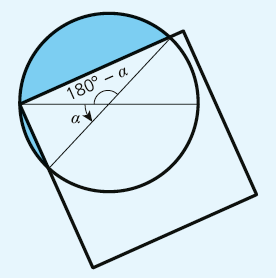
**STEP, MAT, TMUA: Skills for success in University Admissions Tests for Mathematics**

**Full solutions – Part 1**

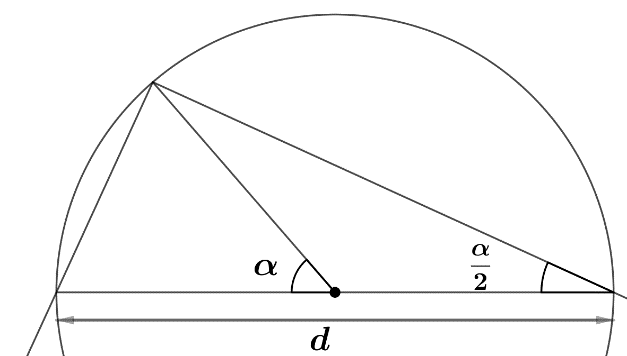
**Chapter 2: Mathematical problem solving in admissions tests**

**Try it out (page 16)**



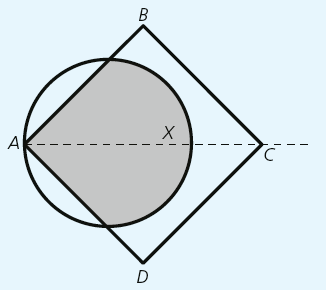
1. The area *X* is the sum of the areas of the semicircle and right-angled triangle that make up the overlap of the square and the circle (it has already been established that the two points of intersection will be the ends of a diameter of the circle.

Let be the angle that the diameter joining the points of intersection makes with the initial horizontal diameter.

The right-angled triangle has one angle of since the angle at the centre of a circle is twice the angle at the circumference.

This gives the right-angled triangle side lengths of and and, hence, an area of which, using the identity

, can be written



2. The area of . Since is constant, it is only that will have an effect on the area. As increases from to and then decreases from there to , it follows that the area will increase until i.e. the square is in the symmetrical arrangement.

The radius of the circle is units so units

The maximum value of

**Chapter 3: Introductory number and algebra techniques**

**Some questions to think about (page 17)**

1. The last digit of is the same as the last digit of . However, since also has last digit , all powers of end in a .
2. The last digit of is . has last digit . has last digit . has last digit . so has last digit . so has last digit , and so on.  
   The last digits of powers of cycle through the sequence .  
   Since , the last digit of is the same as the last digit of .

The answer is .

1. Any positive integer that leaves a remainder when divided by the positive integer can be written as for some integer , however, , and this is far more helpful.  
   For example, any positive integer that leaves remainder when divided by can be expressed as , so is one less than a multiple of .  
   Using similar reasoning throughout, you are looking for the smallest number that is one less than a number that is simultaneously a multiple of , in other words, . Using the prime factorisations , it follows that .
2. Let the two-digit number, be written as , such that and .  
   Reversing the digits gives , written as .  
   Adding the digits (of either number) gives .  
   The sum of these is which, since is an integer, is a multiple of . For it to be a multiple of , must be a multiple of . Since , the only possibilities are and .  
     
   The case leads to the two-digit numbers and .  
   The case leads to the two-digit numbers and .
3. a) Let, for example, the two consecutive integers be and . Then the product can be expressed . Since one of and must be an even number, it follows that the product, , must also be even.  
     
   b) Let, for example, the three consecutive integers be and . Then the product can be expressed . Since exactly one of and must be a multiple of , and at least one must be even, it follows that the product, , must be a multiple of .  
     
   c) Let, for example, the four consecutive integers be and . Then the product can be expressed . Amongst and there is at least one multiple of and two consecutive even numbers. Since the two even numbers are consecutive, one of them must also be a multiple of . It follows that the product, , must be a multiple of .  
     
