

OCR A Level Mathematics Year 2 – Full Worked Solutions

Practice questions: Pure Mathematics 1 (page 125)

1 Perimeter = 3 cm

Formula valid when θ in radians

(i) Arc length $s = r\theta$

So $P = r + r + s = 2r + r\theta = 3$

Perimeter is the total of three lengths and its value is given in the question.

$$\theta = \frac{3 - 2r}{r}$$

Rearrange the equation to find θ then substitute in the formula for Area.

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{3 - 2r}{r}\right)$$

$$= \frac{1}{2}r(3 - 2r) = \frac{3r}{2} - r^2$$

Show all the steps here as the answer is given in the question.

(ii) For maximum area $\frac{dA}{dr} = \frac{3}{2} - 2r = 0$

Calculus is used for maxima and minima problems.

$$r = \frac{3}{4} \text{ so } \theta = \frac{3 - 2\left(\frac{3}{4}\right)}{\left(\frac{3}{4}\right)} = 2$$

The question asks for the value of θ and not area.

$\frac{d^2A}{dr^2} = -2 < 0$ so this is value for maximum area

The question asks you to justify your answer, so show that the value obtained gives a maximum and not minimum point using the sign of the second derivative at the stationary point.

2 Substitute values

(i) When $x = -\frac{\pi}{3}$

$$\cos x = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \text{ and}$$

$$\begin{aligned} \cos\left(x - \frac{\pi}{6}\right) + 0.5 &= \cos\left(-\frac{\pi}{3} - \frac{\pi}{6}\right) + 0.5 \\ &= \cos\left(-\frac{\pi}{2}\right) + 0.5 = 0.5 \end{aligned}$$

Substitute each value in turn into both expressions. Make sure your calculator is set to radians. This is quite straightforward, and so is likely to be a hint to help with the second part.

When $x = -\frac{\pi}{2}$,

$$\cos x = \cos\left(-\frac{\pi}{2}\right) = 0 \text{ and}$$

$$\begin{aligned} \cos\left(x - \frac{\pi}{6}\right) + 0.5 &= \cos\left(-\frac{\pi}{2} - \frac{\pi}{6}\right) + 0.5 \\ &= \cos\left(-\frac{2\pi}{3}\right) + 0.5 = 0 \end{aligned}$$

(ii) Curves cross when

$$\cos x = \cos\left(x - \frac{\pi}{6}\right) + \frac{1}{2}$$

This equation is quite complicated to solve and the question is only worth two marks. Look for a link with the previous part of the question.

The values $x = -\frac{\pi}{3}$ and $x = -\frac{\pi}{2}$ are both negative solutions of the equation. The functions are periodic with period

$$2\pi \text{ so } x = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3} \text{ and}$$

$$x = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2} \text{ are both positive solutions}$$

Use the existing solutions and the periodic nature of the graphs to find other solutions. You only need one of these.

To find the y coordinate,

$$y = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \text{ so point is } \left(\frac{5\pi}{3}, \frac{1}{2}\right)$$

It is really easy to miss this part of the question – notice the question says coordinates.

To check

$$\begin{aligned} \cos\left(\frac{5\pi}{3} - \frac{\pi}{6}\right) + \frac{1}{2} &= \cos\left(\frac{3\pi}{2}\right) + \frac{1}{2} \\ &= 0 + \frac{1}{2} \end{aligned}$$

Build checking strategies into your work wherever possible.

$$\begin{aligned} \text{When } x = \frac{3\pi}{2}, y &= \cos\left(\frac{3\pi}{2}\right) = 0 \\ \text{so point is } &\left(\frac{3\pi}{2}, 0\right) \end{aligned}$$

You do not need both solutions here.

To check

$$y = \cos\left(\frac{3\pi}{2} - \frac{\pi}{6}\right) + \frac{1}{2} = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2} + \frac{1}{2} = 0$$

- (iii) The graph of $y = \cos\left(x - \frac{\pi}{6}\right) + \frac{1}{2}$ is a

translation of $y = \cos x$ by $\begin{pmatrix} \frac{\pi}{6} \\ \frac{1}{2} \end{pmatrix}$

You must use the word translation as well as the vector. Use a vector to fully describe the translation. Take care with the direction of the change in the x direction.

3 $y = (x^2 - 1)^4$

- (i) Using the chain rule

Simplify this before differentiating again.

$$\frac{dy}{dx} = 4(x^2 - 1)^3 \times 2x = 8x(x^2 - 1)^3$$

For a function which is a power of a bracket, it is good to do the chain rule without writing it out in full. Multiply by the power, reduce the power by 1 and multiply by the derivative of the bracket.

To differentiate again, using the product

$$\text{rule with } u = 8x \text{ and } v = (x^2 - 1)^3$$

$$\frac{du}{dx} = 8 \text{ and } \frac{dv}{dx} = 3(x^2 - 1)^2 \times 2x = 6x(x^2 - 1)^2$$

$$\frac{d^2y}{dx^2} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = 8x \times 6x(x^2 - 1)^2 + (x^2 - 1)^3 \times 8$$

It is easiest to simplify here by keeping factors together.

$$\begin{aligned} &= 48x^2(x^2 - 1)^2 + 8(x^2 - 1)^3 \\ &= 8(x^2 - 1)^2[6x^2 + (x^2 - 1)] \\ &= 8(x^2 - 1)^2(7x^2 - 1) \end{aligned}$$

- (ii) Curve is concave downwards when

$$\frac{d^2y}{dx^2} < 0$$

$$8(x^2 - 1)^2(7x^2 - 1) < 0$$

This inequality can be simplified as two of the factors are always positive, so it is the sign of the last factor that determines the sign of the second derivative.

$$8(x^2 - 1)^2 \geq 0 \text{ for all values of } x$$

$$(7x^2 - 1) < 0$$

$$\text{So } 7x^2 < 1$$

$$x^2 < \frac{1}{7}$$

$$-\frac{1}{\sqrt{7}} < x < \frac{1}{\sqrt{7}}$$

The solution of the quadratic inequality is a range of values between the boundary values that are the roots of the corresponding quadratic equation.

4 $f(x) = \frac{x+3}{x-1}$

→ (i) $ff(x) = f\left(\frac{x+3}{x-1}\right) = \frac{\left(\frac{x+3}{x-1}\right) + 3}{\left(\frac{x+3}{x-1}\right) - 1}$

$ff(x)$ is found by using the output of $f(x)$ as the input of the function.

$$= \frac{(x+3) + 3(x-1)}{(x+3) - 1(x-1)} = \frac{4x}{4} = x$$

Clear the double-decker fractions by multiplying top and bottom by $(x-1)$.

$ff(x) = x$ means that the inverse of $f(x)$ is the same as $f(x)$.

$$\text{So } f^{-1}(x) = f(x) = \frac{x+3}{x-1}$$

- (ii) The inverse function is a reflection in the line $y = x$ so this is a line of symmetry for this graph.

You can also find this line of symmetry by sketching the curve using the transformations required in part (iv).

→ (iii) $f(x) = \frac{x+3}{x-1} = \frac{(x-1) + 4}{x-1} = 1 + \frac{4}{x-1}$
so $a = 1$, $b = 4$, $c = 1$

Another way of finding this is to do algebraic division, and the remainder is the numerator of the fraction.

→ (iv) $y = \frac{1}{x}$ to $y = \frac{1}{x-1}$ Translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

To find a series of transformations, look at changing the equation one step at a time.

Take care with the sign of translations in the x direction.

$$y = \frac{1}{x-1} \text{ to } y = \frac{4}{x-1} \text{ Stretch in } y \text{ direction scale factor 4}$$

$$y = \frac{4}{x-1} \text{ to } y = 1 + \frac{4}{x-1}$$

$$\text{Translation } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- 5 $y = a - b \cos x$
Longest day 16.75 hours and shortest is 7.58333.

Use decimal values for time here.

- (i) Maximum value $y_{\max} = a + b = 16.75$

Maximum and minimum values of $\cos x$ are 1 and -1.

Minimum value $y_{\min} = a - b = 7.583333$

Adding equations $2a = 24.3333$
so $a = 12.166667$ and $b = 4.583333$

You could use the equation mode on your calculator to give these answers without writing down a method.

- (ii) 21st June is 20 days later than 1st June which is day 152.

Substitute $x = 172$ into the model.
 $y = 12.166667 - 4.583333 \cos 172$
 $= 16.705395$ which is 16 hours and 42 mins

This is only 3 minutes less than the actual value, so the model is good.

The question asks for a comparison, so the value of y is not enough.

- (iii) (a) The model is periodic with period 360 but the year has 365 days, so a

stretch scale factor $\frac{365}{360}$ parallel to the x axis would change the period.

