**STEP, MAT, TMUA: Skills for success in University Admissions Tests for Mathematics**

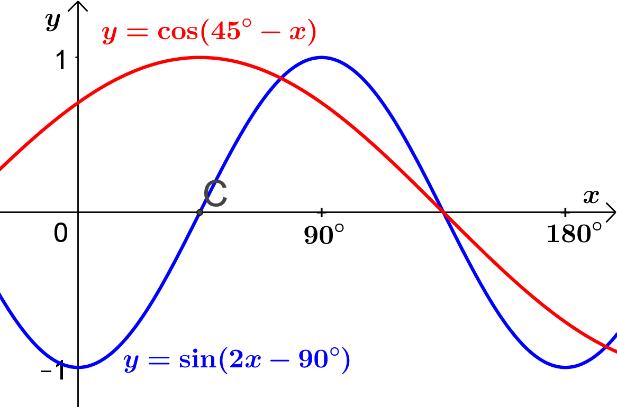
**Full solutions – Part 2**

**Chapter 7: Graph sketching, identification and transformation**

**Two questions to think about (page 80)**

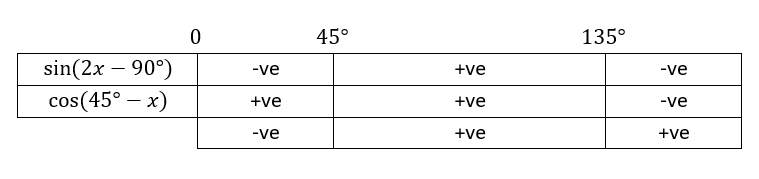
1. Considering where the product is positive or (the will not change where this is).

Sketching the graphs of and is one way of answering this question.



For the given domain is negative for , positive for and negative for . It is for and

For the given domain is positive for and negative for . It is for .



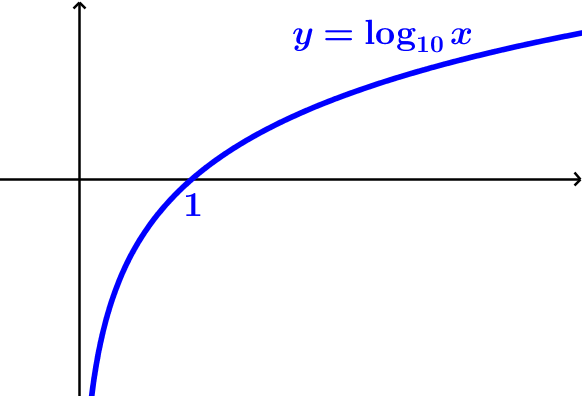
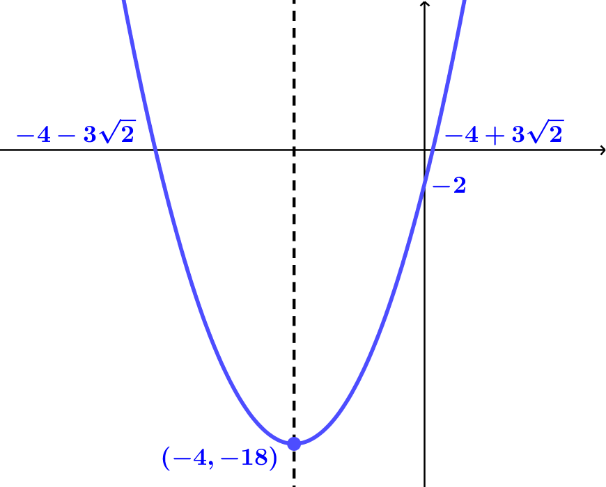
The product is positive (or equal to ) for

This is of the domain so the correct option is E.

1. The graphs of and are sensible starting points.

Completing the square for gives . The sketch will be a parabola with a vertex at . The graph will intercept the axis at .

The graph of will be below the axis for



is undefined for so the graph of will be undefined for

As from the left, (from the positive direction) so and

As from the right, (from the positive direction) so and

has the line of reflection symmetry so the graph of will also have the line of symmetry .

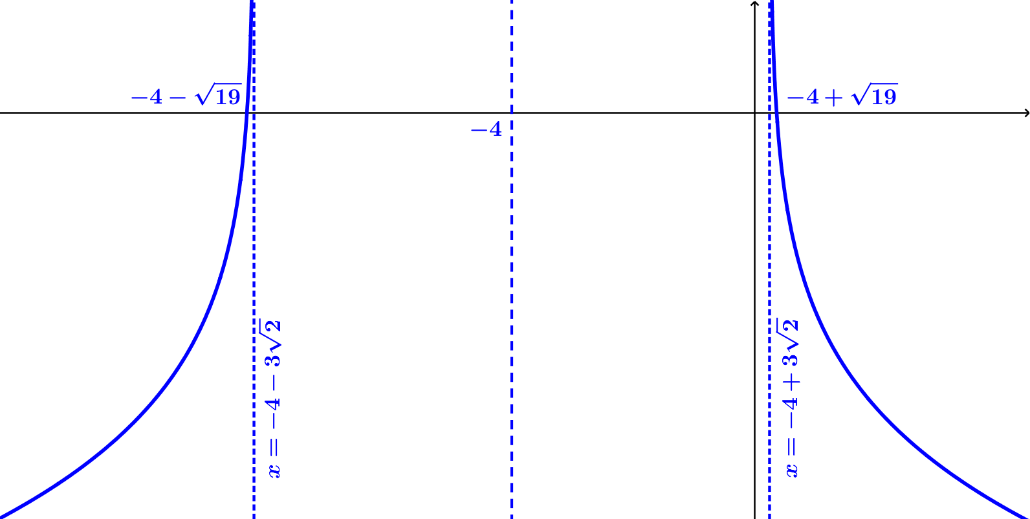
As , so and

As , so and

Since , when , the graph of will intercept the axis.

has solutions

Putting this information together gives the sketch graph



**Try it out (page 82)**

1. The sum and difference of two even functions

If and are even functions, and

Let the function

then

since and are even functions so

The sum of two even functions is an even function

Similarly if

then

The difference between two even functions is an even function

1. The sum and difference of two odd functions

If and are odd functions, and

Let the function

then

since and are even functions

so

The sum of two odd functions is an odd function

Similarly if

then

The difference between two odd functions is an odd function

1. The sum and difference of an even function and an odd function

If is an even function and is an odd function then

and

Let the function

then

since is an even function and is an odd function

The resulting function is neither even nor odd (unless one of the functions is equal to over the given domain)

Similarly if

then

The resulting function is neither even nor odd (unless one of the functions is equal to over the given domain)

1. The product and quotient of two even functions

If and are even functions, and

Let the function

then

The product of two even functions is an even function

Let the function

then since and are even functions so

The quotient of two even functions is an even function

1. The product and quotient of two odd functions

If and are even functions, and

Let the function

then

The product of two odd functions is an even function

Let the function

then since and are odd functions so

The quotient of two odd functions is an even function

1. The product and quotient of an even function and an odd function

If is an even function and is an odd function then and

Let the function

then

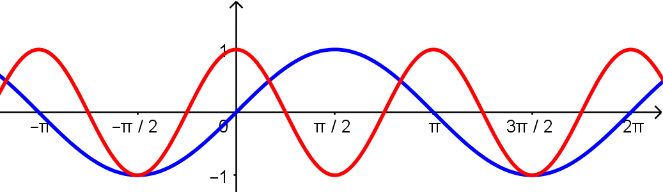
The product of an even and an odd function is an odd function.

Similarly if

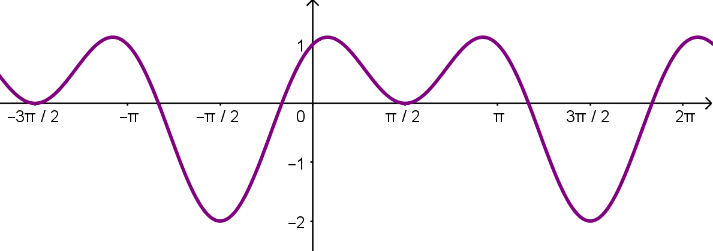
then

The quotient of an even and an odd function is an odd function.

**Try it out (page 82)**



The graph of has a period of radians and the graph of has a period of . Sketching each graph individually on the same axes makes it clear that their sum will repeat every radians so the graph of is periodic with period . The two sketches are simultaneously symmetrical about the lines , , and so on. These are the positive and negative odd number multiples of . In general, any line of the form will be a line of reflection symmetry.



Sketching the sum of the two (by imagining the sum of the values of the previous two sketches) shows that there is no rotation symmetry.

This is an even function since . It will have reflection symmetry in the line . Imagining sketches of the graphs of and should be enough to indicate that the graph has no more symmetry.

Substituting for and for into this will not change it so its graph will have reflection symmetry in the line . If the graph were to be sketched, considering values that result in division by indicates that the line is an asymptote. Rewriting the equation as and considering division by indicates that the lines and are asymptotes. If both and are negative then would be negative so which is not possible for real numbers so the graph does not exist in the third quadrant. This is enough to indicate that all the symmetry has been identified.

The quadratic part of the function, , can be rewritten by completing the square as . This has reflection symmetry in the line so the graph of will also have reflection symmetry in . Imagining the sketches of and should be enough to indicate that there is no further symmetry.

This is an even function since . It has reflection symmetry in . Although is periodic, will not be as does not increase at a uniform rate. Considering where the graph crosses the axis indicates that the “waves” get closer and closer together as increases in both the positive and negative directions.

This is an odd function. Rewriting it as helps to show this. . It has rotation symmetry about . Considering division by indicates that , and are asymptotes. As , the graph approaches from above and as , the graph approaches from below. There is no further symmetry.

**Try it out (page 83)**

Sketch the graph of .

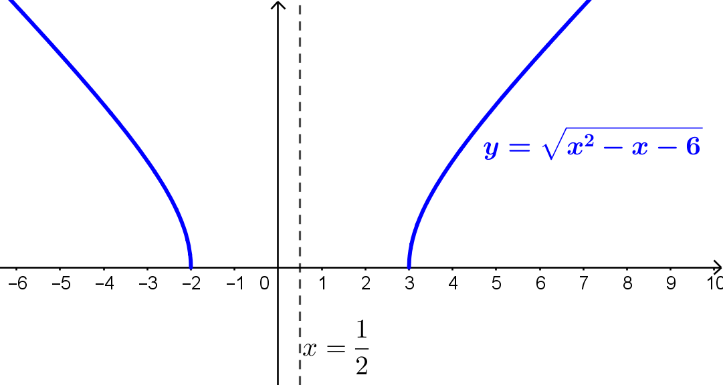
.

for so the graph does not exist between and .

shows that the graph has reflection symmetry in

and both increase as . This gives the part of the graph for . The reflection symmetry can be used to sketch the rest of the graph.

The graph touches the axis when i.e. at and



**Try it out (page 85)**

From the form , for large positive e.g. ,

rewriting the expression as and substituting gives a small positive value. This is greater than indicating that the graph approaches from above for large positive .

For large negative , e.g. , . Using once more gives a small positive value. This is greater than indicating that the graph approaches from above for large negative .

As the graph approaches from the left, i.e. is negative and small:

Consider . Since approaches and will be positive (the product of two negative numbers) this indicates that approaches

As the graph approaches from the right, i.e. is positive and small:

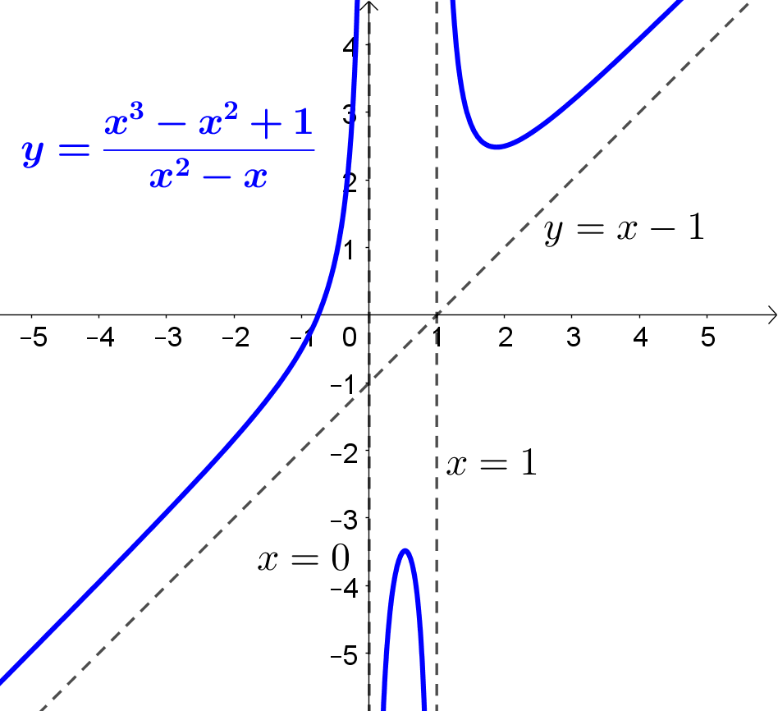
Consider . Since approaches and will be negative ( for ) this indicates that approaches

As the graph approaches from the left, i.e. but is close to

approaches and will be negative ( for ) indicating that approaches

As the graph approaches from the right, i.e. but is close to

approaches and will be positive (and for ) indicating that approaches



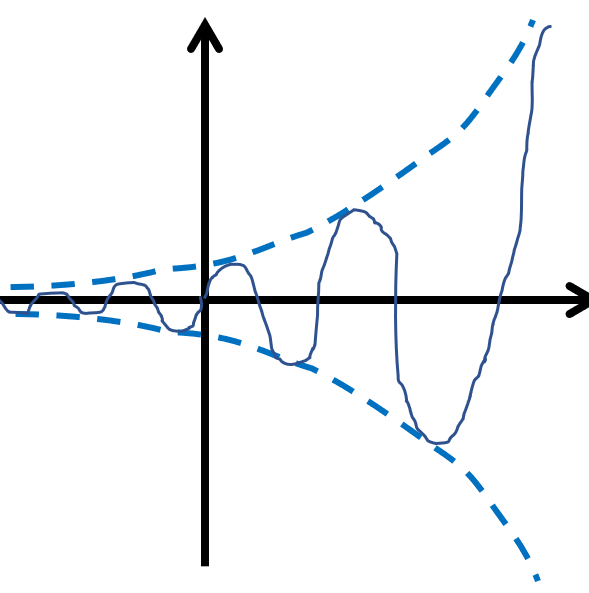
**Try it out (page 86)**

is strictly increasing for and decreasing for so there won’t be any stationary points other than at . The sketch of F shows two stationary points of inflection so it cannot be F, but it could be B.

**Try it out (page 87)**

Consider the graph of where is a real constant. acts as a multiplier stretching the graph of by a scale factor of . Since is constrained between and , will be constrained between and . Using a multiplier of rather than will mean that there is a variable scale factor that increases as and decreases (approaches ) as . will be constrained between and .

This is difficult to sketch with any real degree of accuracy but enough information can be conveyed to give the right idea.

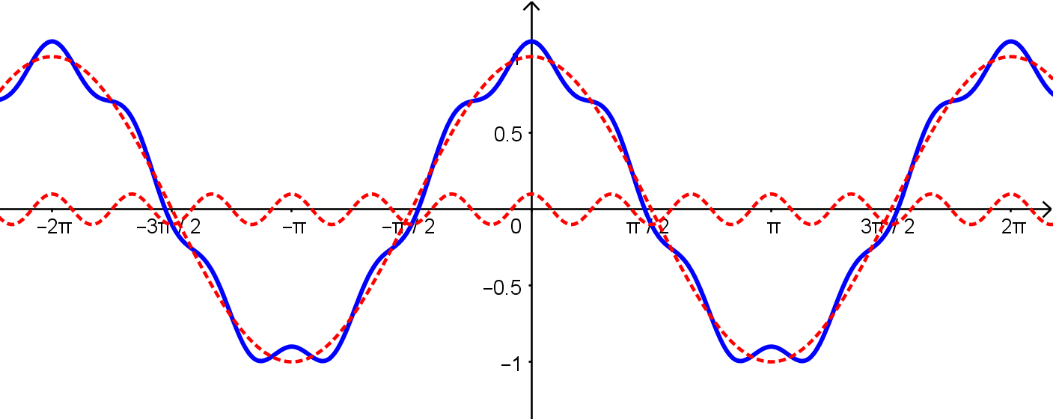


This is another example that is difficult to sketch that accurately but for which the main information can be conveyed.

Carefully sketching the two graphs and and then imagining the sum of their coordinates is a sensible approach to take.

is constrained between and . It has period .

is constrained between and . It has period



**Try it out (page 89)**

Rewriting as and then as indicates that there is a translation in the direction and a stretch in the direction. This eliminates options a), b), c), d) and e) leaving only f) which must be the correct option.

No more work is required but if you wish to confirm this, let .

so the translation occurs before the stretch.

**Try it out (page 91)**

The hint for this question should lead to the equation being rewritten as and, eventually

From the original equation, considering division by indicates that the graph will have vertical asymptotes at and .