   d) Let, for example, the seven consecutive integers be and . Then the product can be expressed . Amongst and there is at least one multiple of , at least one multiple of and at least two multiples of . There are also at least three consecutive even numbers, at least one of which is also a multiple of .  
   It follows that the product, , must be a multiple of . So, in fact, this proves more than was asked for! Since , it follows that the product is also a multiple of .
4. Factorising gives .  
   First notice that, if is a positive integer, is the product of five consecutive integers; one of these is a multiple of , at least one is a multiple of and at least two are even numbers that must be consecutive, so, further more, one of these must also be a multiple of . So the product of five consecutive integers must be a multiple of .  
   It follows that is a multiple of if is a multiple of . However, if is not a multiple of , then it can be written as, for example, or , for some integer .  
   If then both and , so the five consecutive integers actually contain two multiples of , and so, in this case, their product is a multiple of .  
   Similarly, if then both and , so the five consecutive integers also contain two multiples of , in this case, and you reach the same conclusion as for .  
   Since any positive integer can be written as either or , for some integer , it must be true that is a multiple of for all positive integer values of .
5. . Since is an odd integer, it follows that and must be two consecutive integers, one a multiple of and one a multiple of . It follows that the product is a multiple of .

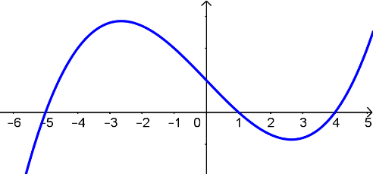
**Try it out (page 18)**

1. , so the last non-zero digit is the last digit of . The last digit of is , of is the same as the last digit of , so , and of is the same as the last digit of which, in turn, is the same as the last digit of , so . It follows that the last digit of , and therefore the last non-zero digit of , is the last digit of , that is .
2. The number has factors.
3. Since and , precisely any factor of that is also a multiple of can uniquely be written in the form where , and . There are such numbers.

**Try it out (page 20)**

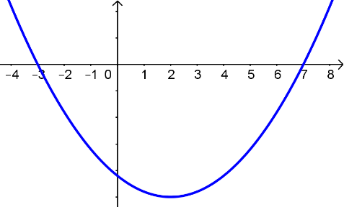
1. . The prime factorisation informs you that there are factors of and distinct pairs of positive integers that multiply together to give . It also gives a method for finding the factors quickly and establishing when a given pair will be two even numbers and therefore lead to an integer solution. Realising that, since and are positive integers, :  
     
   (1. and gives which does not have an integer solution.)  
   2. and gives , so , and : .  
   (3. and gives which does not have an integer solution.)  
   4. and gives , so , and : .  
   (5. and gives which does not have an integer solution.)  
   6. and gives , so , and : .  
   (7. and gives which does not have an integer solution.)  
   (8. and gives which does not have an integer solution.)  
   9. and gives , so , and : .  
   10. and gives , so , and : .  
   (11. and gives which does not have an integer solution.)  
   12. and gives , so , and : .

**Try it out (page 24)**



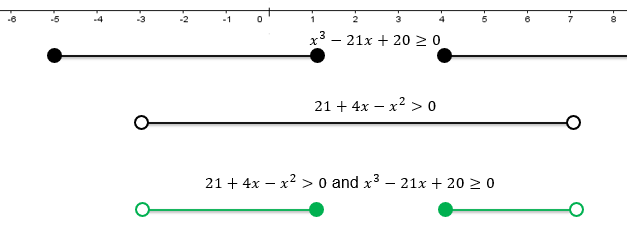
: Let . Since , is a factor of . Factorising gives .

Sketching the graph of shows that precisely when or .

  
  
.

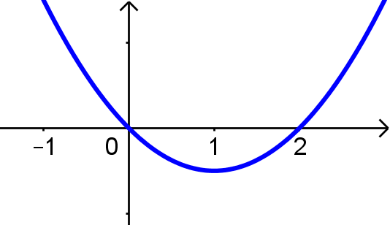
Sketching the graph of shows that precisely when .

Finally, the following diagram shows that both inequalities are both satisfied precisely when

or .  
  