Stretch in the x direction changes the period of the cosine function.

- (b) The maximum value occurs at day 172 and the model at 180, so a translation of 8 units parallel to the x axis would achieve this.

A translation of the cosine function will put the longest and shortest day in the correct places.

- 6 Arithmetic sequence has terms $a - d$, a , $a + d$

The proof works equally well with a , $a + d$ and $a + 2d$ but there is a little more algebra to do.

Assume that $d \neq 0$ and that the terms form a geometric sequence.

Begin by assuming the statement is true and show this leads to a contradiction.

Common ratio ($r =$) $\frac{a}{a-d} = \frac{a+d}{a}$

Each term divided by the previous term gives the common ratio.

Rearrange $a^2 = (a+d)(a-d) = a^2 - d^2$
So $d^2 = 0$ which is a contradiction since assumed that $d \neq 0$

Make very clear what the contradiction you have found actually is and make sure you make the conclusion explicit.

Therefore the terms of the arithmetic sequence cannot form a geometric sequence.

- 7 Geometric sequence is $1, r, r^2 \dots r^8$

Given $1 \times r \times r^2 \times \dots \times r^8 = 262144$

$$1 + 2 + \dots + 8 = 36$$

$$\text{so } 1 \times r \times r^2 \times \dots \times r^8 = r^{36} = 262144$$

$$r = \pm \sqrt[36]{262144} = \pm \sqrt{2}$$

Your calculator may give a decimal answer for r . You may recognise this as $\sqrt{2}$. You can work in decimals here. Use the Answer key or the memory to avoid problems with rounding.

When $r = \sqrt{2}$,

$$S_9 = \frac{a(r^9 - 1)}{r - 1} = \frac{((\sqrt{2})^9 - 1)}{\sqrt{2} - 1}$$

$$= 31 + 15\sqrt{2} = 52.2132$$

Use each value of r to find the sum of terms.

As the power is even, the root needs plus or minus. There is a clue in the wording of the question that two values are needed.

When $r = -\sqrt{2}$,

$$S_9 = \frac{a(r^9 - 1)}{r - 1} = \frac{((- \sqrt{2})^9 - 1)}{-\sqrt{2} - 1}$$

$$= 31 - 15\sqrt{2} = 9.7868$$

- 8 $y = x^4 - 1.5x^2 + 3$

(i) $\frac{dy}{dx} = 4x^3 - 3x$

$$\frac{d^2y}{dx^2} = 12x^2 - 3 = 0 \text{ at a point of inflexion.}$$

$$x^2 = \frac{1}{4}$$

$$x = \pm \frac{1}{2}$$

Points of inflexion occur when second derivative is zero.

When $x = \frac{1}{2}$,

$$y = \left(\frac{1}{2}\right)^4 - 1.5\left(\frac{1}{2}\right)^2 + 3 = \frac{43}{16}.$$

B is $\left(\frac{1}{2}, \frac{43}{16}\right)$

You could work in either fractions or decimals here as the decimals will be exact and not require rounding.

When $x = -\frac{1}{2}$,

$$y = \left(-\frac{1}{2}\right)^4 - 1.5\left(-\frac{1}{2}\right)^2 + 3 = \frac{43}{16}.$$

A is $\left(-\frac{1}{2}, \frac{43}{16}\right)$

(iii) At B when $x = \frac{1}{2}$,

$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) = -1$ so gradient of the tangent is -1 and the gradient of the normal is $+1$

Start by showing that all the angles in ABCD are 90° which makes it a rectangle.

At A when $x = -\frac{1}{2}$,

$\frac{dy}{dx} = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right) = 1$ so gradient of the tangent is 1 and the gradient of the normal is -1

The normal at A is parallel to the tangent at B, so all the angles of ABCD are 90°

The graph is a symmetric graph so the length AC = BC. Therefore ABCD is a square because adjacent sides are equal.

You only need to prove that two adjacent sides are equal – opposite sides are automatically equal because ABCD is a rectangle.

You need to make it very clear that you are arguing from the symmetry of the graph. This happens because the equation of the curve has only even powers.

Alternative method

To show that AC = BC

The alternative method has a lot more algebra and could cost valuable time under exam conditions.

When $x = \frac{1}{2}$

Equation of the tangent is

$$y - \frac{43}{16} = -1\left(x - \frac{1}{2}\right)$$

$$y = -x + \frac{51}{16}$$

Using the equation of a graph $y - y_1 = m_1(x - x_1)$

When $x = -\frac{1}{2}$

Equation of the tangent is

$$y - \frac{43}{16} = +1\left(x + \frac{1}{2}\right)$$

$$y = x + \frac{51}{16}$$

C is the point of intersection of the

tangents. C is $\left(0, \frac{51}{16}\right)$

Length AC =

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{43}{16} - \frac{51}{16}\right)^2} = \frac{1}{2}\sqrt{2}$$

Length BC =

$$\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{43}{16} - \frac{51}{16}\right)^2} = \frac{1}{2}\sqrt{2}$$

AC = BC so ABCD is a square.

Practice questions: Pure Mathematics 2 (page 237)

1 $y = \frac{1}{x}$

$$\int_2^4 \frac{1}{x} dx = [\ln x]_2^4 = \ln 4 - \ln 2 = \ln\left(\frac{4}{2}\right) = \ln 2$$

Evaluate the first area.

$$\int_5^a \frac{1}{x} dx = [\ln x]_5^a = \ln a - \ln 5 = \ln\left(\frac{a}{5}\right) = \ln 2$$

Find the second area in terms of a .

$$\text{So } \ln\left(\frac{a}{5}\right) = \ln 2 \text{ giving } \frac{a}{5} = 2 \text{ so } a = 10$$

Equate the answers

If the values of the natural logs are the same, the values must be the same.

2 $y = 4 \sin x - 3 \cos x$ and $y = 5 \sin x$

(i) $y = 5 \sin x$ goes through the origin since $y = 5 \sin 0 = 0$ and $y = 4 \sin x - 3 \cos x$ goes through $(0, -3)$.

Decide which is which by looking at the value of y when $x = 0$. You could also use any value for x and look at which is the larger value for y .

(ii) Rewrite $y = 4 \sin x - 3 \cos x$ in R, α form.

To be sure that the graphs have the same amplitude, we need to rewrite the first equation. Use the $R \sin(x - \alpha)$ form to easily compare with the other graph.

$$y = 4 \sin x - 3 \cos x = R \sin(x - \alpha) \\ = R(\sin x \cos \alpha - \sin x \sin \alpha)$$

Equate coefficients $4 = R \cos \alpha$ and $3 = R \sin \alpha$

Solve the non-linear simultaneous equations.

Square and add $R^2 = 4^2 + 3^2 = 25$
so $R = 5$

Dividing $\tan \alpha = \frac{R \sin \alpha}{R \cos \alpha} = \frac{3}{4}$

Question asks for 'precisely' so it is better not to find a rounded answer here.

So equation is $y = 5 \sin\left(x - \arctan \frac{3}{4}\right)$

This is a translation of $y = 5 \sin x$ by

$$\begin{pmatrix} \arctan \frac{3}{4} \\ 0 \end{pmatrix}$$

Since the amplitude of the two graphs is the same, no stretch is needed in the y direction.

Take care with the sign of the translation.

3 $(1-x)^{-2}$ The Binomial expansion can be used at once as the bracket begins with 1.

$$\begin{aligned} \text{(ii)} \quad (1-x)^{-2} &= 1 + (-2)(-x) + \frac{(-2)(-3)(-x)^2}{2!} \\ &+ \frac{(-2)(-3)(-4)(-x)^3}{3!} + \frac{(-2)(-3)(-4)(-5)(-x)^4}{4!} \\ &+ \frac{(-2)(-3)(-4)(-5)(-6)(-x)^5}{5!} + \dots \end{aligned}$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \dots \quad \text{(iii)}$$

Substitute $n = -2$ and $[-x]$ for x . Take care to use brackets to make sure the signs are correct.

$$\begin{aligned} \text{(iii)} \quad \text{When } x = 0.1, (1-0.1)^{-2} &= 1 + 2 \times 0.1 \\ &+ 3 \times 0.1^2 + 4 \times 0.1^3 + 5 \times 0.1^4 + 6 \\ &\times 0.1^5 + \dots \\ (0.9)^{-2} &= \left(\frac{9}{10}\right)^{-2} = \left(\frac{10}{9}\right)^2 = \frac{100}{81} \\ &= 1 + 0.2 + 0.03 + 0.004 + 0.0005 \\ &+ 0.00006 = 1.23456 \end{aligned}$$

Substitute $n = 0.1$ into both sides and show where the fraction arises.

Use the whole triangle and the angle AOB.

