**STEP, MAT, TMUA: Skills for success in University Admissions Tests for Mathematics**

**Full solutions – Part 3**

**Chapter 13: Further Trigonometry**

**Try it out (page 148)**

**Exercise 1**

1. , ,

Using the cosine rule

Since for the triangle to exist (also , and )

or

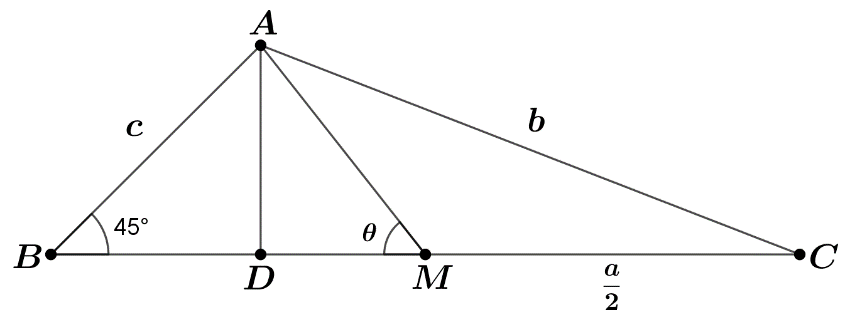
1. Area of triangle

Let then by Heron’s formula

Area of triangle

Note: you prove Heron’s formula is true for question 4.





1. Area of triangle

Area of triangle

1. Using the sine rule for triangle AMB

Using the cosine rule for triangle AMB

Using the cosine rule for triangle AMC

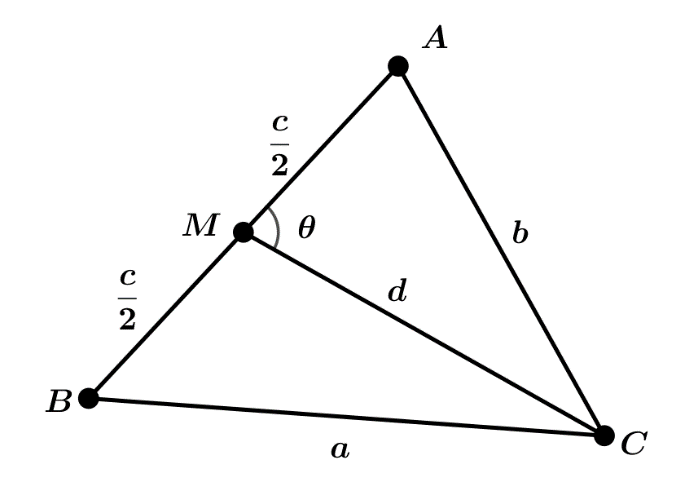
From the cosine rule for triangle ABC

1. , , ,

so

**Exercise 2**

1. Let the area of the triangle be then



Using the sine and cosine rule for triangle to find and

So

In the same way

and

From triangle

But from triangle

giving

So

But by the cosine rule (for triangle )

So

Hence

, and

,

If , and form an arithmetic series then and where is the common difference.

,

and are also three successive terms of an arithmetic sequence.



Using gives

One method for this part requires the use of the identities and

. Both of which can be found from and

. Subtracting one of these two identities from the other gives

and adding the two gives .

If then

* 1. so

To prove

Use the formula for the sum of two sines

But

Hence

* 1. To prove

Use the formula for the sum of two sines and

But

So we have

Now, so we have

Using the difference of two cosines

To prove

Using the formula for the sum of two sines and

But

So we have

Now, so we have

Using the sum of two cosines



where

You may wish to use a calculator to confirm that you have found the solutions to the equation.