**Try it out (page 26)**

* + - 1. The term in the expansion of is , so the coefficient is .  
         The coefficient of is the greatest of all the coefficients precisely when it is greater than the coefficients of both and .  
         The term in the expansion of is , so the coefficient is .  
         The term in the expansion of is , so the coefficient is .  
         Therefore all values of that satisfy both and .  
         . Since is clearly not a solution, and so the inequality is satisfied precisely when . (1)  
         . Since is clearly not a solution, and so the inequality is satisfied precisely when . Notice that could be positive or negative, hence why was factored out instead. Sketching the graph of shows that . (2)

  
Putting (1) and (2) together leads to the conclusion that both inequalities are satisfied precisely when .

* + - 1. Consider how terms can be obtained from where are non-negative integers satisfying . It is clear that has to equal to get the and that, therefore, has to equal to get the . It follows that is the only way to obtain an term, but there are lots of ways of getting this; for example, from the brackets you could choose the brackets from which to take the and then, from the remaining brackets you could choose the brackets from which to take the ( is automatically taken from the final brackets). It follows that appears in the expansion times.  
         Therefore the coefficient of in the expansion of is .

**Exercise 1**

* + - 1. Since has last digit , has last digit ,  
          has last digit , has last digit , has last digit , and has last digit , it follows that the square of any integer must end in one of and . In particular, no square of an integer has last digit (or or ), and therefore has no integer solution.
      2. The 3-digit number can be written . Reversing the digits gives and this can be written .  
         The difference between these two numbers is . Since and are digits and so integers, is also an integer, and is always a multiple of .
      3. Factorising gives . This product contains as a factor , which is the product of three consecutive integers. Every three consecutive integers contain one multiple of and at least one even number. Their product is therefore always a multiple of and, hence, so is .
      4. This is an extension of question 3, where it was established that is a multiple of for any positive integer .  
         It was also shown that . It is still left to show that is a multiple of when is a positive integer greater than .  
         If or are multiples of then the factorization shows immediately that is a multiple of .  
         If none of and are multiples of then either or must be a multiple of .  
         If is a multiple of , it can be written as for some integer, . Then and is a multiple of , since is an integer.  
         If is a multiple of , it can be written as for some integer, . Then and is a multiple of , since is an integer.  
         It follows that if none of and are multiples of then , and therefore are.  
         Combining this with the work for question 3, is a multiple of for any integer , and, in particular, for any integer .
      5. Factorising gives .  
         First consider . This is a product of three consecutive integers, since is a positive integer, and therefore one of the three integers is a multiple of . However, notice that the only prime factor of is ; in particular, cannot be a multiple of . Since this means that one of and must be a multiple of , it follows that is also a multiple of .

1. In prime factorized form, , so has at least and as prime factors, but, since it has factors, it can only have these two prime factors. Therefore has to take the form , for some positive integers and . Then . This has factors. For this to be requires and or vice versa. Either way, .  
   Since , this has .
2. Expressing numbers in their prime factorised form shows that

An integer is a cube number precisely when all of the powers in its prime factorisation are multiples of . So the smallest cube number that contains as a factor is . If then .

1. Using prime factorisation, . That is not a factor of this number but means that has to equal , so . For the number to be an integer, it is required that and . The only positive integer that satisfies both of the inequalities is .  
   With and , .
2. The highest power of in comes from

.  
The highest power of in comes from .  
Therefore the highest power of in the full expansion is .

1. Since , all that matters is the power of and the power when considering how many zero digits are at the end of a number. Also since, when multiplying (sufficiently small) consecutive integers, must be raised to a higher power than the power of .  
   In , there are multiples of , of which are multiples of , so the product ends in zero digits.  
   In , there are also multiples of , of which are multiples of , so the product ends in zero digit. The answer is that there is the same number of zero digits for the two products. However, ends in zero digits. Can you see why?
2. In prime factorized form, . This has factors.
3. It may help to first read the solution to Question 10. As explained there, is to consider the power of in . From the numbers from to inclusive: each multiple of adds to the power of ; of these, each multiple of adds a further to the power of ; of these, each multiple of adds another to the power of ; and so on.  
     