4 Area of triangle AOB =
 $\frac{1}{2}ab \sin C = \frac{1}{2}r^2 \sin \angle AOB = \frac{1}{2}r^2 \sin 2\alpha$

Area of triangle AOM = $\frac{1}{2}OM \times AM$

Right-angled triangle area = $\frac{1}{2}$ base $\times$ height

$AM = r \sin \alpha$ and $OM = r \cos \alpha$ so area
 $= \frac{1}{2}(r \sin \alpha)(r \cos \alpha) = \frac{1}{2}r^2 \sin \alpha \cos \alpha$

Area triangle AOB is twice area of triangle AOM so $\frac{1}{2}r^2 \sin 2\alpha = 2 \times \frac{1}{2}r^2 \sin \alpha \cos \alpha$

Triangle AOB is isosceles.

So $\frac{1}{2}r^2 \sin 2\alpha = r^2 \sin \alpha \cos \alpha$ giving

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$5 \quad \sec^2 \theta + \operatorname{cosec}^2 \theta \equiv \frac{4}{\sin^2 2\theta}$$

$$\text{(i)} \quad \sec^2 \theta + \operatorname{cosec}^2 \theta = \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}$$

Start with the left hand side and work until you achieve the right hand side. If you get stuck in the middle, you can begin from the right hand side also, but this needs a conclusion when you have both sides equal to the same thing.

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta} = \frac{1}{\cos^2 \theta \sin^2 \theta}$$

Look for trig identities

$$= \frac{1}{\frac{1}{4}(2 \cos \theta \sin \theta)^2} = \frac{4}{\sin^2 2\theta}$$

Wherever you see $\sin \theta \cos \theta$, think about the double angle formula.

To solve $\sec^2 \theta + \operatorname{cosec}^2 \theta = 4$

The word 'hence' is an instruction to use the previous answer in your method in (ii).

$$\frac{4}{\sin^2 2\theta} = 4$$

Take care to get both values for $\sin 2\theta$.

$$\sin^2 2\theta = 1$$

$$\sin 2\theta = \pm 1$$

$$\text{When } \sin 2\theta = 1, 2\theta = \frac{\pi}{2}, \frac{5\pi}{2} \text{ giving } \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

It is important to put in the choices of values for 2θ at this stage and not at the end. All values of 2θ to 4π are needed. The bigger values are found by adding 2π .

$$\text{When } \sin 2\theta = -1, 2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}$$

$$\text{giving } \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$6 \quad x^2y + 2y - 4x = 0$$

(i) Differentiate both sides with respect to x

$$\left(2xy + x^2 \frac{dy}{dx}\right) + 2 \frac{dy}{dx} - 4 = 0$$

Use the product rule to differentiate the first term.

$$x^2 \frac{dy}{dx} + 2 \frac{dy}{dx} = -2xy + 4$$

Collect terms containing $\frac{dy}{dx}$ together.

$$(x^2 + 2) \frac{dy}{dx} = 4 - 2xy$$

Factorise and divide by the factor.

$$\frac{dy}{dx} = \frac{4 - 2xy}{2 + x^2}$$

(ii) Maximum point when $\frac{dy}{dx} = 0$,

$$\frac{4 - 2xy}{2 + x^2} = 0$$

$$4 - 2xy = 0 \text{ so } y = \frac{4}{2x} = \frac{2}{x}$$

Solving $\frac{dy}{dx} = 0$ does not give the coordinates straight away. You need to solve this equation simultaneously with the equation of the curve.

Substitute for y in the equation of the curve

$$x^2 \left(\frac{2}{x} \right) + 2 \left(\frac{2}{x} \right) - 4x = 0$$

$$2x + \frac{4}{x} - 4x = 0$$

$$\frac{4}{x} = 2x$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

The diagram indicates that the maximum occurs when $x = \sqrt{2}$.

Use the information on the graph given in the question to distinguish maximum and minimum points. There is quite a lot of extra work needed to find the second derivative. You could also evaluate the derivative on either side of the point to determine which is maximum.

Substitute into the simpler equation

$$y = \frac{2}{x} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

So maximum point is $(\sqrt{2}, \sqrt{2})$

To check, substitute into the other equation $(\sqrt{2})^2 y + 2y - 4\sqrt{2} = 0$

Use checking strategies whenever possible.

$$4y = 4\sqrt{2}$$

$$y = \sqrt{2}$$

Partial fractions with distinct linear factors

$$(i) \frac{x-2}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$x-2 = A(x-3) + B(x+1)$$

Multiply by the denominator.

Substitute $x = 3$, $3-2 = A(0)$

$$+ B(3+1) \text{ so } B = \frac{1}{4}$$

Choose the values of x for which the brackets are zero.

Substitute $x = -1$, $-1-2 = A(-1-3)$

$$+ B(0) \text{ so } A = \frac{3}{4}$$

$$\frac{x-2}{(x+1)(x-3)} = \frac{\frac{3}{4}}{x+1} + \frac{\frac{1}{4}}{x-3} \\ = \frac{3}{4(x+1)} + \frac{1}{4(x-3)}$$

$$(ii) \text{ Area} = \int_0^2 \left(\frac{3}{4(x+1)} + \frac{1}{4(x-3)} \right) dx \\ = \frac{1}{4} \int_0^2 \left(\frac{3}{(x+1)} + \frac{1}{(x-3)} \right) dx$$

It makes the integration slightly easier to take the factor of $\frac{1}{4}$ outside the integral sign.

$$\frac{1}{4} [3 \ln|x+1| + \ln|x-3|]_0^2 \\ = \frac{1}{4} ((3 \ln 3 + \ln 1) - (3 \ln 1 + \ln 3))$$

Modulus is needed here as the limits used give negative numbers for $|x-3|$.

$$= \frac{1}{4} (2 \ln 3) = \frac{1}{2} \ln 3$$

You can check your final answer with your calculator if it has the integral function. The answer on its own without the algebra will not get full marks.

$$8 \quad y = x \sin x$$

$$(i) \text{ Substitute } x = \frac{\pi}{2}, y = \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2} \times 1 \\ = \frac{\pi}{2}$$

The word 'verify' in the question means that you do not have to solve the simultaneous equations. To verify it is the point of intersection, it is enough to substitute the given x value into both equations.

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{2}$ on the line

So P is the point $\left(\frac{\pi}{2}, \frac{\pi}{2} \right)$, which is the point of intersection.

To show it is a tangent to the curve at P the curve must have the same gradient as the line. Gradient of the line is 1.

The question also requires you to show that the line is a tangent, so it is enough to show it has the same gradient as the curve at the point P.

Product rule gives $\frac{dy}{dx} = 1 \sin x + x \cos x$

It is good in simple cases to be able to use the product rule without writing it out in full.

When $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 1 \sin \frac{\pi}{2} + \frac{\pi}{2} \cos \frac{\pi}{2}$
 $= 1 + 0 = 1$ so the line is a tangent at P

Be clear what conclusion you have reached from your working.

(ii) Area is $\int_0^{\frac{\pi}{2}} (x - x \sin x) dx$

The question requires an exact value here. An algebraic method is required.

You could take x as a common factor here and use parts with x and $(1 - \sin x)$.

Use integration by parts for

$$\int (x \sin x) dx \quad \begin{array}{l} u = x \\ \frac{du}{dx} = 1 \end{array} \quad \begin{array}{l} \frac{dv}{dx} = \sin x \\ v = -\cos x \end{array}$$

Take care with signs when integrating trig functions.

$$\begin{aligned} &= x(-\cos x) - \int 1(-\cos x) dx \\ &= -x \cos x + \sin x \end{aligned}$$

$$\begin{aligned} \text{Area is } &\int_0^{\frac{\pi}{2}} (x - x \sin x) dx \\ &= \left[\frac{1}{2} x^2 - (-x \cos x + \sin x) \right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{1}{2} \left(\frac{\pi}{2} \right)^2 - \left(-\left(\frac{\pi}{2} \right) \cos \left(\frac{\pi}{2} \right) + \sin \left(\frac{\pi}{2} \right) \right) \right) \\ &\quad - \left(\frac{1}{2} 0^2 - (-0 \cos 0 + \sin 0) \right) \end{aligned}$$

Make sure you use both limits here – it is possible to get a non-zero value from substituting $x=0$ when cosine function is used.

$$= \frac{\pi^2}{8} - 1$$

Your calculator may have the facility to calculate a definite integral as a decimal. This is a very useful check but it does not take the place of this algebraic method.