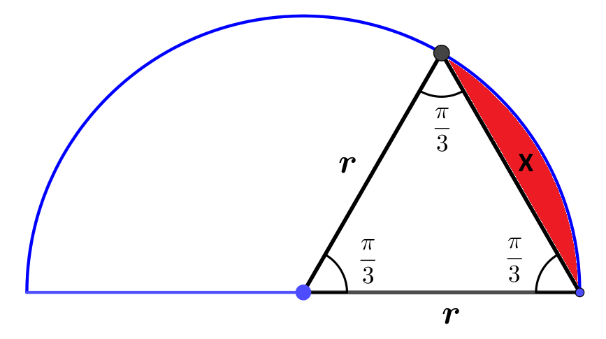
where

If then but is undefined for these values so this is not a solution

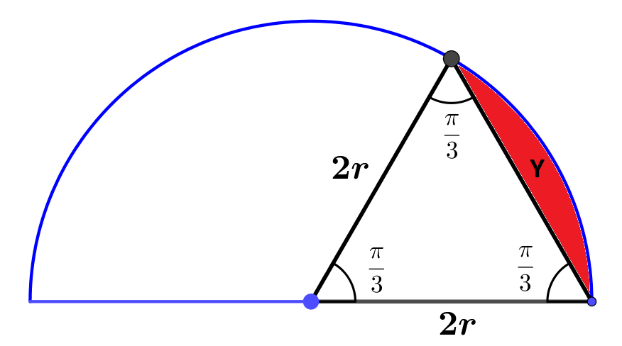
where

**Exercise 3**

1. (a)



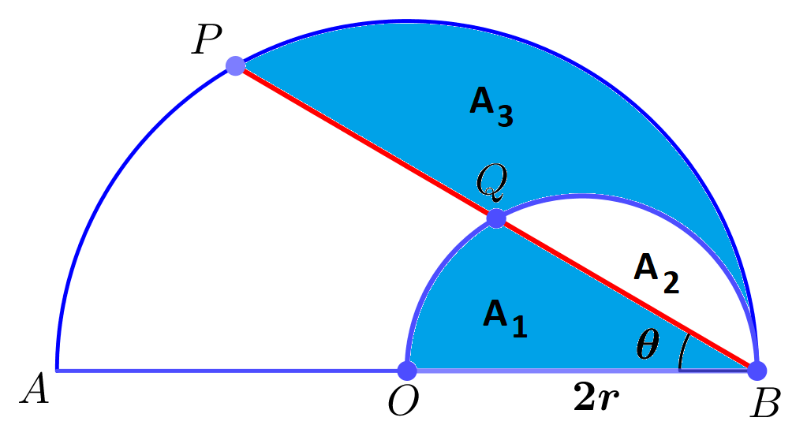
Area

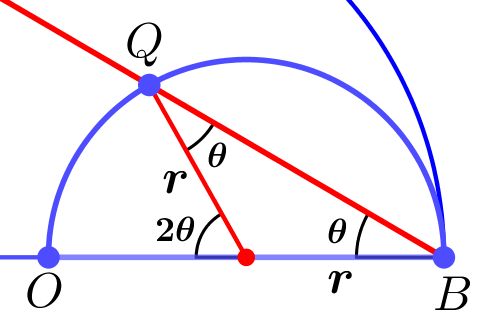


Area

The fraction of the large semicircle that is shaded is

(b)





Shaded area since

Unshaded area

As the large semicircle is an enlargement of the small semicircle with scale factor ,

Since

**A STEP Question**

Divide throughout by since and

Rearrange and divide by since

This needs to reduce to the form .

Hence , and

and

The question says “if and only if ” so, starting with it should be possible to find that , and

If

then

since

So , and

and

, and

Test:

and so the equation can be written in the form

so

,

and

, and

The values of , and are the same as before so the equation can be written in the form using the same values of and .

In this case, so it appears that the equation will be true for all where

This may not be entirely the case since the form required that and . In this case, but when and so these values might not be in the solution set.

They need to be checked to see if they are solutions to the original equation.

So is in the solution set (it satisfies the equation)

So is also in the solution set.