As from the left, so and (as you are subtracting a very large negative value from )

As from the right, so and (as you are subtracting a very large positive value from )

As from the left, so and (as you are subtracting a very large positive value from )

As from the right, so and (as you are subtracting a very large negative value from )

From the form , dividing all terms in the fraction by gives

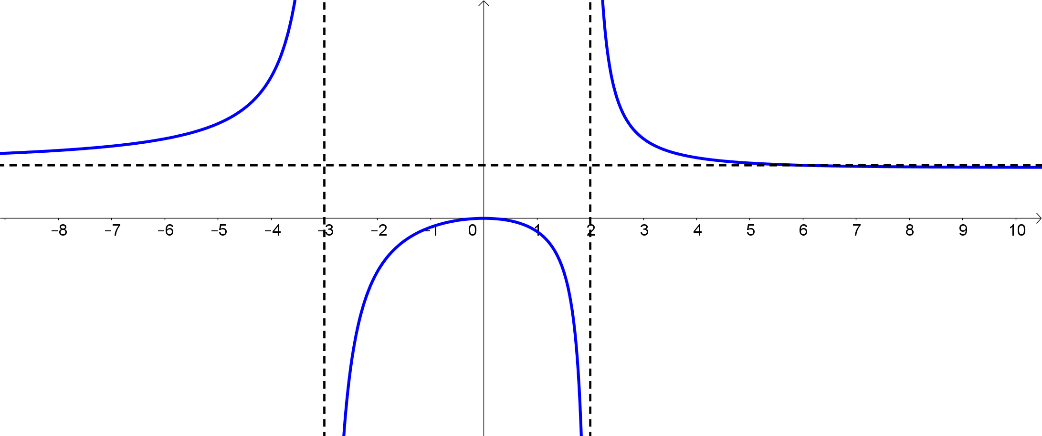
As , and so indicating that is a horizontal asymptote.

As , and so and . The graph will approach from below. It must cross as increases from and then approach it from below.

As , and so and . The graph will approach from above.

When ,

This is enough information to sketch the graph.



**Exercise 1**



so this is neither an odd nor an even function.

has reflection symmetry in the line so has reflection symmetry in

It is not periodic.

so this is an odd function. It has rotation symmetry about

so this is neither an odd nor an even function. Since it has reflection symmetry in the line

so this is an even function. It has reflection symmetry in the line .

is periodic with period radians and so is periodic with period radians.

has reflection symmetry in lines of the form where so also has reflection symmetry in , .

has rotation symmetry about points of the form where so will also have rotation symmetry about , .

resembles but does not reach a maximum of and a minimum of since and and and

so this is an even function. It has reflection symmetry in the line .

is periodic with period radians but so is periodic with period radians (the negative values for will return the same as their equivalent positive values when calculating ).

has reflection symmetry in lines of the form where . Since , has reflection symmetry in , .

The maximum value of will occur when . This is . The minimum value will occur when . This is this is equal to to 2 d.p.

The graph will have rotation symmetry about a point on the curve half way between the maximum and the minimum points. These will be at points ,. Note that the is approximation, the value is actually to 2 d.p.

so this is an odd function and has rotation symmetry about

Swapping and does not change this so the graph has reflection symmetry in the line

Neither odd nor even.

possible symmetry about ,

Translate to make this easy to identify

which is an odd function.

has rotation symmetry about so has rotation symmetry about

Even function. It has reflection symmetry in and no other symmetry

Since is periodic with period , will be periodic with period

Odd function. It has rotation symmetry about and every point where since for ,

It has reflection symmetry in ,

1. Need to show

As is an even function,

hence

As ,

The differential of an even function is an odd function.

To show that the differential of an odd function is an even function:

Need to show that if then

As is an odd function,

Hence

As ,

The differential of an odd function is an even function..



Exists for



Exists for in

Coefficients of : so

Exists for

Exists for

Exists for

1. 1. as ,

as the curve approaches from below. As the curve approaches from above.

This can be confirmed by calculating that the graph crosses the axis once at

There is no vertical asymptote

1. is a vertical asymptote

as from the left (, . As from the right

( .

oblique asymptote

test using large(ish) .

and to compare, multiply both by

so approaches from above

and to compare, multiply both by

- so approaches from below

as from the left (, . As from the right

( .

as from the left (, . As from the right (

.

. As ,

as the curve approaches from above. As the curve approaches from below.

as from the left (, . As from the right (

.

as from the left (, . As from the right (

.

. As ,

as the curve approaches from below. As the curve approaches from above.

as from the left (, . As from the right (

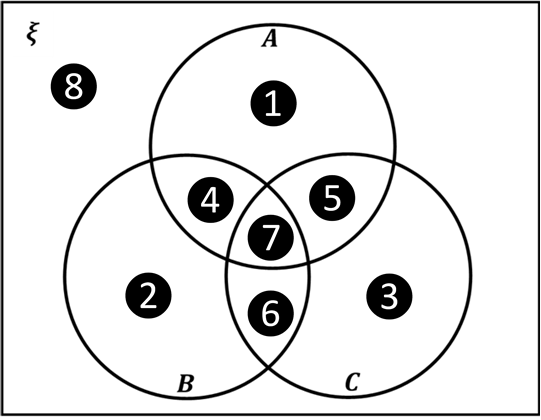
.

as from the left (, .

As from the right ( .

as the curve approaches from above. As the curve approaches from below.





The following answers show one possibility for each region. There are several answers for some regions. You may wish to check your results using a graph plotting app.

* 1. This is not possible
  2. This is not possible

For where is a polynomial of degree (numerator) and is a polynomial of degree (denominator) then

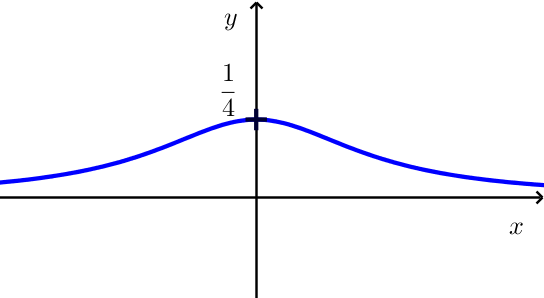
* If there will be no horizontal asymptote
* If the axis will be a horizontal asymptote
* If there will be a horizontal asymptote
* If there will be an oblique asymptote

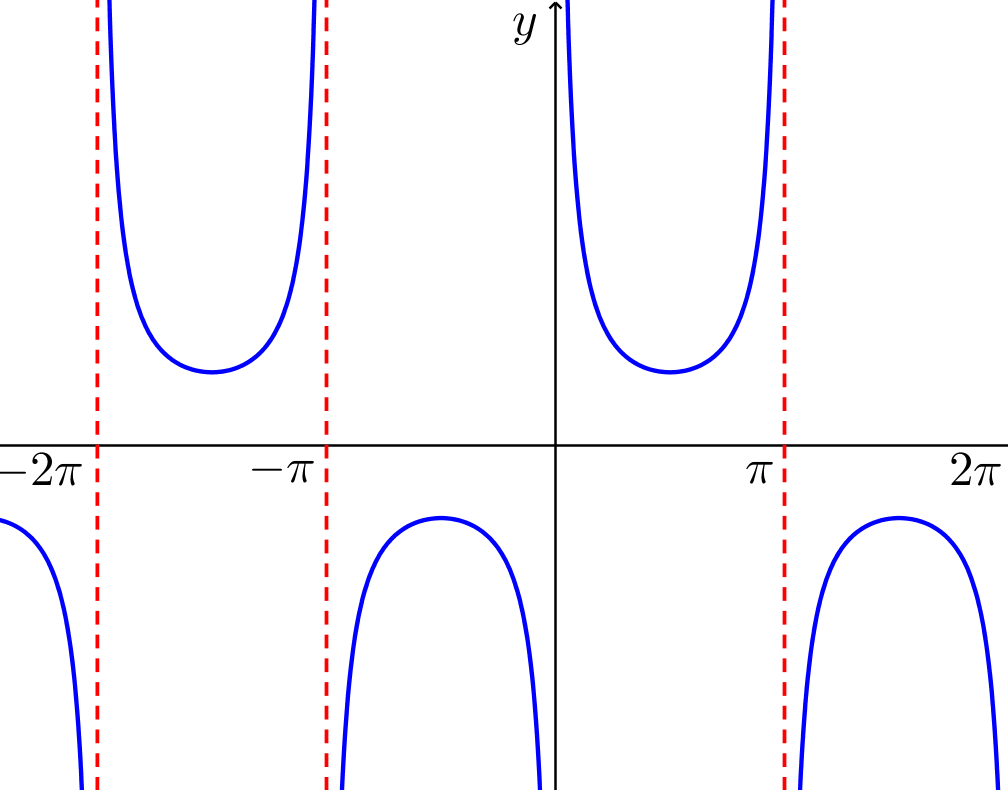
For region 6, the graph would need to have horizontal and oblique asymptotes but not vertical asymptotes. The condition for oblique asymptotes is that and the condition for horizontal asymptotes is . These two things cannot both be true at the same time.

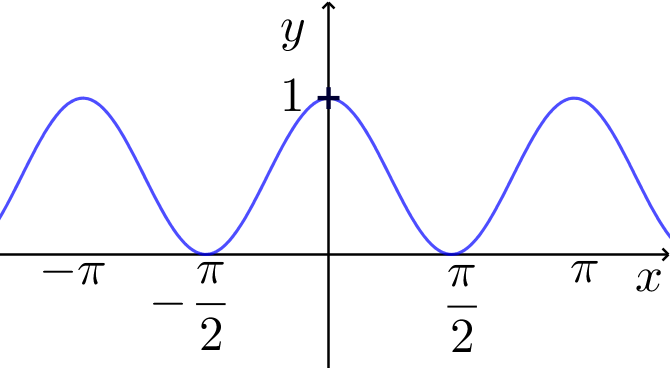
For region 7, the graph would need all three types of asymptote. As region 7 is entirely contained in region 6 it is not possible to find graph for the same reason.

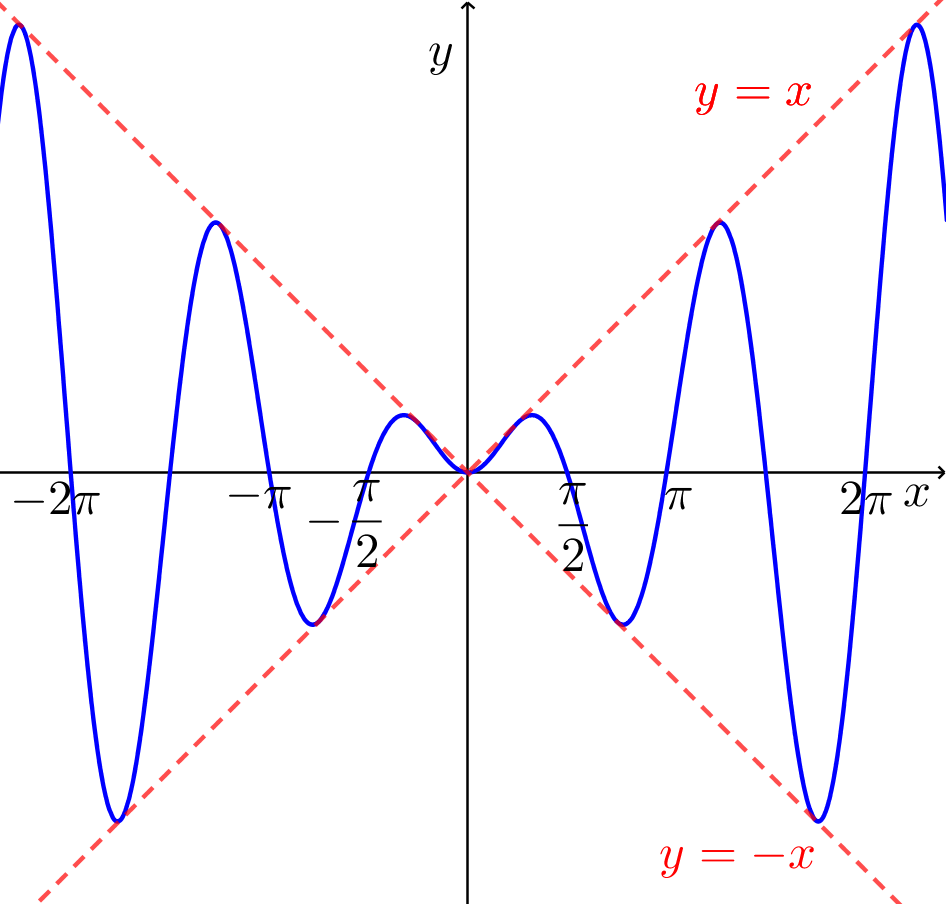
For each region in the Venn diagram below, find an example of a graph of a rational function, where and are both polynomials in .

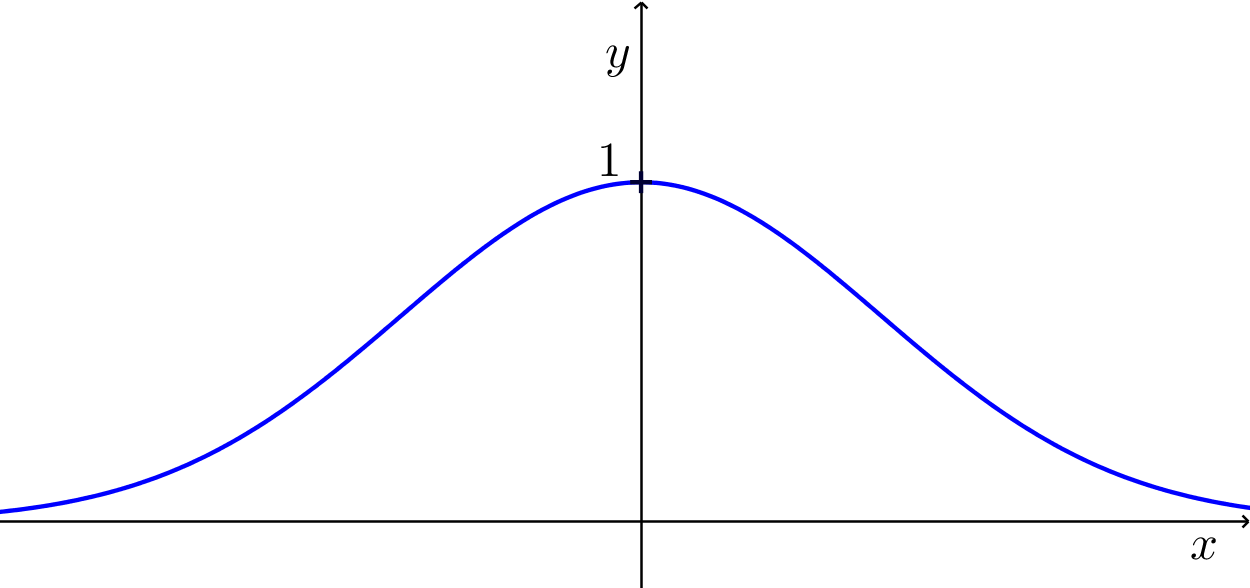












2. Either Reflect in the axis and then translate

or Translate then reflect in the line

1. Either Reflect in the line and then translate

or Translate then reflect in the line

1. Either Stretch parallel to the axis with scale factor 2 followed by a translation by

or Translate then stretch parallel to the axis with scale factor

1. Either Translate then stretch parallel to the axis with scale factor

or Stretch parallel to the axis with scale factor followed by a translation by

1. Either Translate then stretch parallel to the axis with scale factor

or Stretch parallel to the axis with scale factor then translate

Finding the correct order in which to apply a series of transformations can often cause problems for students. Sometimes swapping the order in which two transformations are applied can have a different effect, sometimes the effect is exactly the same. It is far better to imagine the algebraic consequences of applying any particular transformation and use that to guide your thinking.



so at the turning point

so at the turning point

The turning point is at

It is a local maximum as the coefficient of in is negative.

Equating coefficients of : A

Equating coefficients of : B

Equating the constants:

From A:

In B:

giving

giving

and at the stationary points

Using the general shape of a cubic polynomial curve, both x values give turning points.

is a local maximum and is a local minimum

This is an even function indicating one of the turning points will be at i.e. . Considering the shape of the quartic curve, this will be a local maximum.

will have roots where is a value to be found. Each of these roots will come from a repeated factor

Equating coefficients of : so

Equating the constants: so in each case

The other two turning points are and . These are both local minima.

Using the discriminant

For repeated roots

which is not the case as from the initial equation

The turning point is at

By considering the graph shape, this is a local maximum.



or

The graph does not exist for

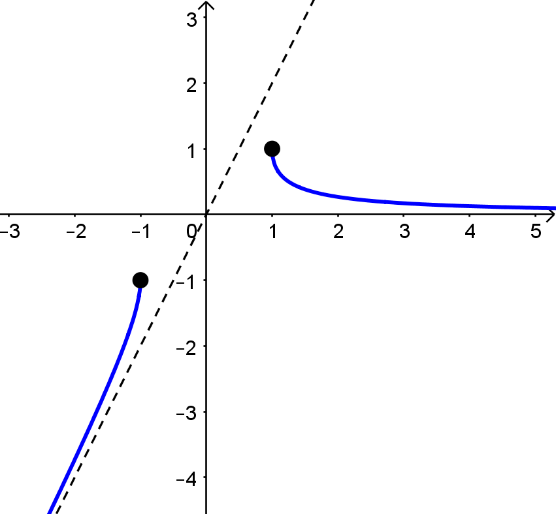
, (since )

, (since )

The graph crosses neither axis.