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$$y = \frac{6e^x}{5 + e^x}$$

(i) Using calculator

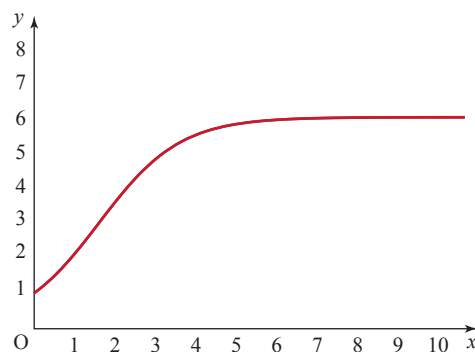


Figure 1

Use the graph drawing facility if you have it. Pick out the most important features – the y intercept and the horizontal asymptote. If you have a scientific calculator, use table mode to find a set of points to indicate the shape of the curve.

Otherwise, the algebra can be rewritten to

$y = \frac{6}{5e^{-x} + 1}$. As x increases, the denominator tends to 1, so y tends to 6.

This alternative version of the algebra is easier to consider as the x appears only once.

(ii) Rate of growth is $\frac{dP}{dt}$. This is maximum when $\frac{d^2P}{dt^2} = 0$

This is the derivative of $\frac{dP}{dt}$ and is zero at a maximum value.

$$P = \frac{6e^t}{5 + e^t}$$

Quotient rule for differentiation

$$\begin{array}{ll} u = 6e^t & v = 5 + e^t \\ \frac{du}{dt} = 6e^t & \frac{dv}{dt} = e^t \end{array}$$

Find the derivatives of the top and bottom of the fraction and use the quotient rule.

$$\begin{aligned} \frac{dP}{dt} &= \frac{6e^t(5 + e^t) - 6e^t e^t}{(5 + e^t)^2} \\ &= \frac{30e^t + 6e^{2t} - 6e^{2t}}{(5 + e^t)^2} = \frac{30e^t}{(5 + e^t)^2} \end{aligned}$$

Simplify before differentiating a second time.

Quotient rule $u = 30e^t$

$$\frac{du}{dt} = 30e^t$$

$$v = (5 + e^t)^2$$

$$\frac{dv}{dt} = 2(5 + e^t)e^t$$

$$\frac{d^2P}{dt^2} = \frac{30e^t(5 + e^t)^2 - 30e^t 2e^t(5 + e^t)}{(5 + e^t)^4} = 0$$

There is no need to simplify this fraction into its lowest terms. We simply need the numerator to be zero.

$$30e^t(5 + e^t)^2 - 30e^t 2e^t(5 + e^t) = 0$$

$$30e^t(5 + e^t)[(5 + e^t) - 2e^t] = 0$$

Take out common factors as much as possible.

$$30e^t(5 + e^t) \neq 0 \text{ so } (5 + e^t) - 2e^t = 0$$

$$e^t = 5$$

$$t = \ln 5$$

Use your sketch to check whether this looks correct. It is the steepest part of the graph.

(iii) Try $P = 7$

A good method to show that P is never 7 is to try and find the points at which $P = 7$ and show it is impossible.

$$P = \frac{6e^t}{5 + e^t} = 7$$

$$6e^t = 7(5 + e^t)$$

$$6e^t = 35 + 7e^t$$

$$e^t = -35$$

Make sure you have a clear conclusion.

e^t is never negative, so P cannot be 7.

Practice questions: Pure Mathematics 3 (page 322)

1 $\frac{dm}{dt}$ is the rate of change of mass.

Rate of change is the time derivative of m .

➔ Inversely proportional $\frac{dm}{dt} = \frac{k}{\sqrt{m}}$

Inversely proportional to a quantity is the same as equal to k over the quantity. k is a constant which can be evaluated from the information in the question.

$$\text{At } t = 0, m = 9, \frac{dm}{dt} = 10$$

$$10 = \frac{k}{\sqrt{9}} \text{ so } k = 30$$

➔ Differential equation $\frac{dm}{dt} = \frac{30}{\sqrt{m}}$

Make sure you state your answer clearly with the value for k substituted.

$$2 \quad \mathbf{F}_1 = (2\mathbf{i} + 3\mathbf{j}), \quad \mathbf{F}_2 = (-\mathbf{i} + \mathbf{j}),$$

$$\mathbf{F}_3 = (10\mathbf{i} - 2\mathbf{j}),$$

$$(2\mathbf{i} + 3\mathbf{j}) + (-\mathbf{i} + \mathbf{j}) + (10\mathbf{i} - 2\mathbf{j}) = 0$$

The vector is zero when both components of the vector are zero.

$$(2a - b + 10)\mathbf{i} + (3b + a - 2)\mathbf{j} = 0$$

$$\text{Simultaneous equations } 2a - b = -10$$

$$a + 3b = 2$$

giving $a = -4, b = 2$

Use the calculator to solve the simultaneous equations if it has that facility.

➔ Check:

$$(-8\mathbf{i} + 6\mathbf{j}) + (-2\mathbf{i} - 4\mathbf{j}) + (10\mathbf{i} - 2\mathbf{j}) = 0$$

$$0\mathbf{i} + 0\mathbf{j} = 0$$

Build checking strategies into your working whenever possible.

$$3 \quad A(1, 2, 0), \quad B(-1, 3, -1), \quad C(3, 0, -6),$$

(i) Column vectors

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix},$$

Each number in the vector is the change as you move from one point to the other.

$$\overrightarrow{BC} = \begin{pmatrix} 4 \\ -3 \\ -5 \end{pmatrix}$$

(ii) Method 1 using scalar product

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}$$

$$= (-2) \times 2 + 1 \times (-2) + (-1)(-6)$$

$$= -4 - 2 + 6 = 0$$

You may have to try more than one scalar product before you get zero. You only need to write one of them down.

So $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ so the triangle is a right-angled triangle.

Make sure you explain why your working makes it a right-angled triangle.

Method 2 using lengths

Pythagoras' theorem can be used to establish that the triangle is a right-angled triangle.

$$|\overline{AB}| = \sqrt{(-2)^2 + 1^2 + (-1)^2} = \sqrt{6}$$

$$|\overline{AC}| = \sqrt{2^2 + (-2)^2 + (-6)^2} = \sqrt{44}$$

$$|\overline{BC}| = \sqrt{4^2 + (-3)^2 + (-5)^2} = \sqrt{50}$$

$$AB^2 + AC^2 = 6 + 44 = 50 = BC^2$$

Be clear how your values establish the conclusion.

So triangle is right-angled by Pythagoras' theorem.

4 $\overline{OA} = \mathbf{u}, \overline{OC} = \mathbf{v}$

(i) $\overline{CD} = \frac{1}{3}\overline{CB} = \frac{1}{3}\overline{OA} = \frac{1}{3}\mathbf{u}$

Build a sequence of vectors leading to $\overline{OE}$.

Use the ratio to establish the fraction of the vector.

OABC is a parallelogram, so the vectors $\overline{CB}$ and $\overline{OA}$ are parallel and the same length, so are equal vectors.

$\overline{OD} = \overline{OC} + \overline{CD} = \mathbf{v} + \frac{1}{3}\mathbf{u}$

Use two vectors that follow on to find the vector $\overline{OD}$.

$$\overline{OE} = \frac{3}{4}\overline{OD} = \frac{3}{4}\left(\mathbf{v} + \frac{1}{3}\mathbf{u}\right) = \frac{3}{4}\mathbf{v} + \frac{1}{4}\mathbf{u}$$

$$\begin{aligned}\overline{CE} &= \overline{CO} + \overline{OE} = -\mathbf{v} + \frac{3}{4}\mathbf{v} + \frac{1}{4}\mathbf{u} \\ &= -\frac{1}{4}\mathbf{v} + \frac{1}{4}\mathbf{u}\end{aligned}$$

(ii) $\overline{CA} = \overline{CO} + \overline{OA} = -\mathbf{v} + \mathbf{u}$

To show E is on the line CA, show that the vector $\overline{CE}$ is parallel to $\overline{CA}$.

$$\begin{aligned}\text{So } \overline{CE} &= -\frac{1}{4}\mathbf{v} + \frac{1}{4}\mathbf{u} = -\frac{1}{4}(-\mathbf{v} + \mathbf{u}) \\ &= -\frac{1}{4}\overline{CA}\end{aligned}$$

Vectors are parallel when one is a multiple of the other.

Vectors are parallel and pass through C, so they must be on the same straight line.

E lies on the diagonal.

5 $f(x) = 4 \sin 2x - x^3$

(i) $f'(x) = 4 \times 2 \cos 2x - 3x^2$

As calculus is needed, your calculator must be set to radians.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{4 \sin 2x_n - x_n^3}{8 \cos 2x_n - 3x_n^2}$$

Avoid problems with premature rounding by using all the figures on your calculator.