The solution set is all values of in

(1)

(2)

(1) + (2)

(2) - (1)

, and as before

Either

for

for

or

for

for

None of these values give or so all can be included in the solution set

**Chapter 14: Calculus – integration and differential equations**

**Try it out (page 154)**

Using and gives

Dividing the top and bottom of the first fraction by gives

Using gives

So

For

Since

So

**Exercise 1**

:

:



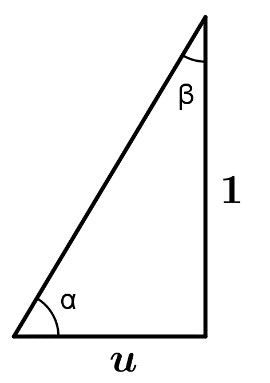




But this is (it’s the same but the limits are reversed). This is what you had to spot (as the question hinted)

If then





This is much harder to spot but

This diagram will help you to remember this.

and

and

So we have the same thing as we had in part (a) with the limits reversed giving

**Try it out (page 165)**

Multiplying both sides by gives

But

so

**Exercise 2**

1. Want to show from

Differentiating both sides of with respect to gives

To show , multiply both sides of by

But so

as required

1. Want to show from

For the quotient rule for differentiation is required

as required.

Differentiating both sides **with respect to**

The DE becomes

but as it can be written entirely in terms of and

Multiplying throughout by gives

Integrating factor

Multiplying through by this gives

Integrating both sides with respect to

There are several equivalent forms of this answer.

Differentiating both sides **with respect to**

The DE becomes

Multiplying throughout by gives

But so the equation can be written as

Integrating factor

Multiplying through by this gives

Integrating both sides with respect to

This can be written in a number of equivalent ways.

To find , the quotient rule for differentiation has to be used

as required

Differentiating both sides **with respect to**

so

The DE becomes

Multiplying throughout by gives

Dividing by gives

Integrating factor

Multiplying through by this gives

Integrating both sides with respect to

, so

Using the chain rule

Differentiating this with respect to (using the chain rule for the left hand side and the product rule for the right hand side) to find an expression involving gives

But so

The right hand side resembles the differential equation but needs to be multiplied by another

The original DE can be expressed as

Substituting and into this gives

But

Aux eqn

Complementary function

Trial function

Substituting this into gives

So

and

The particular integral is

The general solution to is

and when

Since , too

so the general solution is now

When and ,

The solution is

In terms of

Since

The solution in terms of and is

This is a little unusual in that substituting directly for and does not appear to work

so becomes which includes , and .

The key to using the substitution is to eliminate and terms.

Using the chain rule but as ,

So but too hence

Substituting this into the DE gives

Separating the variables gives

Since , this is

Separating the variables for this gives

The general solution is

1. This uses the same ideas as question 8

so

The DE becomes

Multiply throughout by

Integrating both sides with respect to gives

Integrating the right hand side using integration by parts gives

If and when then since

so

If when

so

Squaring gives

1. gives the differential equation

Aux eqn:

Since , and , so and will have real values

Complementary function

Since this is the solution to , this is the general solution

From

Since when .

when

so and

The solution is

**Exercise 3**

**STEP questions**

Let

Limits:

Since

Since

Hence

* 1. To do this and need to be found

Let

Let , . Limits: and

Let

Need to find the value of

Using the idea from part (i)

Since



Using the product rule

For the differential equation

Let so

Substituting this into gives

* 1. Using a similar idea to part (i) for the substitution

seems reasonable – the final part of the differential equation, , provides the strongest hint for this.

Using the substitution in the given differential equation gives

* 1. The two answers point to the solution to

Being

**Chapter 15: Complex numbers**

**Try it out (page 172)**

To prove that if is any integer then

The proof is in three parts

(i) (ii) and (iii)

(i) When is a positive integer, the proof by induction is

Base case: when , .

It is (obviously) true for .

Assuming that it is true for gives

Then for ,

So, by induction the theorem is true for positive integers .

(ii) When is a negative integer

gives

If is a negative integer, let then

by

(iii) When

for all non-zero complex numbers so

**Try it out (page 174)**

Let

**Try it out (page 176)**

**Exercise 1**



Multiplying throughout by

So

Is ?

Let so

So the equation becomes

as required

1. If is a root of then

since

since

Hence

is clearly a solution so is a factor of

Equating coefficients of :

Equating coefficients of :

Equating coefficients of :

So the equation becomes

and either or

so

Converting to principle arguments

The solutions are

1. can be rewritten as

Hence is the root of unity

Let

So

Since is an root of unity, and so the roots are for

There is a little more that needs to be considered here. Since , if then is undefined. This occurs when where is an integer i.e. when

. So the will be solution set will be the one above excluding those for which is not an integer multiple of .

is the root of

Let the root of be where and

i.e.