,

,



Note: although the factor cancels, it is important to realise that for the original function. The coordinate is undefined at

Vertical asymptote

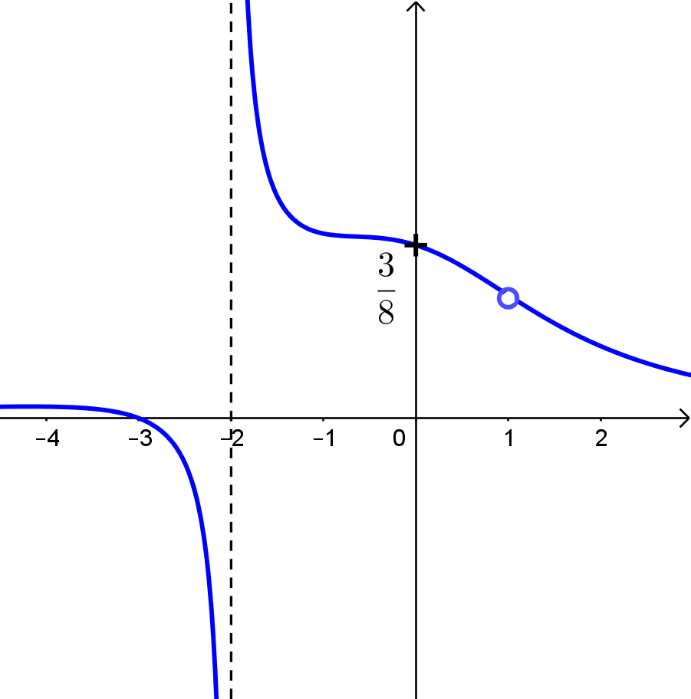
As , ,

As , ,

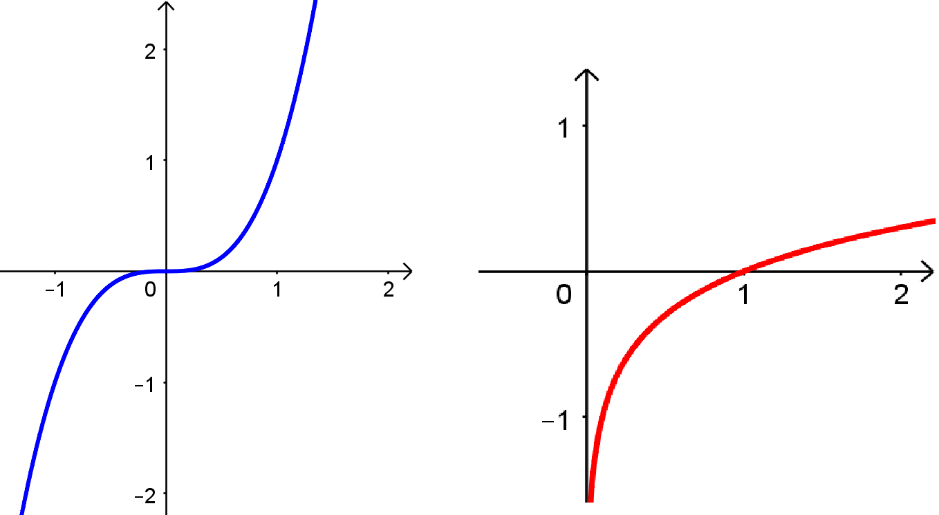
. As , from above in both cases

Intercepts: (or but is undefined for this)

Using calculus to find the turning points gives a factor for which there are no easy factors. There is a local maximum to the left of , but as this is so difficult to find it is not expected.

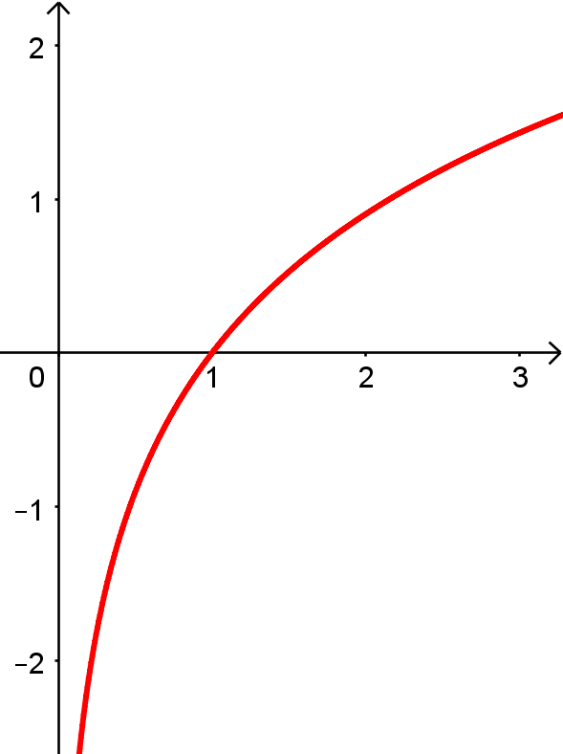


This can be sketched using the graphs of and



so that exists

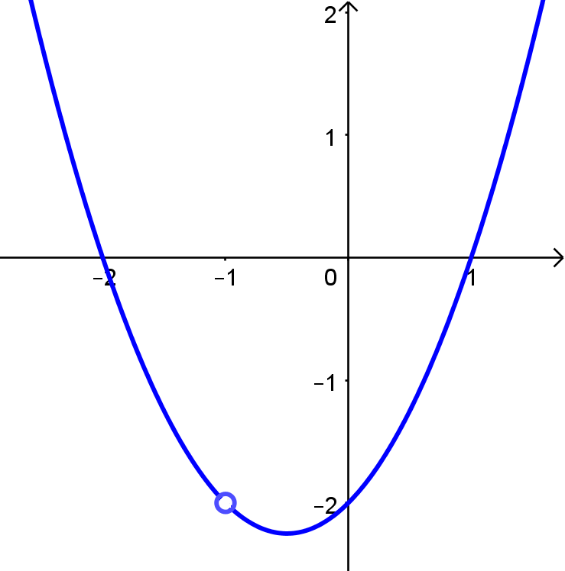
This is a stretch of parallel to the axis with scale factor



is a factor of (using the factor theorem with

Equating coefficients of :

The graph is undefined at



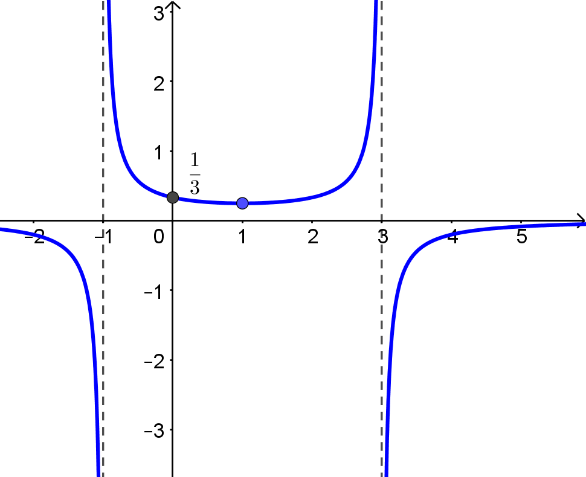
Vertical asymptotes ,

from below in both cases

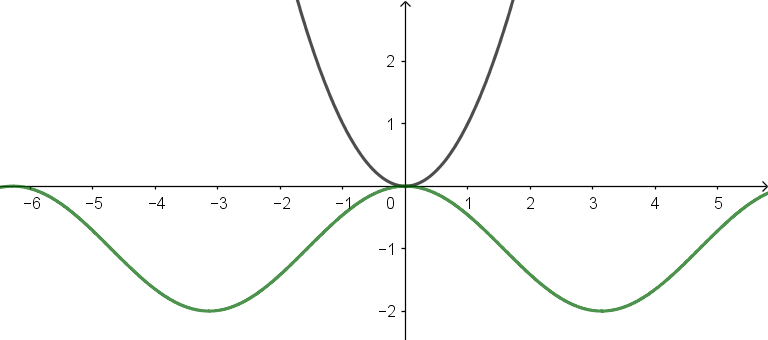
,

(from the original function)

, is a local minimum (from the graph shape)

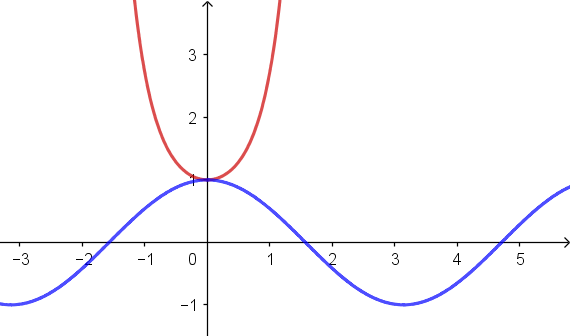


1. Sketching the graphs of and shows that there is only one real solution, .



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.

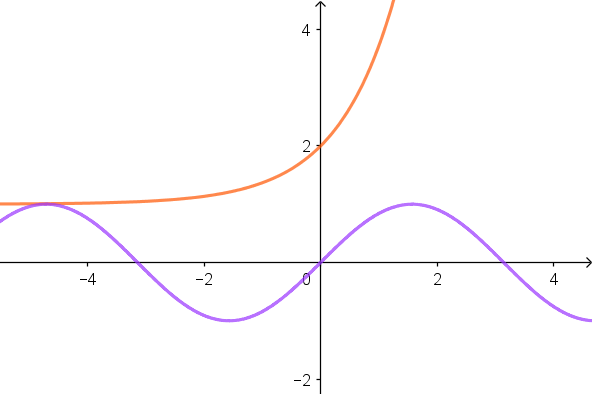


1. How many real solutions does the equation have?

As , from above.

The maximum value of so there are no values of for which

There are no real solutions for

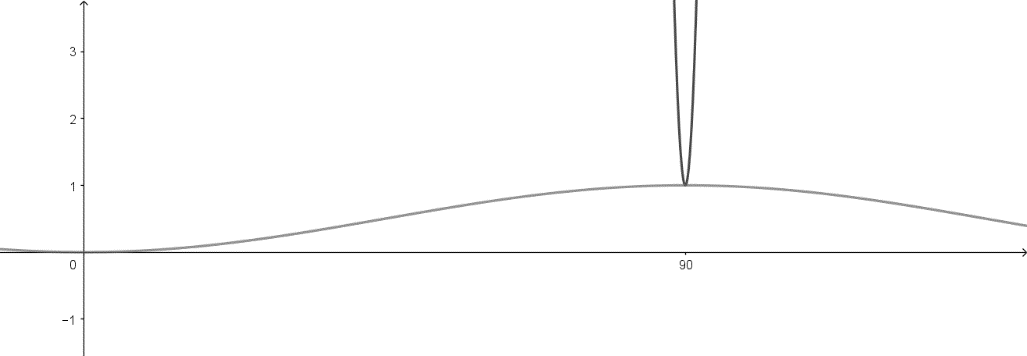


1. Solve the equation .

This is a parabola with a vertex at

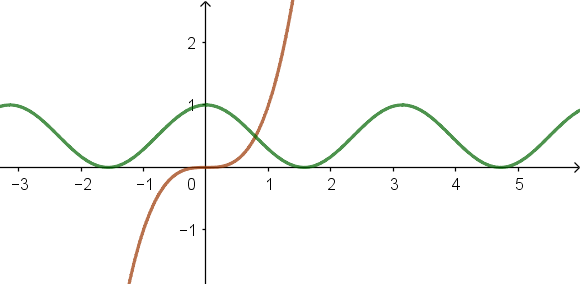
The maximum value of is and this occurs at where

There is only one solution at



1. How many real solutions does the equation have?

Sketching the graphs of and (carefully) shows that here is only one real solution.



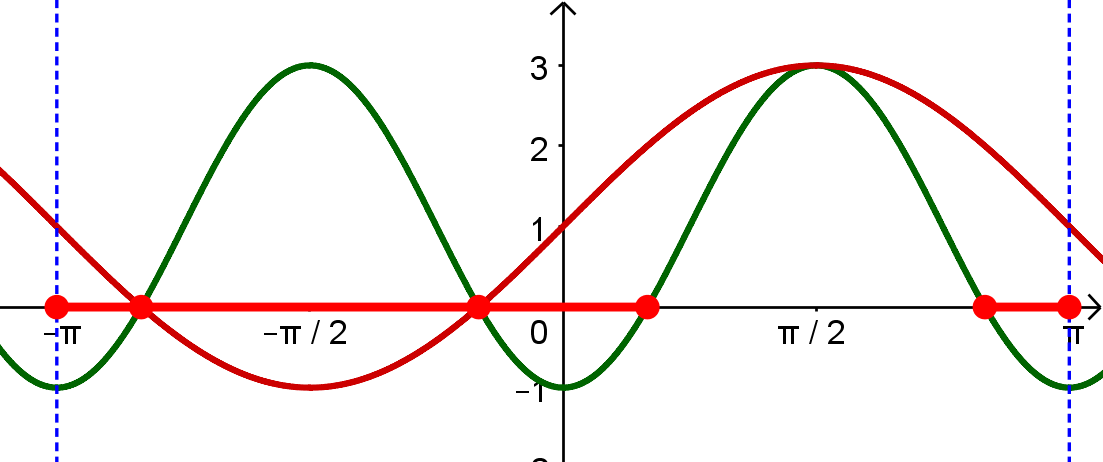
**Exercise 2 Admissions Test Multiple Choice Questions**

**TMUA style questions**

1. Using completed square form:

:

The correct answer is F



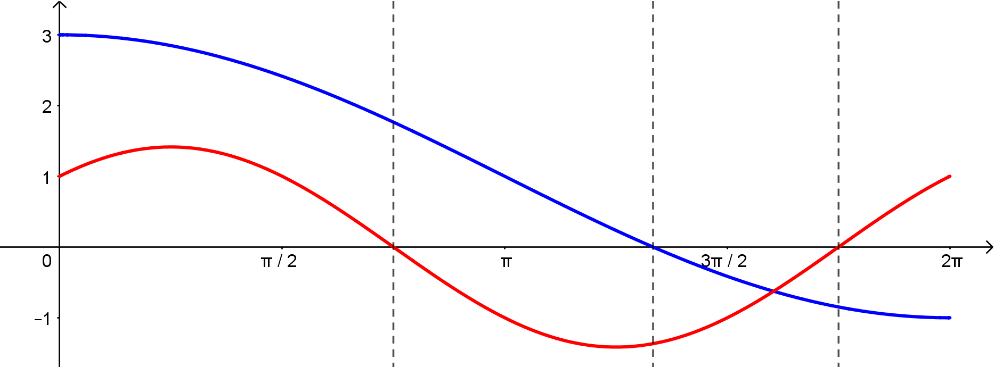
The product of and will be negative or 0 for all of , positive for and negative again for

So for and

The correct answer is B.

1. The graph of can be sketched directly

For , this can be rewritten using

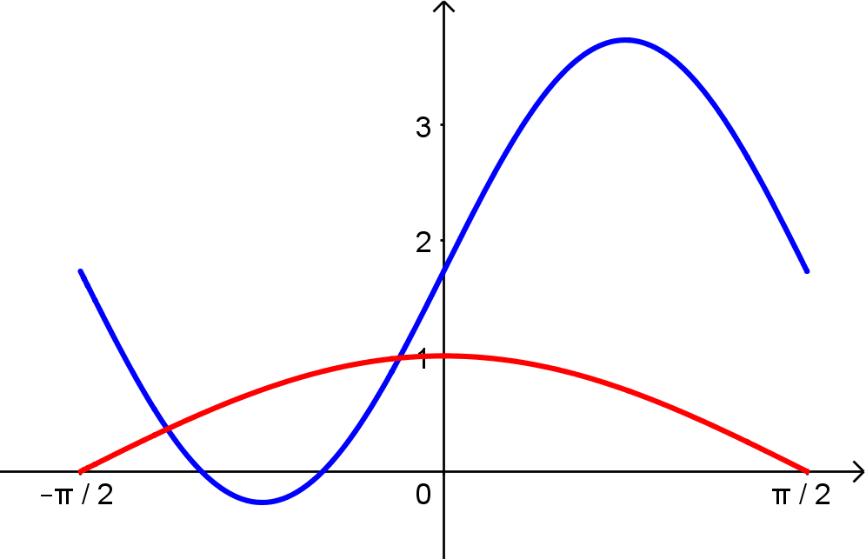


,

Fraction

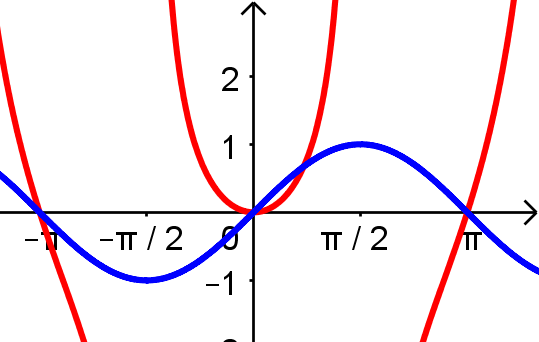
The correct answer is F

1. for



The correct answer is C.

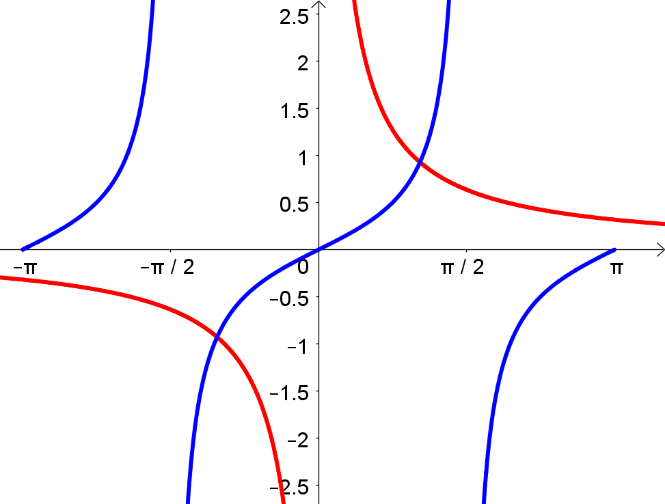
1. Sketching and gives



There are 4 real solutions (being careful to include

The correct answer is E.