The most efficient way of calculating this is to type 1 and press =, then type the Newton-Raphson formula with ANS for x_n . A new iteration is given just by pressing = key.

$$x_1 = 1.416671973 \text{ (1.41667 to 5 d.p.)}$$

$$x_2 = 1.297230123 \text{ (1.29723 to 5 d.p.)}$$

Your answer will disagree with this slightly if you use the rounded value for x_1 .

$$x_3 = 1.288642572 \text{ (1.28864 to 5 d.p.)}$$

$$x_4 = 1.288591797 \text{ (1.28859 to 5 d.p.)}$$

The last two answers agree when rounded to 3 significant figures, so $x = 1.29$

(iii) Answer correct to 3 s.f. if there is a change in sign of $f(x)$

Use the upper and lower bound for 1.29 to 3 s.f. to show there is a change of sign in $f(x)$.

$$\begin{aligned}f(1.285) &= 4 \sin(2 \times 1.285) - 1.285^3 \\ &= 0.042\end{aligned}$$

$$\begin{aligned}f(1.295) &= 4 \sin(2 \times 1.295) - 1.295^3 \\ &= -0.07557\end{aligned}$$

So the root is between 1.285 and 1.295 so $x = 1.29$ to 3 s.f.

6 $f(x) = x^5 - 2x + 3 = 0$

(i) The root lies between -1.5 and -1.4 as

There is a root in the interval where the value of $f(x)$ changes from positive to negative.

$$\begin{aligned}f(-1.5) &= -1.59 < 0 \text{ and } f(-1.4) \\ &= 0.42 > 0\end{aligned}$$

(ii) $f(-1.43) = (-1.43)^5 - 2(-1.43) + 3 = -0.120$

Table mode is useful if you have it on your calculator. If not, use trial and error to find the two values which produce a change of sign.

$$\begin{aligned}f(-1.42) &= (-1.42)^5 - 2(-1.42) + 3 \\ &= 0.0665\end{aligned}$$

So the root lies between -1.43 and -1.42

Make your conclusion clear.

(iii) $f(-1.425) = (-1.425)^5 - 2(-1.425) + 3 = -0.026$

Use the value of the function in the middle of the interval to determine whether to round up or down.

Root lies between -1.425 and -1.42 so the root is -1.42 to 2 d.p.

Use the equation solver on your calculator to check your answer.

7 $x^3 - 2x - 5 = 0$

(i) $x^3 - 5 = 2x$ giving $x = 0.5x^3 - 2.5$ so $a = 0.5$ and $b = 2.5$

Isolate the linear term.

(ii) $x_0 = 1$

$x_1 = -2$

$x_2 = -6.5$

$x_3 = -139.8125$. The sequence is

divergent so will not find the root.

Iteration is not guaranteed to find a root - where the graph is too steep, it gives a divergent sequence.

(iii) $x^3 = 2x + 5$ giving $x = \sqrt[3]{2x + 5}$ so $c = 2$ and $d = 5$

Isolate the cubic term.

(iv) $x_0 = 1$

$x_1 = 1.912931183$

$x_2 = 2.066580768$

$x_3 = 2.090292423$

Use the ANS key on your calculator to avoid problems with rounding.

(v) Rounded to 2 d.p. $x_3 = 2.09$

$f(2.085) = 2.085^3 - 2 \times 2.085 - 5 = -0.106 < 0$

Make sure you use the original function and test whether the value changes sign in the interval which rounds to 2.09.

$f(2.095) = 2.095^3 - 2 \times 2.095 - 5 = 0.005 > 0$

So there is a root between 2.085 and 2.095 which is 2.09 to 2 d.p.

8 $x = 1 + 2t, y = 2t + t^2 - 2 \leq t \leq 2$

(i) At A, $y = 0, 2t + t^2 = 0$ giving $t = 0$ or $t = -2$

A is on the x axis, so $y = 0$.

When $t = 0, x = 1 + 2 \times 0 = 1$

When $t = -2, x = 1 + 2 \times (-2) = -3$

So A is $(-3, 0)$

C is the point at the end of the domain, so $t = 2$

The function is only defined for values of $t \leq 2$.

$x = 1 + 2 \times 2 = 5, y = 2 \times 2 + 2^2 = 8$

C is $(5, 8)$

(ii) $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 2 + 2t$ so

$\frac{dy}{dx} = \frac{2 + 2t}{2} = 1 + t$

$\frac{dy}{dx}$ is found by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$.

At B $\frac{dy}{dx} = 0$ so $1 + t = 0, t = -1$

B is a minimum point when $\frac{dy}{dx} = 0$.

When $t = -1, x = 1 - 2 = -1,$

$y = -2 + (-1)^2 = -1$

Use the value of t to find the co-ordinates of B.

B is $(-1, -1)$

(iii) $2t = x - 1, t = \frac{x - 1}{2}$

Rearrange the simpler equation so that t can be eliminated.

Substitute for $t: y = 2\left(\frac{x - 1}{2}\right) + \left(\frac{x - 1}{2}\right)^2$

$y = x - 1 + \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}$

$y = \frac{1}{4}x^2 + \frac{1}{2}x - \frac{3}{4}$

Use the wording in the question to know how much to rearrange the answer.

9 $\frac{dv}{dt} = -\frac{1}{3}kv(v - 3) \quad k > 0$

(i) Separate the variables

$\int \frac{1}{v(v - 3)} dv = \int \left(-\frac{1}{3}k\right) dt$

You could take the 3 to the LHS at this stage also.

The fraction is too complicated to be integrated as it stands and must be rewritten using partial fractions.

To integrate LHS, partial fractions needed

$\frac{1}{v(v - 3)} = \frac{A}{v} + \frac{B}{v - 3}$

$1 = A(v - 3) + Bv$

Use $v = 0: 1 = A(0 - 3)$ so $A = -\frac{1}{3}$

Use $v = 3: 1 = 3B$ so $B = \frac{1}{3}$

$\frac{1}{3} \int \left(-\frac{1}{v} + \frac{1}{v - 3}\right) dv = \int \left(-\frac{1}{3}k\right) dt$

$-\ln v + \ln(v - 3) = -kt + c$

It is easiest to take out the factor of $\frac{1}{3}$ to make the integration simpler.

Take care with the signs of the natural log terms.

The factors of $\frac{1}{3}$ cancel out.

$$\ln\left(\frac{v-3}{v}\right) = -kt + c$$

Use laws of logs to combine into a single term.

$$\frac{v-3}{v} = e^{-kt+c} = e^{-kt} \times e^c = Ae^{-kt}$$

Use laws of indices to rewrite the exponential term. A is an arbitrary constant in place of e^c .

$$v-3 = Ae^{-kt}v$$

$$v - Ae^{-kt}v = 3$$

$$v(1 - Ae^{-kt}) = 3$$

$$v = \frac{3}{1 - Ae^{-kt}}$$

The question requires this to be rearranged so that v is the subject.

$$\text{When } t = 0, v = 10$$

Use the additional information in the question to evaluate the constant. This step can be done much earlier if preferred.

$$10 = \frac{3}{1 - Ae^{-k(0)}} = \frac{3}{1 - A}$$

$$10(1 - A) = 3$$

$$A = 0.7$$

$$\text{So } v = \frac{3}{1 - 0.7e^{-kt}}$$

Substitute the value of A in to the solution.

- (ii) For terminal velocity, let $t \rightarrow \infty$
so $e^{-kt} \rightarrow 0$ and $1 - 0.7e^{-kt} \rightarrow 1$

Find out what happens to the velocity when t is very large. You can also do this by substituting large values for t in to the expression for v .

So $v \rightarrow 3$ giving terminal velocity 3 ms^{-1}

Practice questions: Statistics (page 418)

- 1 (i) Only 5 countries out of 29 show a difference of more than 50 mins and all the rest less, which might suggest that the figure is not accurate. However, the countries are not equal in size and the countries with the biggest populations are China, where the difference is almost 50 mins, and India, where the difference is much more than 50 mins. If the figure was calculated with the size of the population taken into account it might be nearer to 50 mins.

The question says 'discuss' which suggests that the answer is not clear cut but arguments can be put for both sides of an argument.

50 mins may have been rounded up for the headline so it may be accurate to 1 significant figure.

The context in which the data is used can be useful – make sure you have read all the introduction in the question where this information is to be found.

- (ii) Using 365 days in a year, 50 mins per day is $365 \times 50 \div 60 = 304$ hours
304 hours over 39 days is $304 \div 39 = 7.8$ hours per day which could be a whole working day. This suggests the 39 days per year is consistent.

It is not clear how many hours make a working day – it may vary across the world.

- 2 The diagram is false. 95% of the population lies between $\mu - 1.96\sigma$ and $\mu + 1.96\sigma$. The diagram can be corrected in several ways:

Use inverse normal tables or your calculator to determine this.