Let so

For

Caution is required for this question too as is undefined when i.e. when

where is an integer

Since and are all integers, this can never be the case so the solution set is all of

For

Let and

is a complex number with no real part i.e. it is of the form .

Note if then , if then

If then the argument is undefined (you have the complex number

1. Let and

means that

(this is using

can be written

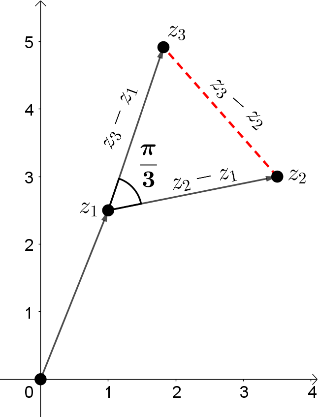
So

1. Let and

So

1. There are two possible arrangements for the three complex numbers and

Anticlockwise

In this arrangement rotating side by gives side

Hence

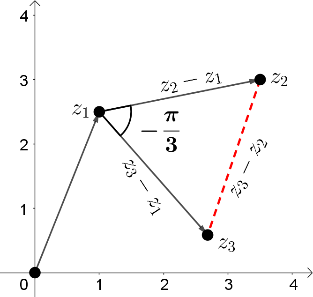
Also rotating side (note the change in direction) by gives

Hence

Multiplying the two together gives

as required

Clockwise



In this arrangement rotating side by gives side

Hence

Rotating side by gives

Hence

Multiplying the two together gives

as before.

**Exercise 2**

**STEP questions**

1. and

From

Since

Since ,

For ,

For ,

These do not have to be simplified.

If then

Multiplying throughout by gives

and this is

Hence satisfies and the solutions to (apart from ) give the solutions to

The fifth roots of unity (excluding ) give the values of

, , ,

For

Hence

So using

Since

and

Hence

Hence can be written as

which is a circle with centre and radius .

1. so

This is a circle with centre and radius

If the two are the same circle then

and

so

For

As , this gives so does not have an imaginary part and the centre of the circle is on the real axis

For

As , this gives so does not have a real part and the centre of the circle is on the imaginary axis



The equation becomes

This is a circle with centre and radius

If this is the same as then

and

giving as before

For

gives

In this case could be real or imaginary.

For

gives so

**Chapter 16: Matrices**

**Try it out (page 185)**



(A)

(B)

Substituting (A) into (B) gives

Equating coefficients of :

Equating the constants:

or

and are invariant lines for

To test if is also a line of invariant points the vector can be used

This is an invariant line rather than a line of invariant points

To test if is also a line of invariant points the vector can be used

This is an invariant line rather than a line of invariant points

(A)

(B)

Substituting (A) into (B) gives

(C)

Let

Either or

If equation (C) gives

Either (giving as above) or or

There are two invariant lines and

To see if is a line of invariant points the vector can be used

is not only an invariant line, it is also a line of invariant points

To see if is a line of invariant points the vector can be used

is an invariant line and not a line of invariant points

If then and is an invariant line or and is an invariant line

To see if is a line of invariant points the vector can be used

Let and

Is ?

is only a line of invariant points for

To see if this is a line of invariant points the vector can be used

Let and

Is ?

So

So, yes, it is an invariant line but not a line of invariant points.

(A)

(B)

Substituting (A) into (B) gives

Equating coefficients of :

Since , or

Equating the constants:

If

So, if then

If

So, if then

and are lines of invariant points

Testing using the vector gives

is an invariant line, not a line of invariant points unless in which case it is a line of invariant points

Testing using the vector gives

is an invariant line, not a line of invariant points unless in which case it is a line of invariant points

If and then could be any value and is an invariant line

Is ?

So, yes, is an invariant line.

If and then could be any value and is an invariant line

?

So, yes, is an invariant line

Summary

For all , and are lines of invariant points.

For , any line of the form is an invariant line and is a line of invariant points.

For , any line of the form is an invariant line and is a line of invariant points.