Two real solutions.

The correct answer is C



,

, which is

, which is

There are only two turning points so there must be three places that the graph of

crosses the axis.

There are three real solutions.

The correct answer is C.

have?

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A |  | B |  | C |  |
|  |  |  |  |  |  |
| D |  | E |  |  |  |



,

,

,

From the shape of the graph, the points are in order , ,

The line (the x axis has to lie between and for there to be four distinct real solutions.

and so

The correct answer is C.

**MAT style questions**

,

,

The correct answer is (b)

1. A counter-example is often a powerful way of eliminating options in a multiple choice question.

For this question, a counter example can eliminate all of the options apart from (e)

Ideally a value for for which the quadratic expression factorises easily and for which can be found and this can be used as the counter-example for (a) to (d)

Either A

or B

where and are positive integers.

For A: and

If , and

If , and

Neither gives a value for that can be used to eliminate several options at once.

For B:

If , and

So, provides a suitable test for eliminating options (a) to (d).

(Note: most of the work behind selecting as a suitable counter-example can be done mentally)

, . This is

, . This is .

Both turning points are above the axis so there is only one real solution.

(a) is not true as the real number does not give three real roots

(b) since is in , this option must be incorrect

(c) since is in , this option must be incorrect

(d) since is in , this option must be incorrect

This leaves option (e) which is not disproved by

1. This question does not require an accurate sketch, just a general idea of the shape of each curve and the points of intersection for each possible pair of curves.

Let curve A be , curve B be and curve C be

Intersection points:

A and B:

Intersect at and

A and C:

Intersect at and

B and C:

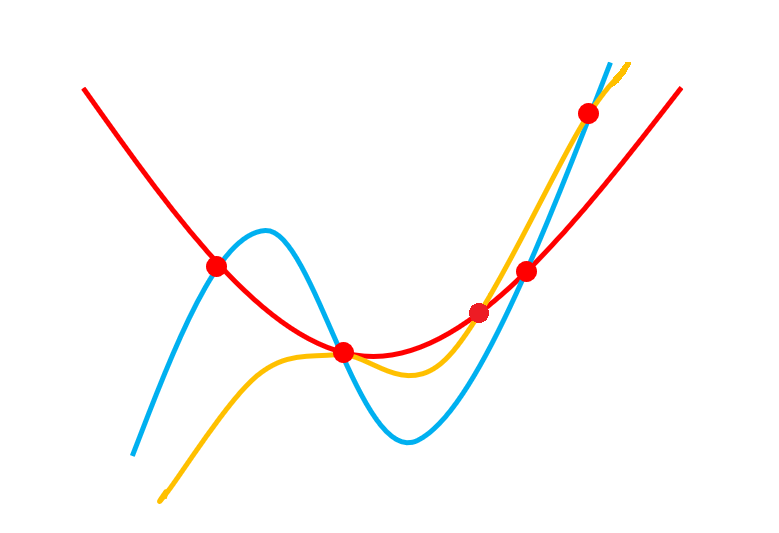
Intersect when , and

There is no need to actually calculate the coordinate for each of these, just to know where it lies in relation to the other points.

Since and i.e. this point is above but below

and i.e. this point is above but below

A rough plot of these points and a sketch of the curves following their general shape allows the number of regions to be counted easily.



There are 10 distinct regions.

The correct answer is (b)

1. is an even function eliminating option (b) immediately

When , . is undefined but as , so the graph should look like it will pass through (even though is undefined when ). This eliminates (c)

When , so the graph will pass through , , and so on since squaring gives

as and since

The graph should cross the axis three times between and (and these points will be reflected in the axis.

This reduces the options to (a) and (d).

The differences between (a) and (d) is that the local maximum for has a greater value for (a) than it does for (b) and that the first intercept for (a) is after for (a) and before for (d).

The easier test is the second of these two. Since , identifying (d) as the correct answer.

Using the factor theorem, is a factor of

Equating coefficients of :

The graph has vertical asymptotes at , and

The only graphs that possibly have these asymptotes are (a) and (e)

At , so the correct option is (e)

This is an even function so (b), (d) and € can be eliminated

or

The graph exists for and

As ,

The correct answer is (a)

1. Testing the point gives so (a) and (d) can be eliminated.

Testing gives . This eliminates (c) as no lines on (c) pass through

Testing gives so € can be eliminated.

The correct answer is (b)

**Chapter 8: Sequences and series**

**Try it out (page 100)**

and so on. In general This is a finite geometric progression with terms. By the usual formula .

**Try it out (page 101)**

By the identity if ,

**Try it out (page 101)**

* 1. and for all positive integers and so for all positive integers
  2. if is even and if is odd. Therefore the sequence could be written as for all positive integers .
  3. and so on. In general This is a finite geometric progression with terms. By the usual formula .
  4. The term the sum of halves. Therefore .
  5. , the first summation is a geometric progression, the second is an arithmetic progression and the final sum is the sum of twos
  6. In sigma notation, the sum is . This can be written as .

**Try it out (page 102)**

for all positive integers is an example. Then and for all positive integers and so the sequence is both increasing and decreasing.

A strictly increasing and strictly decreasing sequence would require that both and that

(and the same for any pair of consecutive terms). This is clearly not possible.

**Try it out (page 102)**

1. Assuming , , and so on. This sequence has period 2 unless . This implies that . So the two values are and .
2. The sequence is  This is periodic with period 6 for all values of 

**Try it out (page 103)**

1. . Therefore .
2. . Therefore  Since the sequence  tends to 0 as *n* tends to infinity so does the sequence .

**Exercise 1**

1. This is an arithmetic progression with first term 1 and common difference 2.

Using the usual formula

.

1. Listing the terms gives 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6, 8. Each positive integer appears twice with the exception of 1 and 2 both of which only appear once.  
     
   If *n* is a positive integer greater than 2 then *n* equals  and also  and so .The terms in the sequence are . Then





So the sequence is periodic with period 5.

1. The terms in the sequence are

.

(Note that these terms are only defined if .)

.

So the value of *k* for which the sequence is periodic with period 4 are  and .

1. .



and so on.

So let’s prove that  by induction.

The result is true when . Suppose the result is true for some integer . In other words, suppose that . Then



So the result is then true when . Hence, by induction, the result is true for all integers .

1. If the arithmetic sequence is then the third term is and the tenth term is . The difference between them is . Therefore

 and .

Taking the third term and subtracting  gives the first term, . Therefore 

The sixteenth term is therefore .

1. General law is .

When  both the left hand side and the right hand side are 1. So the result is true when . Suppose the result is true for some integer . In other words, suppose that .

If  is odd then



If  is even then



Therefore, whether  is odd or even, the statement is true when . Hence, by induction, the result is true for all integers .

1. General law is .

When  both the left hand side and the right hand side are . So the result is true when . Suppose the result is true for some integer . In other words, suppose that .

Then.

So the statement is true when . Hence, by induction, the result is true for all integers .

1. , , ,

Noticing that , a sensible conjecture is that . This is true when  by a direct check.

Suppose the result is true for some integer . In other words suppose that .  
  
Then .

So the result is true when . Hence, by induction, the result is true for all integers 

1. If  is odd then .

If  is even then .

This gives that for any , if then 

Taking in the above gives that for any , .



If then  This gives the result.

**Exercise 2**

**TMUA style questions**

1. The correct answer is B.



Noting that the powers of 10 given are in a geometric series:



So .

1. Using the formula for given in part 1, since  which tends to as ,  tends to  as .

So the answer is E.

1. Let  be the first term of the geometric sequence. The sum of the first 8 terms is . The sum of the first four terms is . So  giving  or  Since  this means that  and the answer is D.
2. The sum of the first 100 terms is . The sum of the first 10 terms is . Therefore  and . Therefore  and  Therefore the answer is F.
3. Using the formula for the sum of a geometric progression the sum to infinity of A is . The other geometric progression has common ratio  and so has sum .  
     
   The information given in the question implies that . This leads to the equation.  can be seen to satisfy this. The answer is A.

**MAT style questions**

1. It can be calculated that the sequence has terms 3, 12, 24, 12, 3, 1.5, 3, 12, 24, 12, 3, 1.5,… So the sequence has period 6. The remainder when 2020 is divided by 6 is 4. This means that .

The answer is d).

1. The first term is 2.

The product of the first two terms is .

The product of the first three terms is .

The product of the first four terms is .

The product of the first 20 terms will be .

Using the formula  this is . The answer is b).

1. By direct calculation the first terms in the sequence are 1, 2, 2, 3, 3, 3,….

For a positive integer ,



Since  there are positive integers  for which .

This means that the first 45 terms of the sequence consists of one 1, two 2s, three 3s,… and nine 9s. The sum of these is

 and the answer is a).







…





From the above it follows that 

So the answer is d).

This can be split as the sum of two summations, these are:

 and .

Using the formula for the sum of a geometric progression in both cases gives

 and .

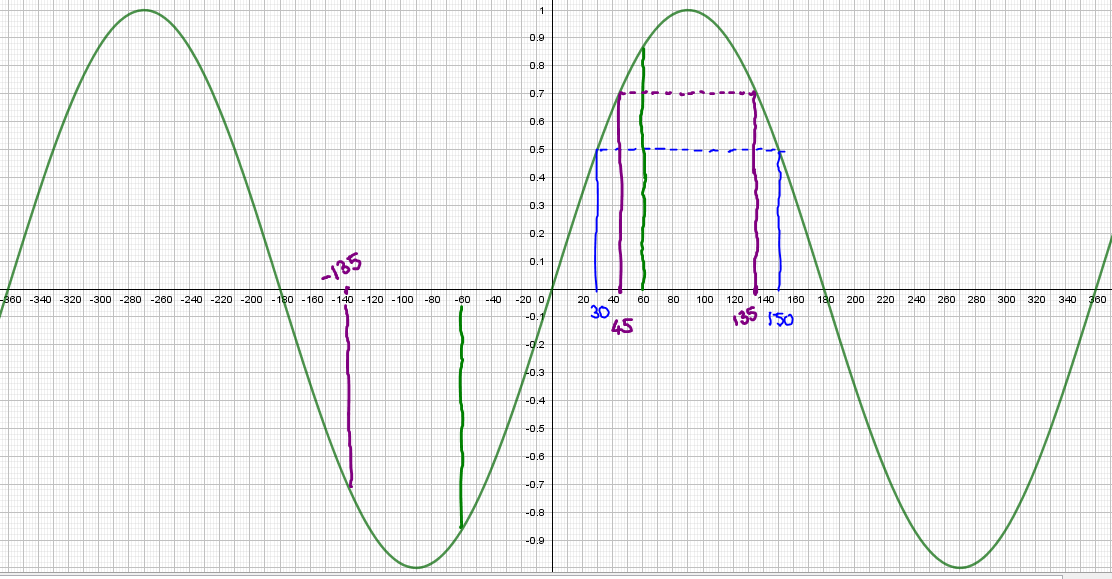
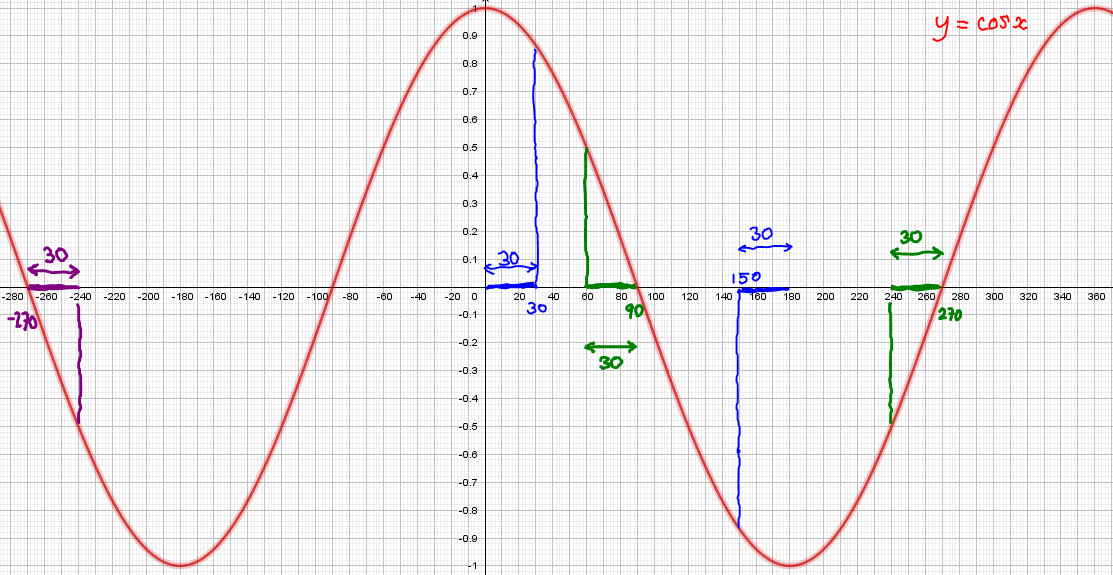
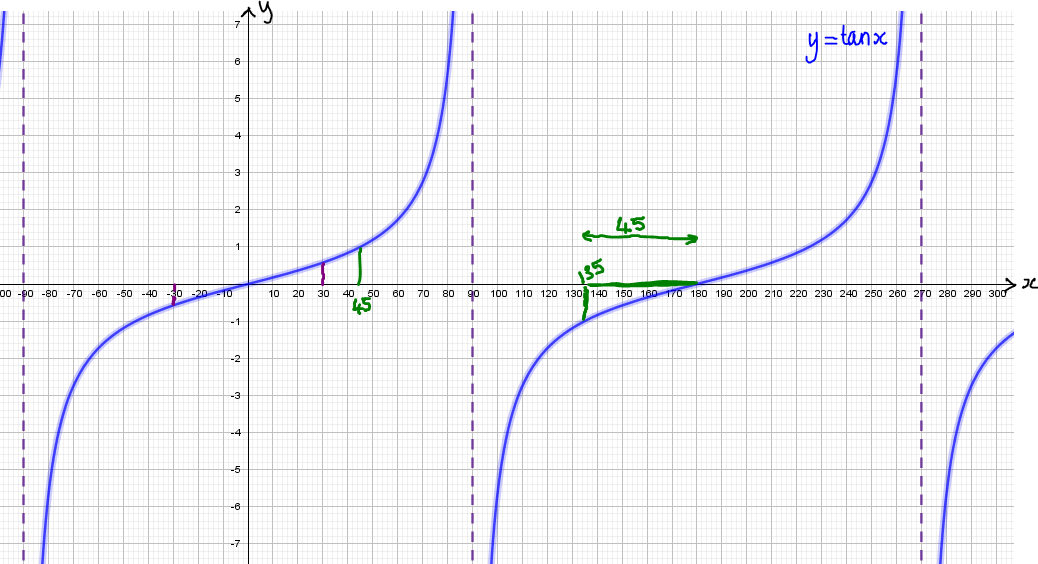
Adding these together gives .

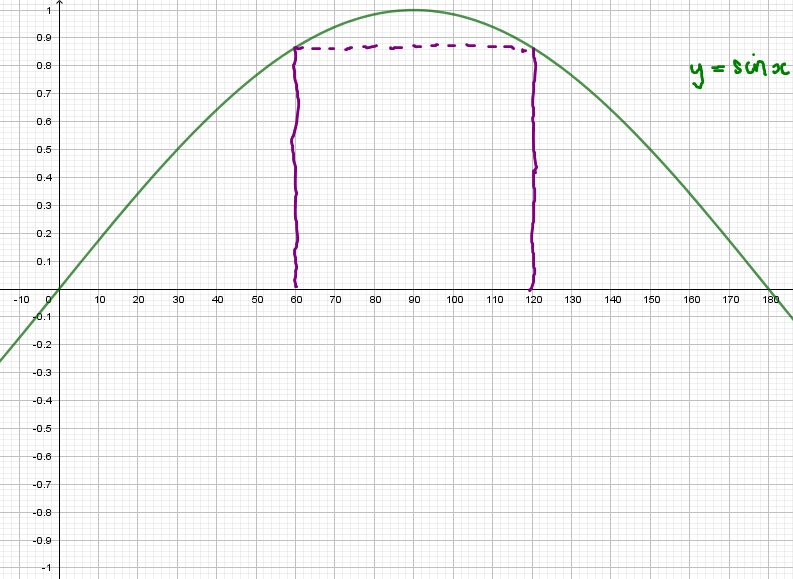
So the answer is d.

