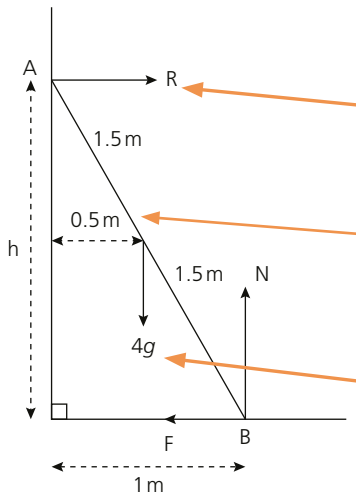
$y = (0.2x - 2)^3 - (0.2x - 2) - 4$

Take care with brackets here. There is no real need to expand.

$y = (0.2x - 2)^3 - 0.2x - 2$

16 A uniform ladder of mass 4 kg which is 3 m long leans against a smooth wall. The bottom of the ladder is 1 m away from the wall and the ladder is on the point of sliding. Find the coefficient of friction between the ladder and the floor.

Draw a diagram showing all the forces and all the information about distances.



It is useful to put the information about distances onto the force diagram.

The wall is smooth so the contact force is at right angles to the wall.

Notice this is half the distance the ladder is from the wall as the triangles are similar.

The weight goes in the middle of the ladder as the question says it is uniform.

Vertical equilibrium gives $N = 4g$

To take moments, find the height $h = \sqrt{3^2 - 1^2} = 2\sqrt{2}$

Take moments about A:

$N \times 1 - 4g \times 0.5 - F \times h = 0$

It is really important to include a force and a distance in every term of this equation even if the numerical value does not change.

$$F = \frac{4g - 2g}{2\sqrt{2}} = \frac{g}{\sqrt{2}}$$

The ladder is on the point of sliding so $F = \mu N$

$$\mu = \frac{F}{N} = \frac{\left(\frac{g}{\sqrt{2}}\right)}{4g} = \frac{1}{4\sqrt{2}} = 0.177 \text{ to 3 s.f.}$$