   Using this idea, the following table shows the first time that ends in zero digits:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Key |  | Multiple of adds a zero digit at the end | | | | | | | |
|  |  | Multiple of adds two zero digits at the end | | | | | | | |
|  |  | Multiple of adds three zero digits at the end | | | | | | | |
|  |  |  |  |  |  |  |  |  |  |
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This shows that Is the smallest number of the form to end in zero digits.

1. Using the law of indices, , it follows that, since , is both a square number () and a cube number ().  
   Since the question asks for two examples, you could give and .
2. Suppose there are two square numbers, and , for integers and , with the largest, that do have a difference of .  
   Then , so .  
   Since is raised just to the power of in the prime factorisation of , it follows that if any two integers multiply to give , then one must be even and one must be odd.  
   Therefore must be even and must be odd, or vice versa. Either way, you conclude that must be the result of adding an odd and an even number, in other words odd. Since is an integer, must be an even number. This contradiction means that it is impossible to write for any integers and .
3. The highest power of in comes from .  
   The highest power of in comes from .  
   However, the subtraction means that is not an term to raise to the power of .  
   The next highest power of in comes from .  
   The next highest power of in comes from . (There are choices for choosing which bracket to take the from. Then, automatically, needs to be taken from the remaining brackets.)  
   Fortunately, subtracting will not cancel the terms. Therefore, the highest power of before raising to the power of is , and the final answer is .
4. Suppose that (Pythagoras’ Theorem) for positive integers .  
   Then . Since are integers, the only possibility is and . Adding these equations gives , so and .  
   The only possible right-angled triangle has sides of length units, units and units.
5. Expressing numbers in their prime factorised form shows that   
   An integer is a cube number precisely when all of the powers in its prime factorisation are multiples of . So the smallest cube number that contains as a factor is , but this would require to equal , which is not possible, since an integer is a square number precisely when all of the powers in its prime factorisation are even.  
   Suppose that for some non-negative integers, and .  
   Then . The smallest value of such that is a multiple of is , the smallest value of such that is a multiple of is and the smallest value of such that is a multiple of is . Therefore the smallest possible value of is .
6. The given inequality can be expressed as a quadratic inequality in :
7. Thinking of as with . That is a factor of is equivalent to the statement that is a factor of .

Therefore, by the Factor Theorem:

,

.

1. Prime factorisation helps once again:

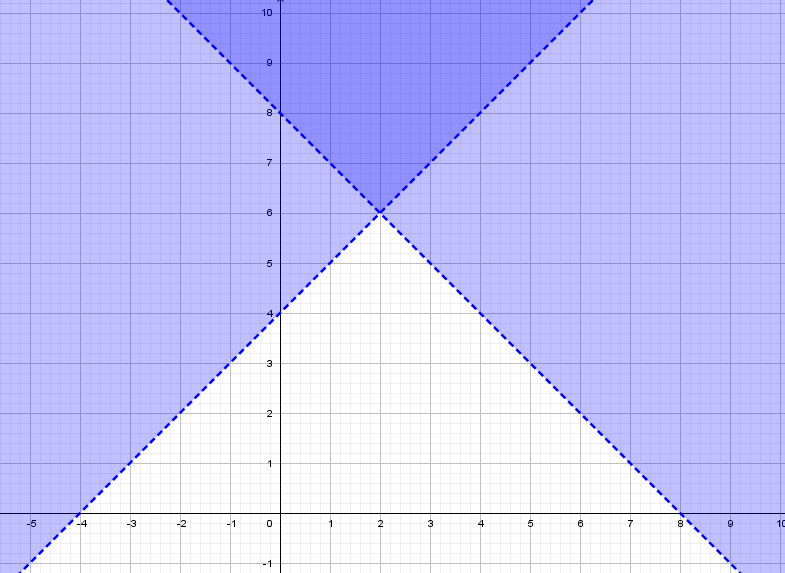
must be an integer that is a multiple of but not of or . That it must be an integer not divisible by forces .  
That it is divisible by but not forces that either and or and .  
Substituting into both inequalities gives and .   
Therefore, it is required that either and or and . Only the latter can be satisfied, and this happens when . Also, . Substituting these values gives:   
The answer is .