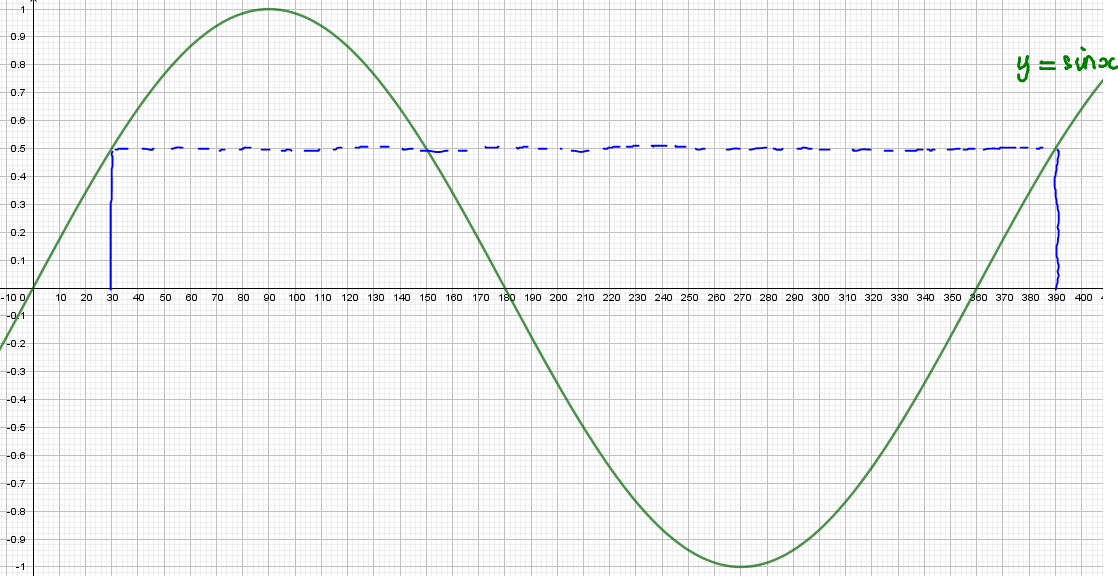
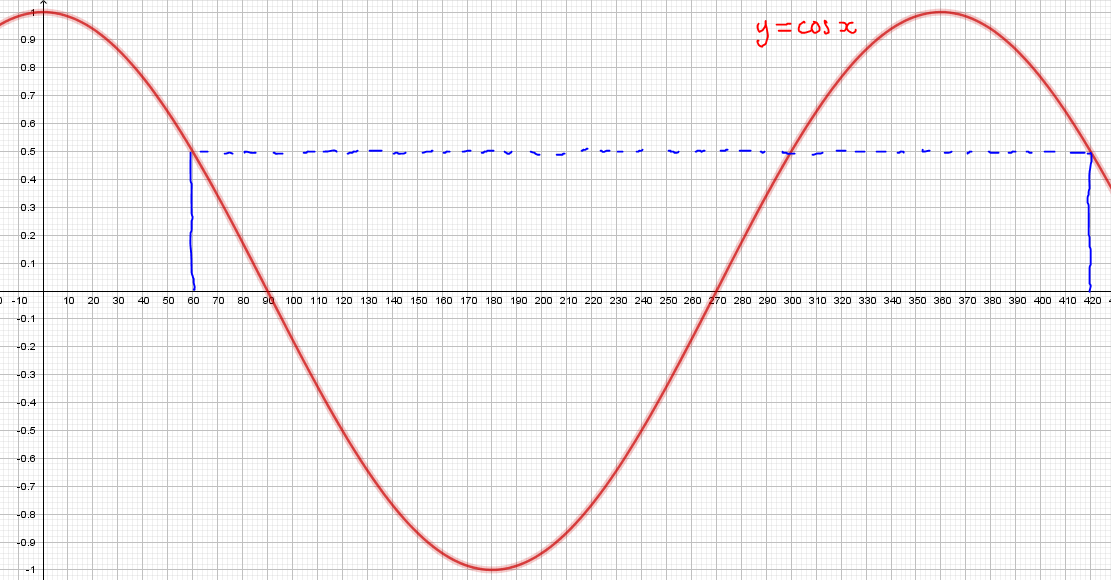
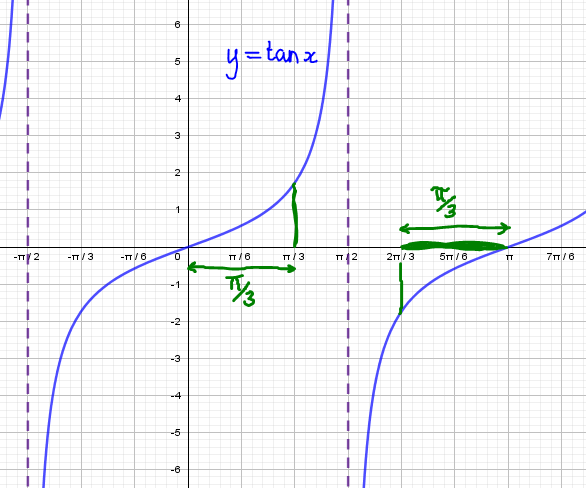
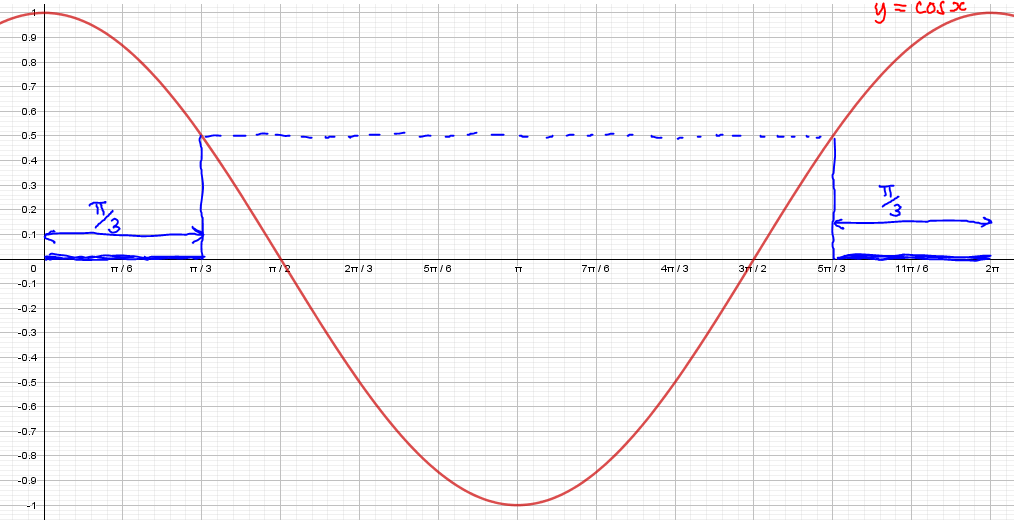
**Chapter 9 Trigonometry**

**Try it out (page 110)**

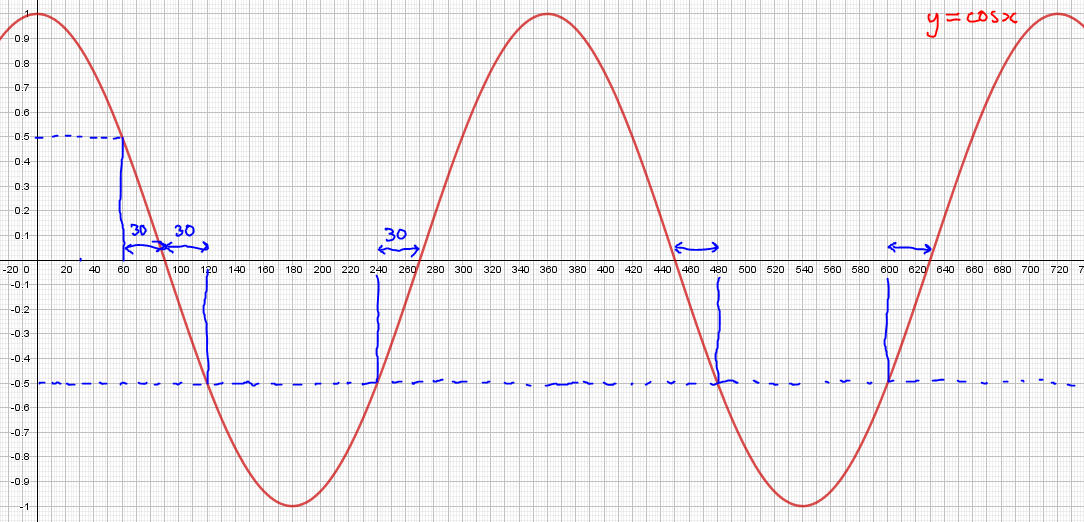
Sketching the graphs of , and . Use your sketches to work out the exact values of the following (do not use a calculator):

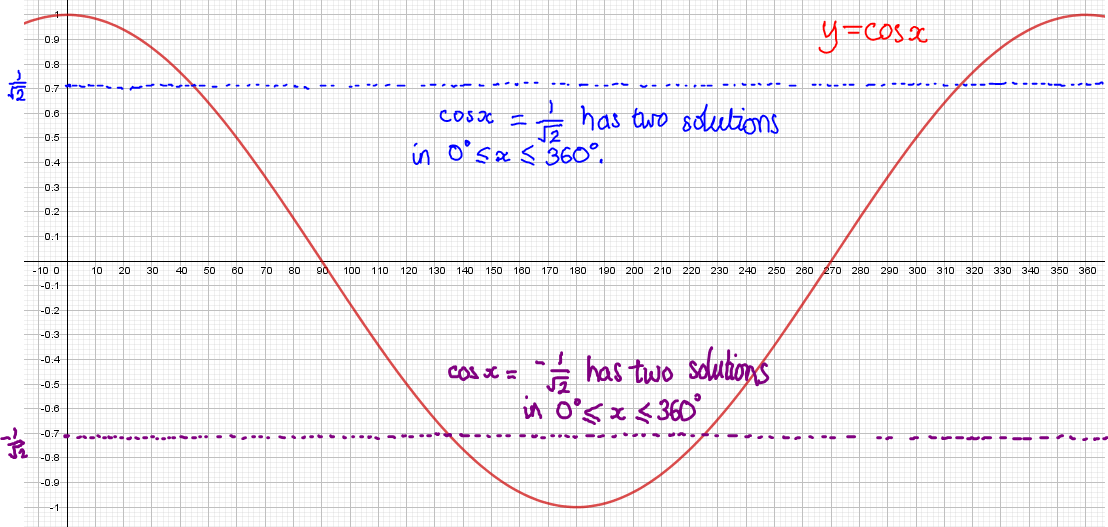
1. **, ,**   
     
     
     
   .  
   .  
   .
2. **, ,**  
     
     
   .  
   .  
   . You could also use the fact that for all .
3. **,**  
     
     
   .  
   .

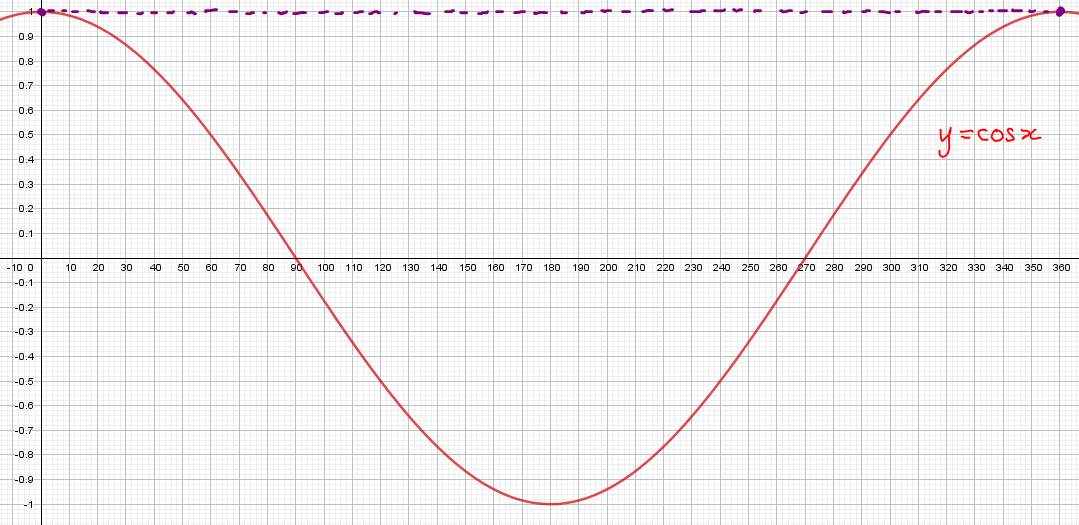
  
  
, so .  
, so .  
  
.

1.   
     
   .  
     
     
     
   .  
     
   So .
2.   
     
   .
3.   
     
   **.**
4.   
     
   .