You would only need one of these in your answer.

- Change the percentages to $\frac{2}{3}$ of the population
 - Change the label on the middle arrow to 4σ
 - Change the arrow to go from the middle to the dotted line.
- 3 (i) The slug population may be low in the winter months or the volunteers may not be so keen to go and look for slugs in the winter.

This data is taken from a large data set that you will have been working on throughout the course. You will have had chance to consider the context in your preparation for an exam.

- (ii) Month 8 (August) seems to be lower than expected (93). This could just be a random variation or it could be a genuine dip in the population. Perhaps weather conditions were not good for slugs that month, or it could be that some volunteers were on holiday so fewer slugs found.

It is good practice to quote the figure as you discuss it.

The figures do not tell us everything – there is some speculation required here.

- (iii) The data seems to be cyclic with a one-year period. This is consistent with an annual life cycle. There could be other explanations, for example that the slugs are not active in the winter and so are not seen.

Make sure you answer the question directly asked. The data is consistent with this idea but it does not prove it.

The population rises for half the year and then returns to a similar value by the end of the year.

The word 'discuss' indicates that the answer is not certain so other points could be made that do not agree with the statement.

- 4 $\rightarrow$ X is the number of times I correctly guess the best checkout.

It is good practice to define the random variable and write down its distribution.

- (i) $X \sim B(10, 0.65)$

More than half the occasions $X > 5$

$$P(X > 5) = 1 - P(X \leq 5)$$

If you are using cumulative Binomial tables, you will need to calculate the probability this way as the tables only give $\leq$ probabilities. If your calculator finds Binomial probabilities, you may be able to find the probability directly and just the answer would be sufficient.

$$= 1 - 0.2485 = 0.7515$$

- (ii) $H_0: p = 0.65$ where p is the probability of choosing correctly.

The null hypothesis is always that there is no change in the probability.

Hypothesis testing is a formal process. Make sure you define the letters you are using.

- $H_1: p < 0.65$ If it is harder to predict, the probability will be less.

It is important to choose carefully the alternative hypothesis to reflect what you are asked to test. This is a one-tailed test.

- (iii) 20 trials. $X \sim B(20, 0.65)$ under the null hypothesis.

Make clear which distribution you are using.

1% significance level

$$P(X \leq 7) = 0.0060 < 1\%$$

Show the evidence for your choice of values – one probability either side of the significance level is sufficient.

$$P(X \leq 8) = 0.0196 > 1\%$$

- $\rightarrow$ So values of k are 0, 1 ... 7

State your answer clearly.

- (iv) 20 visits may be a large enough number to make the test still valid, especially if the visits are spread over a reasonable length of time and occur at different times and days. If they all occur very soon after the change, then it might invalidate the test as the new circumstances may affect the situation, e.g. new staff under training or new attitudes by the staff and customers.

The answer here is not a clear yes or no. Put arguments for both sides.

- 5 Tree diagram may be used here.

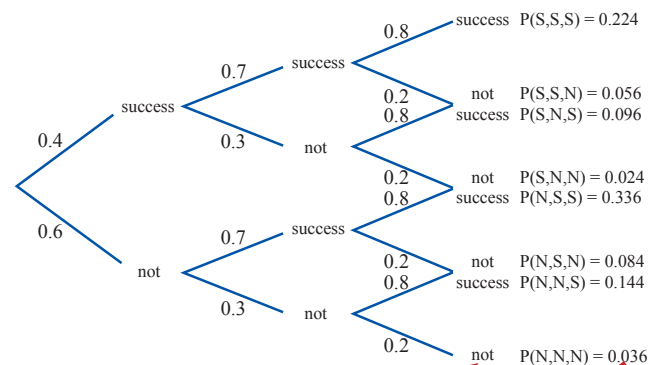


Figure 2

Multiply along the branches.

Check your answers by adding the probabilities to get 1.

- (i) Using the values on the tree diagram

(a) $P(\text{exactly 2 successes}) = 0.056 + 0.096 + 0.336 = 0.488$

Select the second, third and fifth branches

(b) $P(\text{at least one success}) = 1 - P(\text{no successes}) = 1 - 0.036 = 0.964$

The opposite event is much easier to calculate, but you could add seven of the eight values to get this answer.

- (ii) Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The phrase 'given that' is the clue to conditional probability.

$$P(\text{success on 3 tasks} | \text{success on at least one})$$

$$= \frac{P(\text{both events})}{P(\text{given event})}$$

$$= \frac{P(\text{success in all 3})}{P(\text{success on at least one})}$$

$$= \frac{0.4 \times 0.7 \times 0.8}{0.964} = 0.2324$$

Use the context to decide what event is the intersection (both events happen) – in this case at least one success and three successes is the same as three successes.

- 6 Normal distribution – mass of potato is X . $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$

State the distribution of X clearly.

- (i) $P(X > 4) = 15\%$ and $P(X < 2) = 20\%$

The two pieces of information are the basis of a pair of simultaneous equations.

$$P\left(Z > \frac{4 - \mu}{\sigma}\right) = 15\% \text{ and}$$

$$P\left(Z < \frac{2 - \mu}{\sigma}\right) = 20\%$$

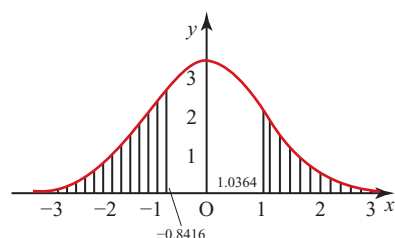


Figure 3

A diagram helps to be clear whether the boundary values are positive or negative. Use inverse normal tables or the statistics function on your calculator to determine the boundary values.

$$\frac{4 - \mu}{\sigma} = 1.0364 \text{ and } \frac{2 - \mu}{\sigma} = -0.8416$$

Solve simultaneous equations

It is enough to use the equation solver on your calculator to solve the equations.

$$\mu + 1.0364\sigma = 4$$

$$\mu - 0.8416\sigma = 2$$

to give $\mu = 2.8963$, $\sigma = 1.0650$
rounded to 3 s.f. gives $\mu = 2.90$, $\sigma = 1.07$

The original data is described as an estimate, so it makes sense to round the answers.

- (ii) $H_0 : \mu = 3.2$ where μ is the population mean yield.

The null hypothesis is always that there is no change - that the yield for the farmer is the same as the research centre.

Hypothesis testing is a formal process. Make sure you define that letters you are using.

$H_1 : \mu < 3.2$ that the yield for the farmer will be less than the research centre.

It is important to choose carefully the alternative hypothesis to reflect what you are asked to test. This is a one-tailed test.

- (iii) **Using critical region**

The significance level is 5% and the test statistic is the sample mean $\bar{X}$.

$$\begin{aligned} \text{Under } H_0: \bar{X} &\sim N\left(\mu, \frac{\sigma^2}{n}\right) \\ &= N\left(3.2, \frac{0.9^2}{100}\right) = N(3.2, 0.0081) \end{aligned}$$

Critical value is

$$3.2 - 1.64485 \times \sqrt{0.0081} = 3.05196$$

Show clearly the value 1.64485 in your working as evidence of using the percentage points of the normal distribution.

$$\text{Actual value is } \frac{306}{100} = 3.06 > 3.05196$$

which is above the critical value, so there is not enough evidence for H_0 to be rejected. There is not enough evidence that the mean has decreased and that the farmer's yield is less than the research centre.

Make a very clear comparison with the critical value.

Learn the script for the conclusions of the hypothesis test. The wording is important.

Using probabilities

Under H_0 the sample mean

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(3.2, \frac{0.9^2}{100}\right)$$

$$\text{Actual value } \frac{360}{100} = 3.06$$

$$P(\bar{X} \leq 3.06) = P\left(Z \leq \frac{3.06 - 3.2}{\frac{0.9}{\sqrt{100}}}\right)$$

For a continuous random variable, there is no difference between < 3.06 and ≤ 3.06 .

$$= P(Z \leq -1.5555) = 0.05991 > 5\%$$

The probability is more than 5% so, so there is not enough evidence for H_0 to be rejected. There is not enough evidence that the mean has decreased and that the farmer's yield is less than the research centre.

Make a very clear comparison of the probability and the significance level.

- (iv) It is assumed that the standard deviation is unchanged from the 0.9 found in the research centre. It is not clear that this assumption is valid as the SD for the farmer's crop may be different.

A value for standard deviation is always assumed. A more sophisticated test is needed if the standard deviation also has to be estimated from the data.

- (v) It is assumed that the trial the farmer uses in the test are typical of all the plants the farmer might grow. If he grows all the trial plants on the best land on the farm, the values for yield will not be representative of the whole farm and the test may not be valid.

There is always a potential problem with any sample that it is not representative of the population.