**Try it out (page 187)**

**Try it out (page 188)**

For the matrix to be singular

or

provided or

**Exercise 1**

is a line of invariant points

(1)

(2)

(1) in (2)

Let so

Either or

If ,

For , .

For , so

is an invariant line

is an invariant point

(1)

(2)

(1) in (2)

Let so

Either or

If ,

For , .

For , so

and are invariant lines

If and

gives which is inconsistent so this cannot be the case.

is an invariant point

(1)

(2)

(1) in (2)

Let so

Either or

If ,

For , .

For , so

and are invariant lines

If and

gives which is inconsistent so this can not be the case.

1. This is best done by using the eigenvalues of

Either or

corresponds to

gives

gives

The invariant line is



1. (subtracting column 1 from column 2)

(subtracting column 1 from column 3)

(subtracting row 2 from row 3)

(subtracting col 2 from col 3)

(subtracting col 1 from col 2)

The matrix is singular and has no inverse.

e)

so there are no values for which the determinant is

An inverse of will exist for all .

(i) The equations are consistent for all real .

(ii) The equations are independent for all real .

If or then the equations are either inconsistent or have infinitely many solutions.

For the equations are

Intersection of and :

From if then substituting into gives

In vector form the equation of the line of intersection is

Intersection of and :

If then from : (1) and from : (2)

Solving these simultaneously by (1) - (2) gives so . From this

Giving the vector equation of the line of intersection as

Intersection of and :

From if then substituting into gives

Giving the vector equation of the line of intersection as

The three lines are parallel.

So, for the planes form a triangular prism.

For the equations are

Intersection of and :

From if then substituting into gives

Vector equation of line of intersection

Intersection of and :

If then from : (1) and from : (2)

Solving these simultaneously by (1) + (2) gives so . From this

Vector equation of line of intersection

Intersection of and :

From if then substituting into gives

Giving the vector equation of the line of intersection as

The three lines are parallel.

So, for the planes form a triangular prism too.

(i) For and , the equations are inconsistent

(ii) For and , the equations are not independent



In both cases . This gives

Note: The determinant has been calculated using the first column rather than the first row. This is slightly quicker as the first column contains a term.

The planes will not meet at a point.

Intersection of and

Let

(1)

(2)



In (1):

Line of intersection

Intersection of and

Let

(1)

(2)

(1)

(2)

In (1)

Line of intersection

Intersection of and

Let

(1)

(2)

(1)

(2)

In (2)

Line of intersection

If , the lines of intersection are

, and

The three lines of intersection are coincident (i.e. along the same line) so the planes would form a sheaf (since they are clearly three different planes).

For all other values of , the three lines are distinct lines (equating any pair of the points from the equations of the lines gives a linear equation with only one solution i.e. ) so the planes form a triangular prism.

1. For , the planes for a triangular prism
2. For , the planes for a sheaf
3. This question is a test of algebraic manipulation as much as of matrix methods.

Provided the equations will have a unique solution

**Exercise 2**

**A STEP question**

1. As this is a line of invariant points,

(1)

(2)

Multiplying (2) by (1) gives

Either or or

Case 1: If , the line of invariant points is

This gives from (1) and from (2)

Since values exist for all points on the line (except the origin),

and in which case

Case 2: If , the line of invariant points is

This gives from (1) and from (2)

Since values exist for all points on the line (except the origin),

and in which case once more

In every case so

Let the invariant line that does not pass through the origin be where

(3)

(4)

These both apply for all (and as ),

and from (3)

and from (4)

So

If ,

If ,

So if does not pass through the origin i.e. the identity matrix (which makes sense)

Case 1: If and

is an invariant point if

i.e. and

(5)

(6)

Multiplying (5) by gives

Multiplying (6) by gives

So

Values exists for all

as required

The line of invariant points in this case is either or (they give the same equation)

Case 2: If and then

Either

or and

If then

So points on are invariant

If and then

So points on are invariant

The conditions imply that where

Theses must be true for all so substituting a simple value of is a sensible strategy

gives

and

So

As ,

gives (to distinguish it from ) and

Substituting into gives

Since this becomes

Multiplying by gives

If then

If , becomes

So and

gives and and as ,

gives and

Since and , so

Hence and so once more.

In all cases,