1. Consider non-negative integer values of and such that and . Since , and must satisfy the simultaneous equations and . Subtracting the first equation from twice the second equation gives , so and .  
   It follows that each term is formed via . The number of ways of doing this, which is also the coefficient of , is ( choices from to select the brackets from which is chosen, followed by selecting from the remaining brackets to select ; is automatically selected from the brackets not chosen). There are five other equivalent and equally valid solutions, depending on what order you consider and : for example, , , , , ... this is not an exclusive list!0

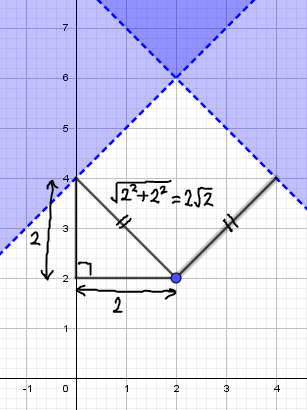
**Exercise 2 Admissions Test Multiple Choice Questions**

**TMUA style questions**

1. For the expansion to give a quartic polynomial, is necessary.  
   Looking at the constant term of shows that . Since is positive and real, is necessary.  
   The term in the expansion of is , so , since is positive. (Seeing the possible answers and the appearance everywhere of , does suggest the possibility of .)  
   The expansion is .  
   From this, .  
   The answer is B.
2. Thinking of the ways in which terms can be obtained from :  
    happens times;  
    happens times.  
   It follow that the coefficient of is .  
     
   The coefficient of in is .  
     
   So .  
     
   The answer is E.
3. A quick sketch is worth doing. The line passes through with gradient and passes through with gradient . By inspection or simultaneous equations, the two lines intersect at . The co-ordinates of all points in the unshaded area of the following image satisfy both inequalities.



Considering the four statements in turn:

* + - 1. : **not true** for every point in the region (for example, ).
      2. : **true** for every point in the region (note that does not quite lie in the region).
      3. , meaning inside, or on the circumference of, a circle of radius and centred at . The sketch below shows that the centre of the circle is never less than distance from either of the lines:  
           
           
           
         Since the dotted lines are not included in the region, the given statement must be **true** for every point in the region.
      4. , meaning all points below the quadratic curve given by the equation . This passes through and , so lies completely in the region: **true** for every point in the region.

The points in the region satisfy 2., 3. and 4. only. The correct answer is L.



and . Both of these inequalities are satisfied precisely when . This is also the solution to the inequality, . The correct answer is G.

1. The term in the expansion of is , so the coefficient is .  
   The term in the expansion of is , so the coefficient is .  
   Therefore, or .  
   If , then both polynomials become constants ( and , respectively) and there is no solution to the question.  
   The answer is B.
2. The term in the expansion of is , so the coefficient is . The term is , so the coefficient is .  
   Therefore, . Dividing both sides of the equation by gives .  
     
   The answer is B.
3. First search for the term in the expansion of . The highest power of in this expansion is , and from there the powers descend by two each time (from both losing an and gaining a . It follows that the term is formed as follows: . Therefore, the term in the expansion of is .  
     
   The answer is A.
4. The highest power of in this expansion is , and from there the powers descend by twelve each time (from both losing an and gaining a . It follows that the constant term is formed as follows: . The answer is G.
5. For any positive integer, , the expansion of begins , and (assuming ).  
   It follows that the term in the above expression is , and that this has coefficient . This sum is groups of where . Therefore the coefficient of is . The answer is C.
6. Although there are other, much longer, available methods for answering this question, the examiners do drop a hint that there must be an easier approach on this occasion. In particular, for example, why do they write instead of ? This must be done for a reason. If you can quickly spot a value that squares to give , then square rooting simply reverses the process. Considering :  
     
    and , so it is worth checking the expansion of … you are in luck!  
    and , so it is worth checking the expansion of … you are in luck, again, and grateful not to have ‘piled in’ to this question!  
     
   Therefore, .

The answer is D.