7 DRV $P(X = r) = \frac{k}{r}$ for $r = 1, 2, 3, 4$

(i) $\sum_{r=1}^4 P(X = r) = \frac{k}{1} + \frac{k}{2} + \frac{k}{3} + \frac{k}{4}$

$= \frac{k(12 + 6 + 4 + 3)}{12} = 1$

Total probability is one.

Use common denominator and take k as a common factor.

$\frac{25k}{12} = 1$

$k = \frac{12}{25}$

- (ii) Vertical line graph

The random variable is discrete so a vertical line graph is the best option.

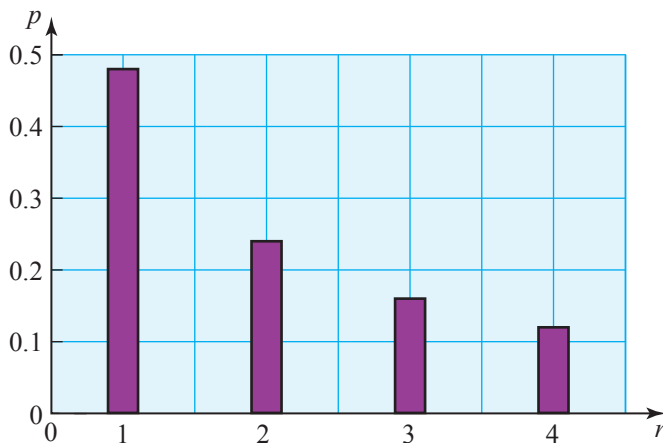


Figure 4

(iii) $P(X_1 = X_2) = P(1, 1) + P(2, 2) + P(3, 3) + P(4, 4)$

List the ways in which equal scores can happen.

$= \left(\frac{12}{25}\right)^2 + \left(\frac{6}{25}\right)^2 + \left(\frac{4}{25}\right)^2 + \left(\frac{3}{25}\right)^2$
 $= \frac{205}{625} = \frac{41}{125}$

As the scores are independent, the probabilities for each value can be multiplied. Add the probabilities for the four different ways an equal score can happen.

$P(X_1 \neq X_2) = 1 - \frac{41}{125} = \frac{84}{125}$

The event that the scores are unequal includes $X_1 > X_2$ and $X_1 < X_2$ which are equally likely. Divide the answer by 2 to get the required probability.

So $P(X_1 < X_2) = \frac{1}{2} \times \frac{84}{125} = \frac{42}{125}$
 $= 0.336$

- 8 (i) Difference in means in wingspan is

$134.1 - 128.6 = 5.5$ mm with standard deviations of 3.28 and 2.96, so difference is approx. 1.7 times the standard deviations.

Make sure you evaluate the quantities under discussion.

The key point is to compare the difference in means with the variability of each population separately.

Difference in means in weight is $111.2 - 108.4 = 2.8$ g with standard deviations of 11.8 and 13.7, so difference is approx. 0.2 times the standard deviation.

Avis is correct as the difference in weights is very little compared with the variability of the weights within the two separate populations, but the difference in wingspan is much more than the variability in each population.

- (ii) Females may have a greater spread of values than the males because some of the females measured may be ready to lay eggs. Weight for males will be more consistent.

Any sensible answer about the context could get the marks here. It is important to think about the situation from which the data arose and not just the figures themselves.

- (iii) The data for wing length will have been

rounded to the nearest mm, so only integer values appear in the data set. There is a relatively small number of values so the scales on the axes make the gaps more noticeable in the x directions.

Rounding of the data as it is collected means that there are only integer values in the data set and the eye perceives the gaps in the data. The horizontal data is also striped, but the scales mean that it is less noticeable in the other direction.

- (iv) The graph is roughly elliptical and suggests that the data is bivariate normal. This means that the hypothesis test for the PMCC is appropriate.

A roughly elliptical shape suggests that the population has both wingspan and weight normally distributed which is a necessary condition for the hypothesis test to be valid when x and y are both random.

- (v) If Avis were doing a hypothesis test for correlation, the critical value would be 0.000001. The actual value is 0.47 so there is very strong evidence that there is correlation between wingspan and weight. There is a very tiny probability that correlation in the sample of 0.47 would be obtained by chance from a population with no correlation.

The value is so small because the data set is so large.

- (vi) It is likely that Avis would find correlation for the male birds also, but it is not certain without doing the test. It is not usually a good idea to combine two samples together as this can lead to spurious results but as the sample sizes are similar and the data had been collected together and separated out later, it would be possible.

Discuss' in the question suggests that you should give arguments both ways.

Practice questions: Mechanics (page 529)

1 Method 1

Weight = $0.5g$

The weight of the medal is mass $\times g$.

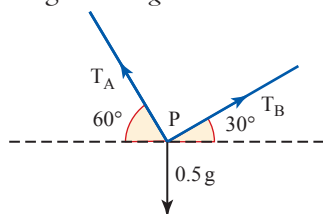


Figure 5

Resolve horizontally and vertically

The force diagram must include all the forces with arrows in the correct direction and labels.

These equations are obtained by using total force in the given direction = 0. This arrangement of the equations is easiest to use if you solve the simultaneous equations directly on the calculator.

$$-T_A \cos 60 + T_B \cos 30 = 0$$

$$T_A \sin 60 + T_B \sin 30 = 0.5g$$

Solve simultaneous equations (pen and paper method)

Calculator method is acceptable if your calculator will solve these. Give your answers to more significant figures and round as evidence of a calculator method.

$$-T_A \cos 60 \sin 60 + T_B \cos 30 \sin 60 = 0$$

$$T_A \sin 60 \cos 60 + T_B \sin 30 \cos 60 = 0.5g \cos 60$$

Add the equations

$$T_B \cos 30 \sin 60 + T_B \sin 30 \cos 60 = 0.5g \cos 60$$

$$T_B = \frac{0.5g \cos 60}{\cos 30 \sin 60 + \sin 30 \cos 60} = 2.45$$

Substitute for T_B

Best to use ANS key on the calculator to avoid rounding issues.

$$T_A = \frac{T_B \cos 30}{\cos 60} = 4.24$$

$$T_A = 4.24 \text{ N}, \quad T_B = 2.45 \text{ N}$$

T_A has no component in the direction of T_B and vice versa.

Method 2 – note the strings are perpendicular

Weight = $0.5g$

Resolve along the strings

$$T_A = 0.5g \cos 30 = 4.24 \text{ N}$$

$$T_B = 0.5g \cos 60 = 2.45 \text{ N}$$

Method 3 Using the triangle of forces

Weight = $0.5g$

Diagram with arrows forming a loop

It is important that the arrows follow each other round the triangle.

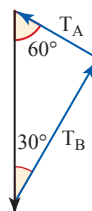


Figure 6

Basic trigonometry can be used in the right-angled triangle.

Third angle in the triangle is 90°

$$T_A = 0.5g \cos 30 = 4.24 \text{ N}$$

$$T_B = 0.5g \cos 60 = 2.45 \text{ N}$$

- 2 (i) Contact force = weight = $72g\text{ N}$

Consider the forces acting on Juan. The contact force acts upwards on him and downwards on the plank.

Mark the distance of the weight of the plank in the middle of the plank as it is uniform.

(ii)

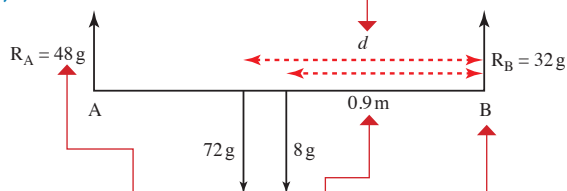


Figure 7

The reaction force is $48g$ as the scales show 48 kg .

Mark the distance of Juan's centre of mass from his feet – this is what the question asks for.

The reaction at B must be included but it does not need to be evaluated if moments are taken about B.

- (iii) Take moments about B

Choose to take moments about the force where the unknown force acts. It is important that every term has a force and a distance or no method marks will be given.

$$48g \times 1.8 = 72g \times d + 8g \times 0.9$$

$$86.4 = 72d + 7.2$$

$$d = \frac{86.4 - 7.2}{72} = 1.1$$

Distance of centre of mass from his feet is 1.1 m .

- 3 (i)

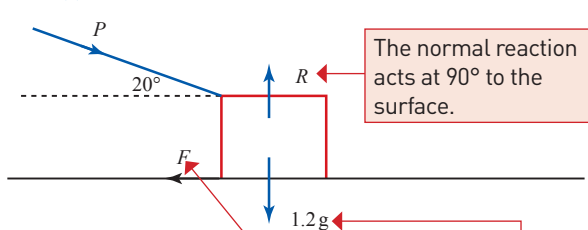


Figure 8

The normal reaction acts at 90° to the surface.