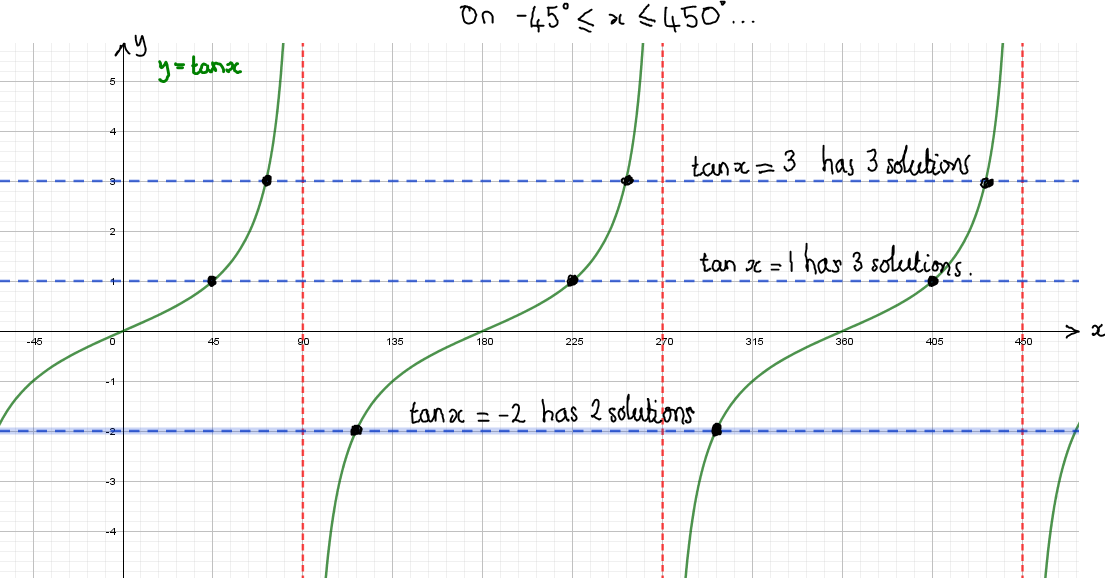
**Try it out (page 110)**

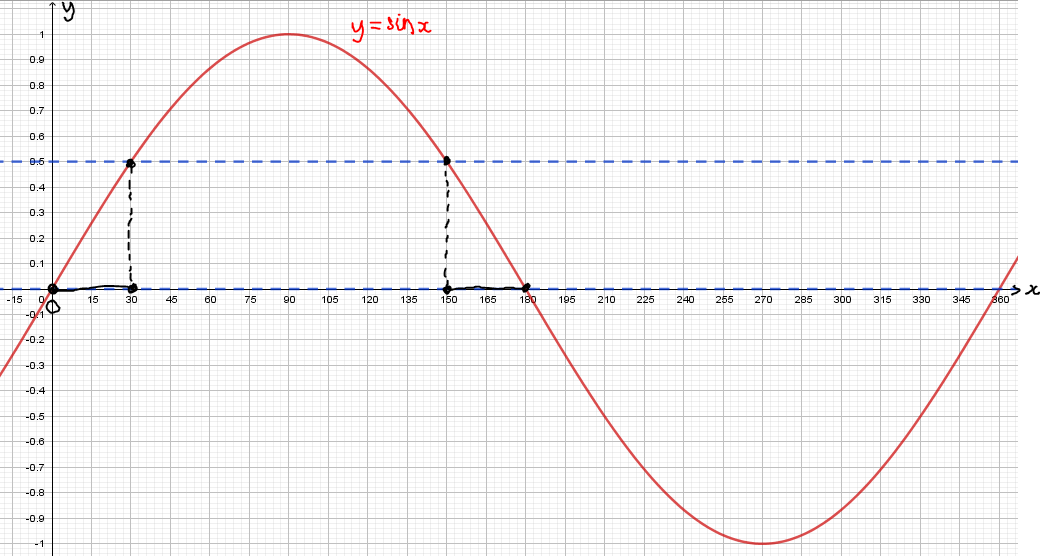
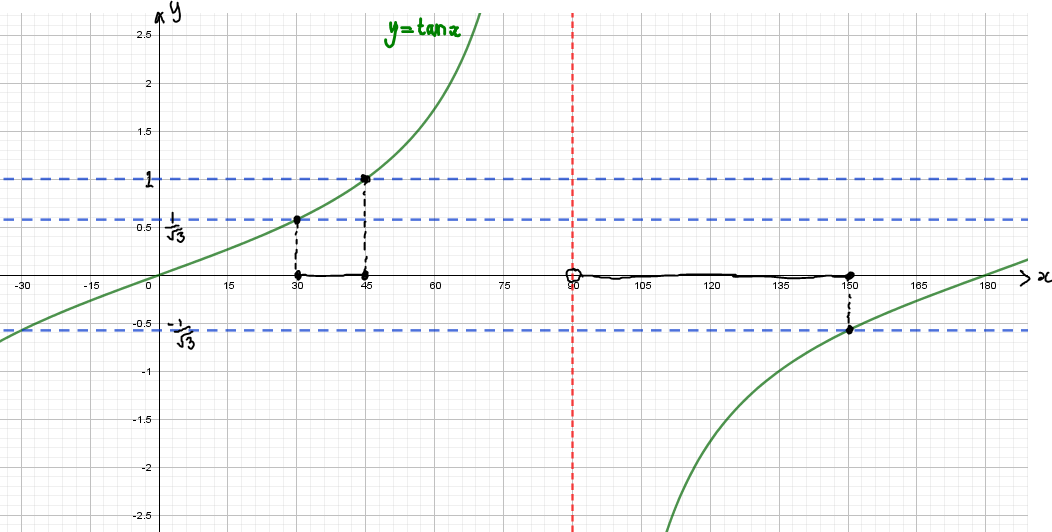
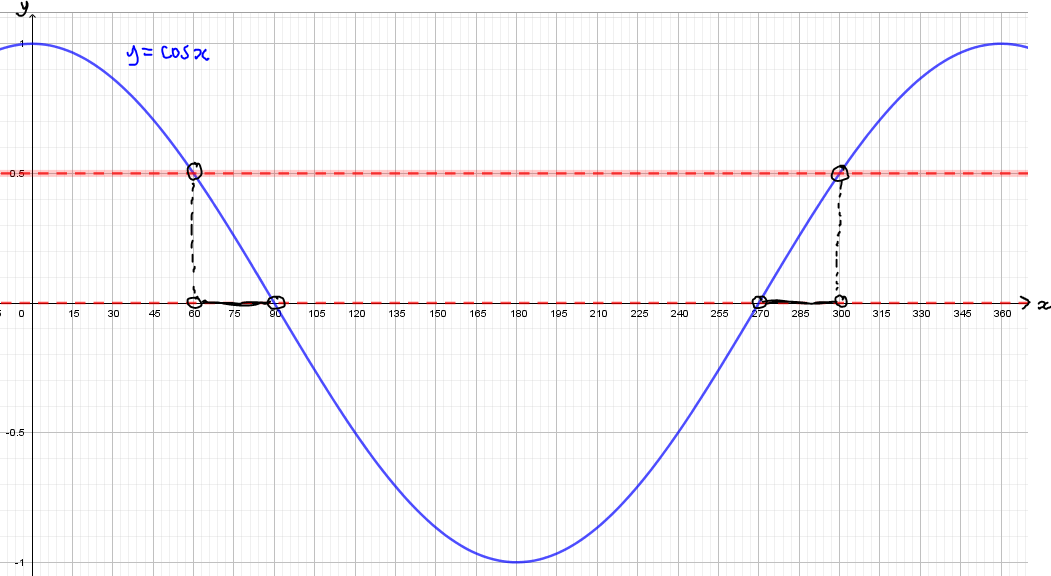
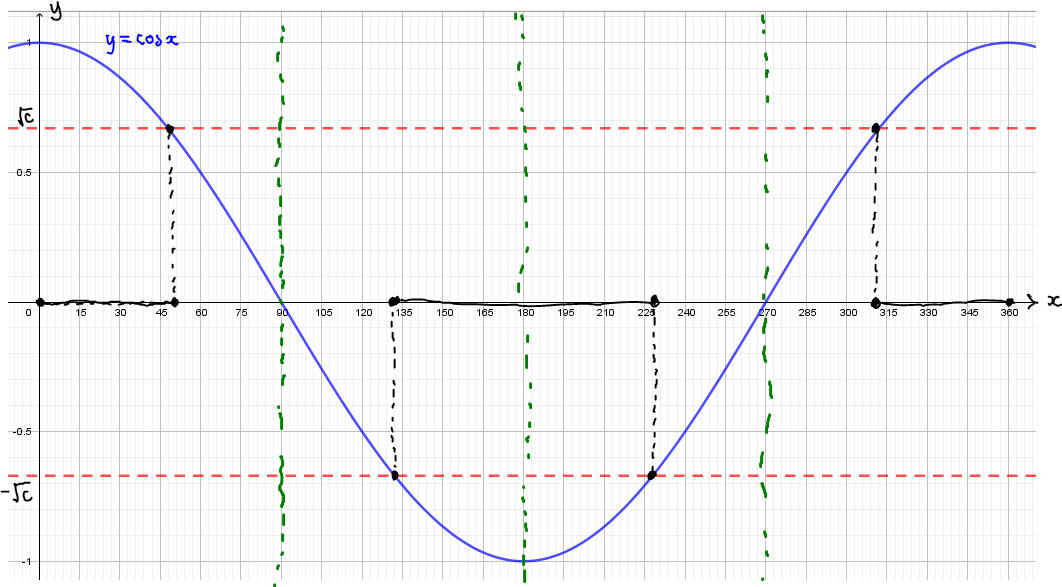
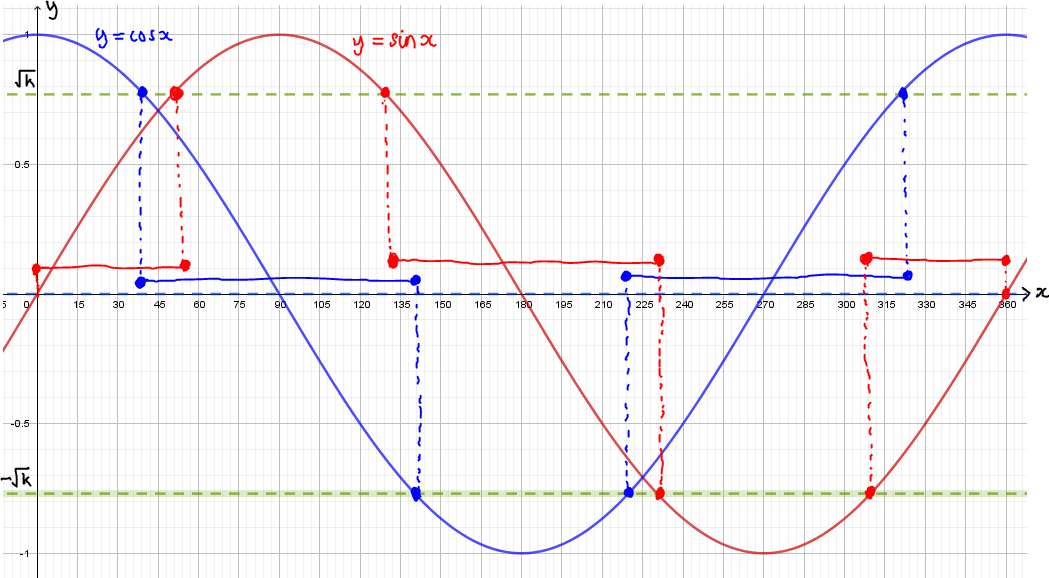
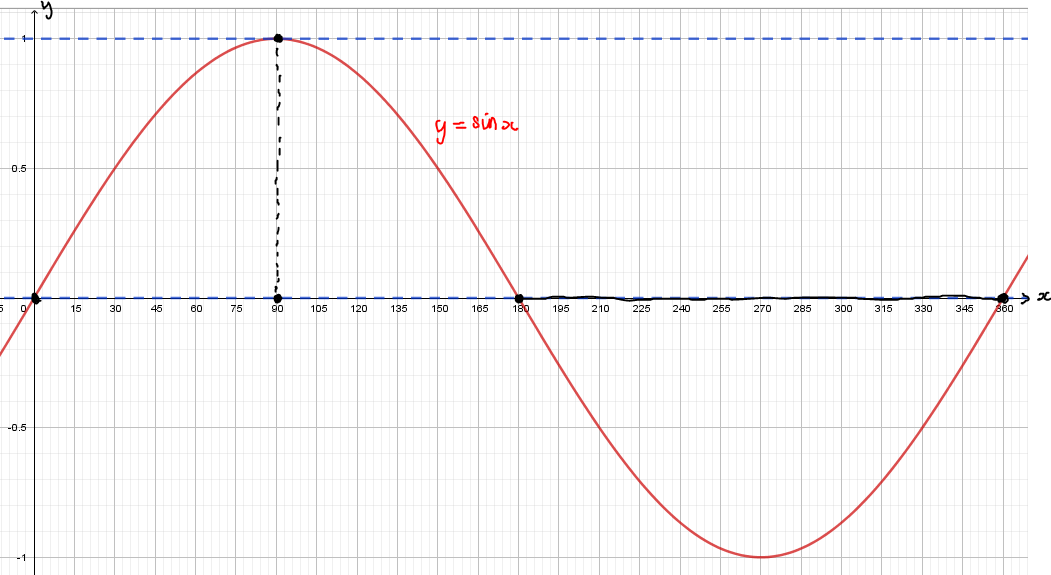
1.   
     
   For , or .  
   All the question asked for was how many solutions the equation has in the given range. The answer is two solutions.
2. , .  
     
     
   For , 2, , or .  
   So, for , , , or .  
   All the question asked for was how many solutions the equation has in the given range. The answer is four solutions.
3. for all values of . In particular, the equation has no solutions for any interval of values of .

1. .  
     
     
     
   As the diagram above shows, has two solutions in the interval , and so does the equation . Therefore, the equation has four solutions in the interval .

1. or .  
     
   Since , has no solutions.  
   As the diagram below shows, has two solutions in the interval (yet no solutions in !).  
     
     
     
   Therefore, the equation has two solutions in the interval .
2. .  
   Thinking of this as a quadratic equation in , and realising that the discriminant of this equation, ‘’, is negative, it follows that there are no real value solutions for the value of . Therefore, the equation also has no solutions.
3. , it follows that . Since , it follows that has no solutions.

**Exercise 1**

1. In order to compare values, using the identity , either express them all as the sine of an angle, or express them all as the cosine of an angle. Since is an increasing function on , it is easier to choose this option.  
     
    and .  
     
   .
2. , .  
   . (That is because is an increasing function on .)  
     
   It follows that is the largest value. The answer is b).
3. , since .  
   So or .  
   Since and , has two solutions and has no solutions in the interval . There are two solutions in total.
4. Notice that if then the left-hand side of the equation equals. By the Factor Theorem, this means that is a factor:  
   So, , or .  
   As the following diagram shows, in the interval , and each have three solutions, whereas has just two solutions. There are eight solutions in total.  
   

1. For what fraction of the interval is the inequality satisfied?  
     
   Since , it follows that . In the interval , this is satisfied precisely when either or . This is of the interval .  
     
   
2. The left-hand side equals zero when , so is a factor of the left-hand side:  
   Since or , it follows that or . Knowing that and , a sketch shows that the original inequality is satisfied for precisely when or .  
     
   
3. Since or , it follows that or . Since , has no solutions. If , however, a quick sketch shows that, for , either or .  
     
   
4. It would not make sense for to be a negative number since then the inequality would always be true, so let be a positive number. or .  
   A quick sketch helps greatly:  
     
     
     
   Appreciating the symmetry of the curve leads to the realisation that the answer to the question would be the same if it were just on the interval and that, since is one third of and , must equal .  
   It follows that .
5. It only makes sense for to be a positive number., and similarly .  
   A quick sketch helps. For the value of used in the illustration below, the interval of values that satisfy do overlap a little with the interval of values that satisfy .  
     
     
     
   As the value of lowers, so does the amount of overlap between the two sets of solutions, each overlap becoming a single point (at and ) precisely when . Therefore, is the smallest value of such that and have solutions in common.
6. or or . The solution to this on the interval is , , or .  
     
   

**Exercise 2**

**TMUA style questions**

1. can be rewritten as

, , .

for

when

i.e. (Note: no other values in give )

For ,

for all

for and



The values occur in groups of four. Each group of four sum to zero . Since there are terms to be added, the last term must be the first term of the next group of four, .

The answer is B.

1. When , , so the solutions of the equation are the same as the solutions of the equation .   
   The graphs of both and pass through the origin, but the gradient of the tangent to at is zero, whereas the gradient of the tangent to at is positive ( when ). So, as increases from zero, the graph of is below the graph of until it passes the first vertical asymptote; moreover, since, between successive asymptotes and above the -axis, both graphs have increasing gradient, with tending to infinity before reaching the next asymptote, it follows that the graphs intersect precisely once.  
   On the graph of the asymptotes are at , , , etc.. Thinking of as a one-way stretch parallel to the -axis scale factor (followed by a one-way stretch parallel to the -axis, scale factor ), the asymptotes on this graph, in the interval , are at , , and .  
   By the reasoning given above, the equation , and therefore the equation , has precisely one solution on each of , , and . (There is no solution in , since on this interval the graph of is still below the -axis.) There are four solutions in total.

The answer is E.

1. Subtracting times the first equation from the second equation gives:

Since for all values of , any solution, , to any equation of the form , for some real number , in the interval , must be paired with a second solution, , and therefore adding all of the solutions must equal zero. There is no need to find the value of nor solve the resulting equation.

Adding times the first equation to the second equation gives:

So, on , then  
 or , so or .  
Adding these up gives .

The answer is D.

Area Area

Let so

Cosine rule for triangle : (1)

Cosine rule for triangle : (2)

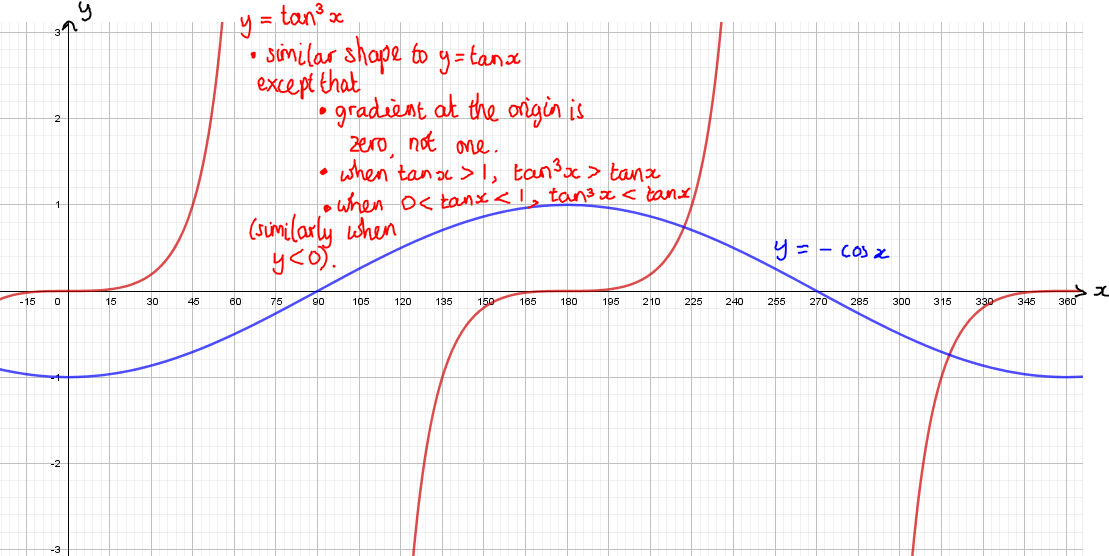
Equating

Substituting back into (1) gives

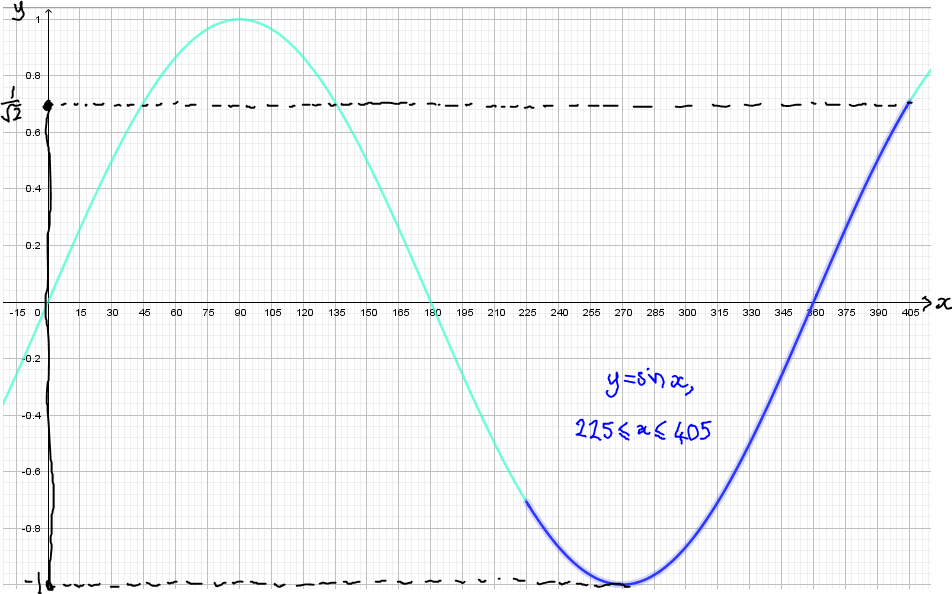
Multiplying top and bottom by gives

The answer is C.

**MAT style questions**

1. If then , so the solutions of are the same as the solutions of .  
   A quick sketch of the graphs of and helps at this stage.  
     
     
     
   For , , and for , . For , and are both positive, with decreasing and increasing from zero towards infinity, meaning that the curves intersect precisely once.  
   For , and are both negative, with decreasing and increasing from negative infinity to zero, meaning that the curves intersect precisely once.   
   For and , either is undefined, or it is but the graphs clearly do not intersect.  
   In total, there are two solutions on the interval .

The answer is c).

1. . In a similar way, if is even then and can never equal zero, since and never equal zero for the same value of . This immediately rules out a), b), d) and e). The answer is c), simply by deduction that it cannot be any of the other answers!
2. For , . A quick sketch of the graph of shows that   
     
     
   So .  
   It follows that .

The answer is d).

Here is another example that it is probably quicker to answer by simply eliminating the wrong answers. The most obvious triangle that satisfies is an equilateral triangle, with . This is handy because and are both common knowledge.  
  
If :

. Since . So , and a) is not possible;  
  
. Since . So , and b) is not possible;  
  
, so this eliminates both d) and e).  
  
The only possible answer is c).

The circle has centre and radius .  
The angle that makes with the -axis is .  
This means that angle is (since corresponding angles are equal) and angle is (angles in triangle add up to ).  
Therefore, angle is (angles on a straight line add up to ) and angles and both equal (since so triangle is isosceles).  
Therefore angle is .

. (Let ; then, by Pythagoras’ Theorem, ).  
So and , and . The answer is b).

**Chapter 10 Logic and proof**

**Try it out (page 118)**

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| F | F | F | T |
| F | T | T | F |
| T | F | T | F |
| T | T | T | F |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
| F | F | T | T | T |
| F | T | T | F | F |
| T | F | F | T | F |
| T | T | F | F | F |

**Try it out (page 119)**

For , if is odd then is prime.