Friction acts along the surface to oppose likely motion.

The weight acts through the centre of mass.

- (ii) Resolve vertically

The component of P in the vertical direction changes the value of R .

$$R = P \sin 20 + 1.2g$$

$$F_{\max} = \mu R = 0.4(P \sin 20 + 1.2g)$$

The box is on the point of sliding, so the maximum value for friction is needed.

On the point of sliding when

The horizontal forces are equal and the friction is maximum.

$$P \cos 20 = F_{\max}$$

$$P \cos 20 = 0.4(P \sin 20 + 1.2g)$$

$$P(\cos 20 - 0.4 \sin 20) = 0.48g$$

$$P = \frac{0.48g}{\cos 20 - 0.4 \sin 20} = 5.86$$

The equation has P on both sides. Collect terms to solve the equation. Your calculator may have an equation mode which could be used to solve this equation.

$$P = 5.86\text{ N}$$

- 4 (i) Stationary when both components of $\mathbf{v} = 0$

$$-0.4t + 8 = 0$$

$$t = 20$$

Choose the simpler of the two components to work with first.

When $t = 20$,

$$-0.03(20)^2 + 0.8(20) - 0.4 = 3.6 \neq 0$$

You could solve the quadratic equation to find when the x component is zero and check if the values coincide with the y component.

So boat is never stationary

Make your conclusion clear.

$$(ii) \quad r = \int \mathbf{v} \, dt = \begin{pmatrix} -0.01t^3 + 0.4t^2 - 0.4t + c_1 \\ -0.2t^2 + 8t + c_2 \end{pmatrix}$$

Integrate velocity to get displacement. Integrate each component separately. There is a constant in each direction.

The boat starts at the origin, so $c_1 = c_2 = 0$

Use the initial position of the boat to find the constants.

$$r = \begin{pmatrix} -0.01t^3 + 0.4t^2 - 0.4t \\ -0.2t^2 + 8t \end{pmatrix}$$

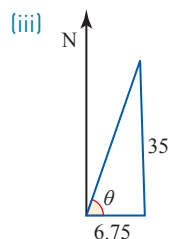


Figure 9

When $t = 5$,

$$\vec{r} = \begin{pmatrix} -0.01 \times 5^3 + 0.4 \times 5^2 - 0.4 \times 5 \\ -0.2 \times 5^2 + 8 \times 5 \end{pmatrix} =$$

$$\begin{pmatrix} 6.75 \\ 35 \end{pmatrix}$$

The bearing is linked to the position vector, so find the position of the boat first.

$$\theta = \arctan\left(\frac{35}{6.75}\right) = 79.1^\circ \text{ Bearing} = 011^\circ$$

The angle in the triangle is not the angle between the North line and the position of the boat.

- (iv) Travelling NE when components of velocity equal (and positive)

$$-0.03t^2 + 0.8t - 0.4 = -0.4t + 8$$

$$-0.03t^2 + 1.2t - 8.4 = 0$$

$$t = \frac{-1.2 \pm \sqrt{1.2^2 - 4 \times (-0.03) \times (-8.4)}}{2 \times (-0.03)}$$

$$t = 30.95, 9.045$$

You can use your calculator equation mode to solve the equation if you have it. Make sure you give both roots and select the one that is useful in this question.

$30.95 > 20$ so time 9.0 s

- 5 (i) In equilibrium for the bucket:

Look at each part of the system separately.

$$T = mg$$

In equilibrium for the block, resolve up the slope

$$T = 1.5g \sin 40$$

Tension is the same throughout the string so $1.5g \sin 40 = mg$

This is because the string is light and the pulley is smooth.

$$m = \frac{1.5g \sin 40}{g} = 0.964 \text{ kg}$$

- (ii) Resolve downwards for the bucket (Newton's second law)

$$1.7g - T_1 = 1.7a$$

The tension will change from its value in the equilibrium situation.

Resolve up the slope for the block

$$T_1 - 1.5g \sin 40 = 1.5a$$

Add equations

Solve the simultaneous equations.

$$1.7g - 1.5g \sin 40 = 3.2a$$

$$a = 2.25$$

Acceleration is 2.25 m s^{-2}

- (iii) To find velocity when the bucket reaches the ground

The problem solving aspect of this question means you have to realise that the velocity of the system when the bucket hits the ground and the new acceleration once the tension is removed must both be found.

$$s = 0.8$$

$$u = 0 \quad v^2 = u^2 + 2as$$

$$v = ? \quad v^2 = 0^2 + 2 \times 2.25 \times 0.8 = 3.6$$

$$a = 2.25 \quad v = 1.90$$

$$t =$$

When the bucket reaches the ground, the string goes slack and the tension is removed. Resolve along the slope.

$$-1.5g \sin 40 = 1.5a$$

$$a = -g \sin 40$$

$$s = ?$$

$$u = 1.90 \quad v^2 = u^2 + 2as$$

$$v = 0 \quad 0 = 1.90^2 - 2 \times (g \sin 40) s$$

$$a = -g \sin 40 \quad s = \frac{1.90^2}{2g \sin 40} = 0.287$$

The initial velocity for this is the final velocity found above.

Block travels another 28.7 cm

- 6 (i) Vertical motion

Begin by finding the time taken to reach ground level again.

$$s = 0 \quad s = ut + \frac{1}{2}at^2$$

$$u = u \sin \alpha \quad 0 = u \sin \alpha t - \frac{1}{2}gt^2$$

$$v = \quad 0 = t\left(u \sin \alpha - \frac{1}{2}gt\right)$$

$$a = -g$$

$$t = ? \quad t = 0, \quad \frac{u \sin \alpha}{\frac{1}{2}g} = \frac{2u \sin \alpha}{g}$$

Resolve the initial velocity into horizontal and vertical components.

Horizontal motion when $t = \frac{2u \sin \alpha}{g}$,

$$R = (u \cos \alpha)t = u \cos \alpha \left(\frac{2u \sin \alpha}{g} \right) \\ = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

Horizontal velocity is constant under the standard modelling assumptions.

(ii) Negligible air resistance

(iii) $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

Use the hint given in the question.

$$R = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 (2 \sin \alpha \cos \alpha)}{g} \\ = \frac{u^2 \sin 2\alpha}{g}$$

$\sin 2\alpha$ takes maximum value of 1 when $2\alpha = 90$

$$R_{\max} = \frac{u^2}{g} \text{ when } \alpha = 45$$

(iv) Using $\alpha = 30$ to find time and range

$$\text{For B2 time } t = \frac{2 \times 10 \sin 30}{g} = 1.02$$

You need to interpret which quantity is required for each cell. Only pen and paper calculations are required, not the spreadsheet formulae.

For C2 range

$$x = \frac{2 \times 10^2 \sin 30 \cos 30}{g} = 8.84$$

(v) Maximum range

$$\frac{u^2}{g} = 7.1$$

$$u = \sqrt{7.1 \times 9.8} = 8.34$$

Speed to 1 d.p. is 8.3 m

Be careful to round your answer to 1 decimal place as required in the question.

(vi) The measurements give the maximum range when the angle is 30° . Changing the value for u will still have the maximum range at 45°

$$(vii) \mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} 10 \cos \alpha - 3t \\ 10 \sin \alpha - 9.8t \end{pmatrix}$$

Speed is the magnitude of the velocity vector, so the displacement vector needs to be differentiated.

$$\text{When } t = 0 \quad \mathbf{v} = \begin{pmatrix} 10 \cos \alpha - 0 \\ 10 \sin \alpha - 0 \end{pmatrix}$$

Initial speed occurs when $t = 0$.

$$\text{Speed} = \sqrt{(10 \cos \alpha)^2 + (10 \sin \alpha)^2} \\ = \sqrt{100(\cos^2 \alpha + \sin^2 \alpha)} = \sqrt{100 \times 1} \\ = 10 \text{ m/s}$$

(viii) To find T in new model y component of $\mathbf{r} = 0$

It would be enough to say that the y component is the same in both models, so the time taken would also be the same.

$$10 \sin \alpha t - 4.9t^2 = 0$$

$T = \frac{2u \sin \alpha}{g}$ which is exactly the same equation as in the standard model in part (ii).

(ix) For E5, we need horizontal component when $t = 1.44$

$$x = 10 \times 1.44 \cos 45 - 1.5 \times 1.44^2 = 7.07$$

There is no need to calculate this time – it is given in cell B5 of the spreadsheet.

(x) The vector model is an improvement. It fits all the measurements better than the standard model and it gives the maximum range (of 7.47) for 40° which is very similar to the measured value (of 7.4 m), which also occurs at 40°

Be clear to justify your answer – a yes/no answer is not enough to get the mark here.