A counter example for this is since and

**Try it out (page 120)**

It is worth noting that this statement is false before you start.

Converse:

This **is** true for all values of greater than , squaring will give a value greater than .

Contrapositive:

This is false. The contrapositive is only true if and only if the original statement is true (the contrapositive can be used to prove a statement).

A counter example here is

Negation: does not imply .

This is actually true as the original statement was false.

**Try it out (page 121)**

1. if is true there are no values greater than that are not also greater than .
2. only if is false. There are values that are not greater than that are greater than e.g. .
3. is a square if is a rectangle is false. It is possible for a rectangle not to be a square (if adjacent side lengths are not equal).
4. is a square only if is a rectangle is true. The definition of a rectangle is that it is a quadrilateral with four internal right-angles. A square has to have these.
5. if and only if is false. If then and could be anything. This could be made true by specifying that .
6. if is false. A counter example is which gives . Think of some more counter examples.
7. only if is true. All of the values for which lie below .
8. if and only if is false. A counter example is for which

**Exercise 1**

* + - 1. The statement does not say “behind every odd number there is a vowel” so turning the over would not prove anything conclusive.

The statement also does not say “behind every consonant there is an even number” so turning the B over would not prove anything conclusive.

Turning the A over would either show that the statement is false, if there is an even number on the back, or *provide evidence* (but not prove) that the statement is true.

Turning the over would either show that the statement is false, if there is a vowel on the other side, or *provide evidence* (but not prove) that the statement is true.

To prove that the statement is true, both the A and the 2 cards would have to be turned over.

The minimum number of cards is .

* + - 1. Let stand for Moves, for Salute and for Paint, We are told that and .

1. We are given . The contrapositive of is . Since a conditional statement and its contrapositive are logically equivalent we have is true and (so is true) and finally so the recruit should salute it.
2. We are given is true but we cannot deduce or from this. If the object is not moving, the recruit should paint it again (just following orders). If the object is something e.g. a vehicle that was painted when it was stationary and has subsequently started moving then it should be saluted.
   * + 1. The part of the statement that says is equivalent to . Siân will go out if it is the negation of this i.e. it is dry and warm so Max will have convinced Siân by saying that it is dry and warm.
       2. It is possible for the car headlights to be on in the daytime even if it is not raining.

You can deduce nothing without additional information.

This may seem like a slightly pointless question but, in mathematics, it is important to be able to ascertain when you can make an inference and when you cannot.

* + - * 1. This is correct
        2. This could be false. It does not say that you can’t put up an umbrella when it isn’t raining.
        3. This is the sort of statement that can cause a great deal of argument! Even though the initial statement is not true, the logic is correct. It is possible to come to a false conclusion but correct logical argument if an initial statement is incorrect (as it is here).
        4. Other animals than cats purr. The statement is only made about cats and it does not say “all animals that purr are cats.” The purring animal could be something different. A quick internet search will show that mongeese, bears, badgers, foxes, hyaenas, rabbits, squirrels, guinea pigs, tapirs and ring-tailed lemurs also purr.
        5. We may choose not to ski even if it is not windy. The statement doesn’t say that we have to ski.

**Exercise 2**

To prove that for any positive integer , if is prime then is prime.

Assume that is not prime, i.e. where and are positive integers.

If is not prime (i.e. ) then is divisible by and is therefore not prime so for to be prime, has to be prime.

The converse of this statement is

If a positive integer is prime then is prime.

This can be proved false by a counter example.

If then which is divisible by and (as well as and )

If then or

Using the contrapositive

Let statement A be and statement B be

or is equivalent to

is the same as

So the statement can be rewritten If then and are not true simultaneously.

The contrapositive is

if and then .

Proving the contrapositive will prove the initial result

If and

Let

So the statement is true for real numbers and .

1. The contrapositive for this is

If and then

since and are real numbers

This statement is not true as the contrapositive is not true. It is possible for to be equal to .

**Exercise 3**

**TMUA style questions**

* + - 1. A table of when Alice and Marcus are telling the truth and lying will help

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Mon | Tue | Wed | Thur | Fri | Sat | Sun |
| Alice | T | T | T | L | L | L | T |
| Marcus | L | L | L | T | T | T | T |

The possibilities are that

* 1. both are telling the truth
  2. Alice is telling the truth and Marcus is lying
  3. Alice is lying and Marcus is telling the truth
  4. both are lying
  5. Sunday is the only day where both tell the truth. If both are telling the truth then on Saturday both would be lying. This is not the case so it is not Sunday
  6. T/L occurs on Monday, Tuesday and Wednesday. If Alice is telling the truth and Marcus is lying then the previous day has to be L/T. None of these days has a previous day as L/T so it is not Monday, Tuesday or Wednesday.
  7. L/T occurs on Thursday, Friday and Saturday. If Alice is lying and Marcus is telling the truth then the previous day has to be T/L. The only day where this happens is Thursday (T/L on Wednesday)

It is Thursday.

The correct answer is D

* + - 1. A simple counter example can reduce the options

If , and then

and

Here so A and B can be eliminated.

This suggests that but it needs to be confirmed for all cases

Let and

This gives and

The correct answer is C

* + - 1. If jar B is false then

B contains £2

B says that C contains £10 so C does not contain £10

C is true so C also does not contain £50

D is true so E contains £20

E is true so A contains £5

This leaves jars C and D with two options, either £10 or £50 but C does not contain either of these (as discussed above)

B must therefore not be false otherwise C would also have to be false and as precisely one jar is false, it cannot be B that is false. B must be true.

If jar C is false then jar C contains £50 however, as B is true and states that C contains £10, this means that C cannot be false without B being false. C must be true.

If jar D is false the D contains £2.

As B is true, C contains £10 and this confirms that C is true (it does not contain £50).

If jar D is false then jar E does not contain £20

If jar E is true then A contains £5

So far the information indicates that A contains £5, C contains £10 and D contains £2. The remaining amounts are £20 and £50. Since D is false and E does not contain £20, it follows that B contains £20 and the £50 must be in jar E.

It looks like the answer is E

As a final check, jar E can be assumed to be false in which case jar E contains £2.

Jar A does not contain £5.

Jar D is true and states that E contains £20 but A also states that jar E contains £2. Both A and D are true so E cannot be false.

The £50 is in jar E.

Either or

The only fully correct option is II so the answer is C

* + - 1. and are factors of (\*) is true. So and are factors of is a sufficient condition for (\*).

However, and are factors of is false (for example, 2 and 3 are factors of 12 but So and are factors of is not a necessary condition for (\*).

The correct answer is B.

* + - 1. If then .

Since then this is not possible. As is an integer, it follows that

If then

in this case as

gives

gives

If then but

If then

gives

If then but

If then

gives but so this signals that no further options are possible.

The only possibilities for are , and

For I, the values , and provide a counter example.

For II, in every case or

For III, in every case

The correct answer is G.

* + - 1. For I since, for , is an increasing function then as ,

For II since , .

Let

So

For III (since all are positive, multiplying by will not change the inequality)

A suitable counter example is , ,

For these values

I and II only.

The correct answer is E.

* + - 1. The solution does not appear to have any obvious mistakes.

From the final solutions, .

Substituting this into the left hand side of A gives

The solution is inconsistent with the first equation.

The correct answer is G.

**Chapter 11 Getting started with longer questions**

**Try it out (page 128)**

terms:

terms:

terms:

Constant:

The equation becomes

and

**A question to think about (page 128)**

Coefficients

constant:

For

Check

so

For

Check

so

The equation can be rewritten using the found identity as

so A

The equation can be rewritten as

B

Let and

A becomes

B becomes

A – 2B gives so and

and

So and

There are four pairs of simultaneous equations, each giving a possible solution pair for and

If and

Subtract

is one solution pair

If and

Subtract

is another solution pair

If and

Subtract

is another solution pair

If and

Subtract

is one solution pair

The four pairs of solutions are , , and

**Try it out (page 129)**

Let

So is also a factor

Factorisation like this should be checked to ensure that each term has the correct sign.

**Try it out (page 130)**

Equating coefficients of

:

:

:

: so (using the coefficient of )

:

Constants:

Using and the coefficients of and

, . Adding these equations gives ,

Check using the coefficient of :

so (as )

gives

**Try it out (page 131)**

where

This is the binomial expansion

**Try it out (page 131)**

where is a prime number and n is a positive integer

, (not prime)

,

The only prime number that is one less than a square number is 3.

All subsequent values of will give the product of two values greater than one which would mean that p is not prime.

Either

Or so or (but

is a positive integer)

For ,

For ,

The only prime number that is one less than a cube number is .

Either

or

i.e.

If is a positive number then

**Try it out (page 132)**

For

**Exercise 1**

For each of the questions, it is important to have some sort of idea what the final factorisation will look like. It is usually useful to find the equivalent terms to the ones that cancel in the expansion above (). For example seeing the terms and might indicate that the terms and will form part of the factorisation. If you are not sure what to look for, in the following examples, the key stage to identifying the correct factorisation is in manipulating the expression to give the line before the final factorisation. Look at each answer and try to link each one to . It is not easy but once you have the idea you should be more confident in spotting these factorisations.

1. c

This could be done directly from (c) but, imagining that you had not seen the previous question, here is the full working:

As the question implies, you should be looking for the most efficient method you can come up with for each of these.



This initially looks like it will be a cyclical factorisation but multiplying it out does not give the required form to enable it to be factorised by comparing to . A different method has to be used.

Looking at the expression, substituting , and into it should all result in the expression equating to . This can be done mentally but here is the written method:

Let :

is a factor.

Let :

is a factor.

Let :

is a factor.

Note that any of these factors could be written “the other way round” e.g. could be

. At this stage, the best thing to do is to be consistent in the approach. The

factorisation will be checked by comparing coefficients later on and any problems with

signs can be dealt with at that stage.

So far the factorisation is but this is clearly not equal to the

original expression which will have terms such as and (ignoring signs for the

moment). The factorisation so far has terms and so it needs to be multiplied by

. In the similar vein there needs to be a multiplication by to give the and

terms and a multiplication by to give the and terms.

At this stage the factorisation seems to be based on .

At this stage the coefficients of terms should be compared to correct any places where

sign changes need to be made.

The coefficient of should be and is currently .

The coefficient of should be and is currently .

The coefficient of should be and is currently .

The coefficient of should be and is currently .

The coefficient of should be and is currently .

The coefficient of should be and is currently .

All of these signs can be corrected by “swapping” the terms in the first three factors

giving the factorisation

The final stage is to check that no additional terms are introduced by the result you have

reached.

That seems like a long process when each stage is written out but a lot of it can be done

mentally.

This should be recognised as the binomial expansion of

The first check should be to see if this is a binomial expansion.

The first term is .

Taking a factor of from the first term, from the second term and from the third gives

The binomial coefficients indicate that this is the expansion of

Using the factor theorem, , and must be factors as substituting

, and into the expression all result in .

Try and compare coefficients

The coefficients of and should both be and are .

The coefficients of and should both be and are .

No additional terms are introduced by

The correct factorisation is

Note that this could be written

Taking out a factor of gives

Writing the powers of 81 in terms of 9 (indicated by ) gives

Writing as , as , as and as gives

This is the binomial expansion for

Note: since , this could also be written as

This appears to be based on but a little more evidence is needed.

By considering the terms the expression can be written as

From the idea that it might be based on and the form that has just

been found, should be a factor of the final result (so should and

). This is by no means obvious but can be tested.

Working with the and and taking out a factor of gives an

additional term is needed to make this but this is not present in

the expression. It can be added to the expression provided a “balancing” is also

included.

The same thing can be done with the and terms

The next part is slightly more obvious

The final part of the factorisation can be done by realising that and

should also be factors and that

The complete factorisation is

Note: this is by no means obvious. A great deal of thought has to be put into difficult

expressions like this. It helps to have a suspicion about the form of the final result.

**Exercise 2**

**MAT style questions**

(i)

, ,

(ii) For , the product of the roots .

If and are distinct integer values then is an integer and will not be an integer.

(iii) For

From (i) . For this equation i.e.

If , and are all positive integers then , and giving

Since the equation cannot have three positive integer solutions

(iv) For

so

Without loss of generality let

If , which is not possible as so

If , so the two possibilities are and

(increasing further would give )

If , giving only one possibility

(increasing further gives )

If , but this means that must be greater than one of or

The only possibilities are

, , giving i.e. and

, , giving i.e. and

, , giving i.e. and

(v) For

There are limited possibilities for this (but more than for part (iii))

Without loss of generality let

Each of these are the roots of a distinct equation so there are 13 distinct equations with four positive integer solutions.

(i) Multiplying out the brackets (carefully) gives

(ii) The coefficients of , , and must be equal to .

is one condition (from )

gives the other condition (from )

from gives the same condition as that from

from gives the same condition as that from

as

(iii)

Equating coefficients of

:

:

:

:

:

Const:

So is a factor of

(iv) Assuming that the other two quadratic factors are and

Equating coefficients of

:

:

or

gives and gives

Hence the other two factors are and

(v)

The only real solution is

(i) The first part of this question is covered in the earlier “try it out” section.

The answer found was

and

(ii) For this to be the case

A

B

From A substituting into B gives

This is quadratic in and has real roots when i.e. as required

(iii)

For this equation, and

for these values so the substitution for (i) can be used

When ,

When ,

For , :

and

For , :

and

Using and

The equation becomes

Using and giving results in so

and the same result follows.

(iv) For

Using the factor theorem, is a factor

Equating coefficients of : so

Giving (a repeated root) or

For , let and

Hence and the roots are (repeated) and

Note: this is question 5 in the book but I have asked for the current question 4 to be removed as it does not work.

(i) satisfies both equations so

A

B

A – B gives

provided

Substituting this into A gives

Multiplying throughout by gives

(\*)

Care is needed for this step. “If and only if” is used. Simply showing that substituting into B gives an equivalent expression is not enough. You need to show that substituting into the left hand side of B gives .

Substituting into the LHS of B gives

Using a common denominator of gives

As from (\*), this is equal to .

There will be a common root if and only if

If and the equations have common roots then for a common root

C

D

From C in D so

Both roots are common.

Is consistent with this?

If it becomes giving

so

The condition still holds but if , both roots will be the same.

(ii) Let the common root be

So

E

F

E : G

F – G:

H

Using part (i) if is a common root then E and H will have a common root

: ,

: ,

as required. Since E + H gives the cubic F, it follows that F is satisfied too.

and

For a common root

is a factor since

Equating coefficients of :

or

, or

**Chapter 12 Number Theory and Combinatorics**

**Try it out (page 136)**

has to be a positive integer.  
If then so .  
If then and is not an integer.  
If then and cannot be a positive integer.  
  
It follows that and is the only solution.

has to be a negative integer.  
If then and is not an integer.  
If then and is not an integer.  
If then so .  
If then and is not an integer.  
If then and is not an integer.  
If then so .  
If then and is not an integer.  
If then and cannot be a negative integer.  
  
It follows that there are two solutions, and , and and .

It helps to think of the line with equation if , and if . The problem stated above is equivalent to asking for what real numbers and does the line travel infinitely far in the top-right quadrant of the co-ordinate axes ( and ). This happens precisely when the line has negative gradient, when the line is horizontal and with a non-positive y-intercept, or when the line is vertical and with non-positive -intercept.  
  
If , the gradient of the line is : . Also, since includes an term for any values of and , it follows that no horizontal line, for some real number, , can be written in this form. However, gives the vertical line , which has non-negative -intercept when .  
  
It follows that the equation has finitely many positive integer solutions precisely when either or and .

Since is an integer and and are positive integers, must be a factor of . Also, .  
  
: if then . This is the only positive integer solution.  
: if then . This is the only positive integer solution, since would lead to .  
: if then and if then . These are the only positive integer solutions, since would lead to .  
: if then , if then , if then , if then and if then . These are the only positive integer solutions, since would lead to .  
  
There are nine solutions in total, namely and .

1. The great thing about powers is that they get large very quickly! and . However, so a pause before starting this question would help you realise that there are up to separate cases to check. Just in case you need convincing that you cannot afford the time to check each case separately, I did exactly that… and it took a long time! See the unnecessarily long table at the end of this solution!  
     
   As well as checking cases systematically, you need to think of ways to cut down the number of cases that you need to check. One way would be to think of last digits. The number has last digit and every positive integer power of has last digit . Therefore, for the sum to work, has to have last digit . The last digits of are and , and the last digits of are and .  
     
   The following are the only ways for to end in last digit and be less than . In each case, you just need to subtract from and see if you are left with a power of .  
     
   ……:  
     
    but is not a power of .  
    and is a power of 5. This gives a solution: and .  
    but is not a power of .  
     
   ……:  
     
    but is not a power of .  
    and is a power of . This gives a solution: and .  
     
   …(power of ) will not end in a .  
     
   ……:  
     
    but is not a power of .  
    and is a power of . This gives a solution: and .  
     
   So there are exactly three solutions, the three found above, and this conclusion is reached after only needing to check cases, as opposed to blindly going through over cases!
2. : (since and are positive integers). There is another to be made up from any number of s and s ( is too large). is not a factor of so there needs to be at least and, in fact, exactly one . This leaves which is . So .  
     
   : , so no solutions are possible.  
     
   Any further cases would involve at least one of and increasing by one, and therefore they would also always lead to a sum greater than .  
     
   There is just the one solution, and .

**Try it out (page 137)**

The solutions of are the same as the solutions of . Since is a positive integer and is an integer (possibly negative), the only possibilities are:  
  
 and , giving and ;  
 and , giving and , but this not allowed since must be a positive integer.  
  
The only positive integer solution is and .

**Try it out (page 138)**

Let be a positive integer that is not an power. Suppose that is rational, so that where and are integers and . Raising each side to the power of gives and from this .

Now, the trick is to spot that, since , powers are precisely those numbers in the prime factorisation of which every prime is raised to an power that is a multiple of . Therefore the prime factorisations of and consists of only prime numbers raised to powers that are multiples of , whereas at least one prime number in the prime factorisation of is raised to a power that is not a multiple of (since is not an power). Pick one of those primes and call it . Then is raised to a power that is not a multiple of in but is raised to a power that is a multiple of in . Two numbers are equal if and only if their prime factorisations are the same. Therefore and cannot be equal and the initial assumption, that is rational, cannot be true; therefore, must be irrational.

**Exercise 1**

1. Let for integers and . Suppose that is rational, so that for integers and . Then . So long as , , which is rational, contradicting the fact that is irrational. If then which is also rational, giving the same contradiction.  
   Therefore, cannot be rational; it must be irrational.
2. could be rational; for example, and (which is irrational, thanks to part a)) gives , which is rational.  
   However, could also be irrational; for example, and gives , which is irrational.
3. Let for integers and . Suppose that is rational, so that for integers and . Then , giving , which is rational ( and the numerator and denominator are integers), contradicting the fact that is irrational. Therefore, cannot be rational; it must be irrational.
4. could be rational; for example, and (which is irrational, thanks to part a)) gives , which is rational.  
   However, could also be irrational; for example, and gives , which is irrational (thanks to part a) applied several times).
5. could be rational; for example, gives , which is rational.  
   However, could also be irrational; for example, gives , which is irrational.

1. Squaring both sides gives .  
   Squaring both sides again gives , or . Since must be a positive integer, the same must be true of , meaning that the numerator, which is always positive, must be divisible by both and ; this happens precisely when is an odd multiple of . Let , where can be any positive integer. Then . Adding the two equations gives , so . Substituting this into the first equation gives , so .  
   So and . These are both positive integers whenever . This gives an infinite family of potential solutions.   
   Checking that these satisfy the original equation, and , so all the solutions we found, that is infinitely many, satisfy the original equation.  
   The answer is E.
2. Squaring both sides gives .  
   Squaring both sides again gives . This is an integer precisely when is an odd number, so that is the square of an odd number. In the interval , is only positive for , so there are only five cases to check.  
     
   : , which is the square of an odd number. This leads to the solution or or .  
   : , which is not the square of an odd number.  
   : , which is the square of an odd number. This leads to the solution or or . Eliminate since has to be a positive integer.  
   : , which is not the square of an odd number.  
   : , which is not the square of an odd number.  
     
   So the only possible positive integer solutions are with , with and with . It needs to be checked that they satisfy the original equation.  
     
   , : and ., : and .  
   , : and .  
   So all of the possible integer solutions do also satisfy the original equation.
3. Let where and are the digits of . Reversing the digits forms . Since is also a two-digit number it follows that must be true.  
   Then is a multiple of . This is also a multiple of precisely if is even, and this in turn is true precisely if is odd, and, therefore, if is odd.
4. Let where and are the digits of . Reversing the digits forms . Since is also a two-digit number it follows that must be true. Squaring the first number gives and squaring the second number gives . The difference between these two numbers is . Since is an integer, the difference must always be a multiple of . If the difference is not a multiple of then must be an odd integer. This happens precisely when either is even and is odd, or when is odd and is even.
5. Let , where and are digits with . Then , which would mean that is necessary for this to also be a three-digit number. Therefore, . Since is an integer then always a multiple of . It is also a multiple of precisely when is a multiple of , and this in turn happens precisely when is a multiple of . This is equivalent to the necessary and sufficient condition that .
6. Suppose that are three digits such that . The six two-digit numbers that can be formed by concatenating two of the three integers are and . The sum of these six integers is .
7. Any four-digit number can be expressed as , where and are digits with . Then the number formed by reversing the digits can be expressed as . Adding these two numbers gives . Since , it follows that , which, since is an integer, must be a multiple of . It is also a multiple of precisely when is a multiple of and this happens precisely when is a multiple of . Since and are integers with , the only way this is possible is if ; this is a necessary and suffucient for the sum of the two numbers to be a multiple of .
8. Suppose that is an -digit number. Let where , the last digit, is any integer from to inclusive, but is a positive integer consisting of digits. Then and , so , which is a multiple of . Therefore is a multiple of if and only if is. (1)  
   If is a multiple of then for some integer . The prime factorisation of each side must include at least and . Since and are not factors of , they must each be a factor of , meaning that , where is an integer. Also, if is a multiple of , say for some integer , then , where is an integer. Therefore is a multiple of if and only if is. (2)  
   Putting (1) and (2) together gives the desired result, that is a multiple of if and only if is a multiple of .

**Try it out (page 142)**

* + - 1. Conditioning on the highest value coin, there 4 ways to make £1.50 using £1:

(£1, 50p), (£1, 20p, 20p, 10p), (£1, 20p, 10p,10p,10p), (£1, 10p, 10p, 10p, 10p, 10p)

Notice how the final two ways are obtained from the second one by replacing a 20p piece with two 10p pieces each time. We could encode this as (£1, 50), (£1, 20, 20, 10)+2 meaning that there are two further solutions obtained from (£1, 20p, 20p, 10p).

Continuing in this way you obtain:

(50, 50, 50), (50, 50, 20, 20, 10)+2, (50, 20, 20, 20, 20, 20)+5, and (20, 20, 20, 20, 20, 20, 20, 10)+7

Giving ways in total.

Saranya needs to make exactly £1.50 using only 50p, 20p and 10p coins so her successful combinations must be a subset of the 22 ways already listed. With the extra restrictions, Saranya’s only solution involves choosing two 50p coins, two 20p coins and a 10p coin in any order. You can work out the probability of success directly by taking the product of the probability of a successful sequence, e.g. (50, 50, 20, 20, 10), and the number of ways this can occur. If all the coins were distinguishable, the number of permutations of 5 coins is 5! But each sequence includes two 50p and two 20p coins you need to divide this number by . So the required probability is .



.

Following the hint, a squad of size m can be chosen from n players in ways. For each of these ways there are then ways to choose a team of size k. So there are ways to choose a squad of size m and then a team of size k. Alternatively, you can choose the team first in and then, in each case, choose the remaining squad players in ways giving ways in total.

Let the -set be . Conditioning on the elements :

There are subsets of that do not contain 1.

There are subsets of that contain 1 but not 2.

There are subsets of that contain 1 and 2 but not 3.

…

There are subsets of that contain (note that this is just the set ).

Since this counts each possible subset of size exactly once, the sum of these terms is .

**Try it out (page 144)**

1. The only motif patterns of lengths 1 and 2 are 1, 11, and 12 so and .
2. Listing the motifs in “numerical order” gives:

1111 1112 1121 1122 1123 1211 1212 1213 1221 1222 1223 1231 1232 1233 1234

Hence .

1. Using part (ii): .
2. for all (the sequences are , and respectively).
3. Each pattern of length 5 that uses 3 symbols is either formed by appending 3 to the end of a pattern of length 4 that uses 2 symbols (in ways) or by appending any of 1,2,3 to the end of a pattern of length 4 that uses 3 symbols (in ways). Since these patterns are all distinct, .

To calculate N(8,2) generalise the idea used in part (v) to obtain: ; and Using parts (iii) and (iv) gives

.

Similar calculations give , , , and . Finally, and adding these terms gives .

**Exercise 2**

Letting denote a 4-digit number, there are 2 choices for ( must be 5 or 6 so that ), then 4 choices for , 3 for and 2 for . So there are 4-digit numbers greater than 5000. Any 5-digit number is greater than 5000 and there are of these (note that ), hence there are 144 such numbers in total.

A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Since , all of the 5-digit numbers are divisible by 3. The 4-digit numbers divisible by 3 are those that do not contain a 4 or a 5. Considering choices for the digits of , there are numbers that do not contain a 4 and numbers that do not contain a . So there are numbers out of that are not divisible by 3, hence the probability that a randomly chosen number is divisible by 3 is .

Notice that if 1 is not in the first position then it is immediately preceded by a number greater than itself. You should find there are 4 sequences with the 1 in position 2; 6 sequences with the 1 in position 3; 4 sequences with the 1 in position 4 and just 1 sequence with the 1 in position 5. So there are 15 such sequences with the 1 not in position 1. If the sequence has a 1 in position 1 then the remaining sequence of 4 must contain exactly 1 descent. These can be listed systematically as follows:

12354 12435 12453 12534 13245 13425 13452 14235 14325 14523 15234

So there are 26 sequences of length 5 with one descent.

It is possible to generalise the previous argument to obtain a recurrence relation for sequences of length . However, there is a more elegant approach which is to observe that if the set is partitioned into two subsets, of size and say, with the numbers in each subset ordered numerically and then joined to form a sequence of length , then this sequence will have exactly descent (with the number in position less than that in position ) unless the partition happens to be { and . For example with and , the partition generates the sequence 25134 whereas the partition generates 12345 which is not valid. Therefore the number of valid sequences with the number in position less than that in position is . Summing over all possible gives:

Using the fact that gives the formula for the total number of sequences of length with exactly one descent to be .

* + - 1. You need the number of non-negative integer pairs such that and . Conditioning on : and similarly , , . No more solutions are possible because but we require . Each of the possibilities leads to multiple solutions, for example with you obtain ). Continuing in this way, there are passing combinations.

For questions, a similar consideration of cases gives that for and 9, and 41 respectively. This gives a total of 115 passing combinations.

* + - 1. Without considering rotations and reflections, the number of distinct arrangements corresponds to the number of ways of choosing two of the nine smaller triangles in which to place the counters. This number is given by .

If rotations and reflections are considered equivalent then these 36 arrangements are partitioned into subsets where any two are related by a rotation or reflection. There are 9 such subsets (three of size 6 and six of size 3) which can be found by taking an arrangement and generating the other arrangements obtained by rotating or reflecting. The subsets of size 3 correspond the arrangements where a rotation is also a reflection.

* + - 1. Let to denote the items of clothing (where and stand for left and right, and stand for sock, boot, inner glove and outer glove respectively).

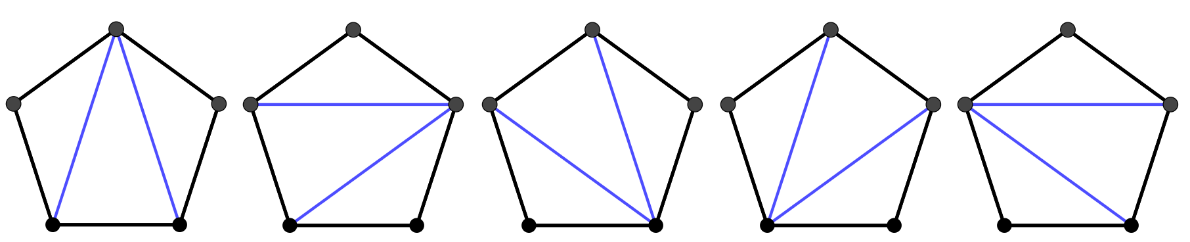
(a) You require the number of permutations of these 8 items such that (where means that occurs to the left of ). There are positions for but having chosen these you must have . For each of these there are positions for but having chosen these you must have , similarly there are positions for and then must fill the final two positions (with ). Therefore the total number of ways for Pogo to dress himself correctly is:

(b) You now require that the final two positions are (in either order). The number of positions for Pogo’s socks and boots (which must occur in the first 6 positions) is now . The inner gloves must be placed in the remaining two places (in either order). Once all the other items have been put on, the two outer-gloves must be put on (in either order). So the total number of ways is now .

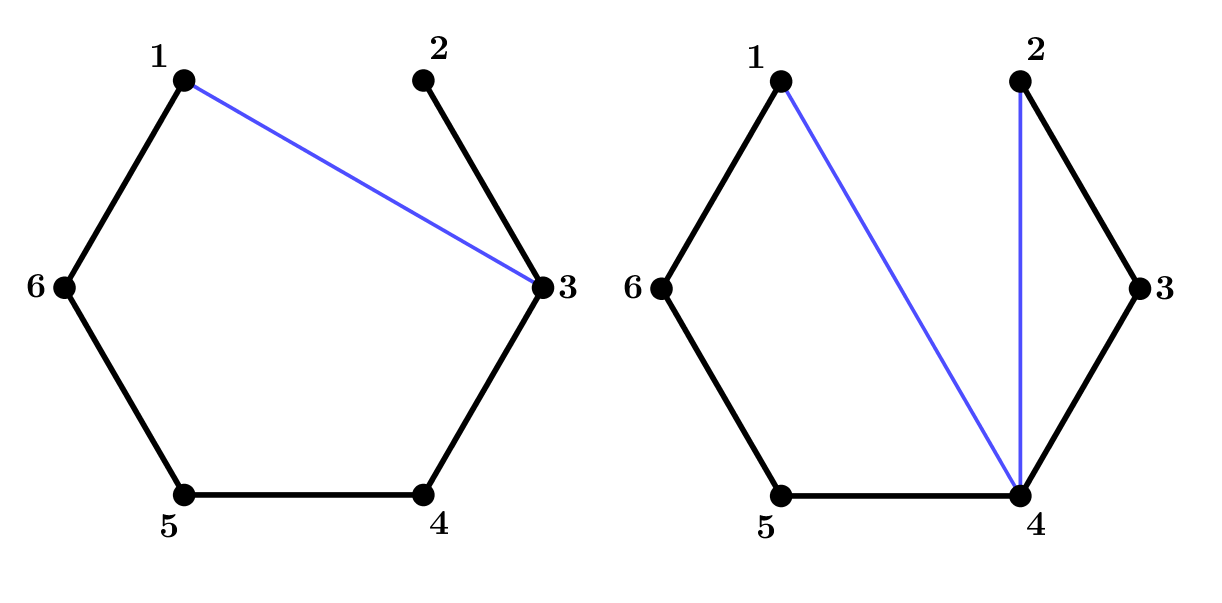
**Exercise 3**

**A MAT Style question**

* + - * 1. is the number of ways of dividing a triangle into 1 triangle using 0 diagonals – this is clearly just 1.

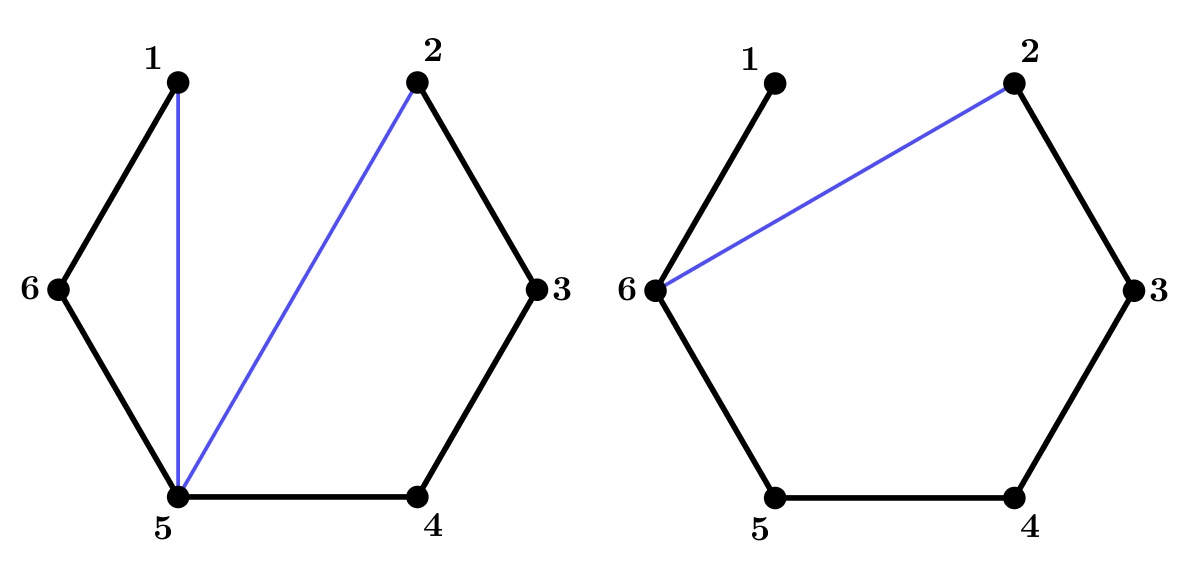


* + - * 1. There are four triangles that can be removed , , and



Removing leaves a pentagon which has possible triangle arrangements and a space which has possible arrangements (considering to be 1). This is arrangements in total.

Removing leaves a quadrilateral with arrangements and a triangle with arrangements. This is arrangements in total.



Removing leaves a triangle with arrangements and a quadrilateral with arrangements. This is arrangements in total.

Removing leaves a space which has arrangements and a pentagon which has arrangements. This is arrangements in total.

Hence

1. is needed. This is the number of ways of splitting a quadrilateral up into triangles using one diagonal. This is as there are only two distinct diagonal lines that can be drawn.

1. For a polygon with sides, and vertices numbered in order to there are points for which the removal of side will make a triangle. Removing the triangle that joins to point 3 leaves a space and an sided polygon. This will give arrangements. Removing the triangle that joins to point 4 will leave a triangle and an sided polygon. This will give arrangements. This pattern will repeat giving until for even or for odd (i.e. the “half-way mark”) after which arrangements will reduce in the same pattern i.e. , .

This gives the number of arrangements for as

Hence

1. For an octagon, so

from part (iv)

, , ,