**MAT style questions**

1. (a): . The left-hand expression is a square number precisely when is, but this is not true for all integer values of and ( and , for example).  
   (b): There is no obvious reason to suspect that the given expression should be a cube number for all integer values of and , so it is worth looking for a counterexample. The value of the expression when is , and this is not a cube number. So this is also not true for all integer values of and .  
   (c): is a square number, so is a square number precisely when is. In particular, ‘ is not a square number’ is not true for all integer values of and ( and , for example).  
   (d): To show whether this can be written as a square number, you need to expand, simplify and then see if it is possible to factorise.  
   , showing that this is a square number for all integer values of and .  
   (e): Always consider that counterexamples may be easy to find. For example, if then , which is a cube number. This shows that ‘ is not a cube number’ is not true for all integer values of and .  
     
   The answer is (d).
2. Since , .  
   Differentiating gives precisely when or .  
   When , and .  
   It is crucial to read the question carefully and realise that and must be positive integers. The closest integers to are , giving , and , giving .  
   When and , .  
   When and , .

The correct answer is (c).  
  
(There is at least one other possible method. Following a similar trend of thought to that which lies behind linear programming, consider all values of and for which has the same value, say . Converting to Cartesian variables, the question is equivalent to finding points with integer co-ordinates where the line intersects the curve . Raising the value of causes the curve to stretch parallel to the -axis; eventually the line becomes tangent to the curve. Where the curve touches the line, the gradient of the curve must be , that is, the gradient of the line. This gives a third equation in addition to the equations of the line and the curve. You can use these to eliminate and to obtain an equation just in terms of just . This leads to and then you proceed as shown in the solution above.)

1. Realising that there is a common factor on both sides of the inequality is key:  
   .  
   So either and and and .  
    and gives and either or .

and gives and either or .  
This is option (c), and no other options fully describe the above solution.

1. This is a question that is worded to make you think that you cannot do it. There is no immediately obvious method to apply to these inequalities, so you have to consider that you may not need to!  
   The key piece of information that you are only asked to find integer pairs. There are only integer values of in ( and ) and integer values of in ( and ). Therefore, there are only integer pairs to be considered in the first place. The correct answer has to be less than or equal to . The conclusion is that the correct answer is (a) simply because it cannot be (b), (c), (d) or (e).

is an integer precisely when . The answer is (c).

1. The highest power of in is and the highest power of in is , so the highest power of in is .  
   The highest power of in is and the highest power of in is , so the highest power of in is .  
   It follows that the highest power of in is .

The answer is (e).

1. if . It follows that the inequality in the question is valid precisely when . The answer is (b).  
   (Note that would imply that the inequality in the question is correct, but the converse of this, which is what you want, is not true.)
2. Noting that there is a common factor on both sides of the inequality:  
   .  
   The graph of is a quartic curve that crosses the -axis at and . The coefficient of is positive so as . It follows that the graph is below the -axis precisely when or .

The answer is (a).

1. Expanding, simplifying and attempting to factorise:  
   .  
   Since is positive for all values of , it follows that . The answer is (b).
2. Consider the coefficient of one term (say the ) divided by the coefficient of the next term, for ascending powers of :  
   , for .  
    since . Then . It follows that but , and that, therefore, the term has the largest coefficient. The term is . (Realising that only one answer is a multiple of , it is not necessary to spend time calculating the coefficient). The answer is (d).

**Chapter 4: Geometry and coordinate geometry**

**Try it out (page 32)**

For a circle, the right hand side needs to be positive (and hence .

When the “circle” consists of a single point.

**Exercise 1**

1. There are 3 possible squares. The square with diagonal has midpoint The midpoints of the squares with side lie on the line through perpendicular to . Line has gradient , so the line through the centres has equation .
2. A quick sketch shows that the circles intersect so the answer to part (i) is 0. The points furthest apart lie on the line through the centres of the circles. The equations rearrange to give and . Hence the line through the centres, and , has equation . This line intersects when , giving the points ) and ). Similarly, the line intersects when , giving the points ) and ). From your sketch you can see that ) and ) which are distance apart.
3. . Intersects with when

. This gives or . When , . When , so the vertices of the trapezium are at and . The area of the trapezium is sq units.

1. The points and have coordinates and respectively (note that and ). The points and lie on the positive quadrant and and It follows that ), and ).

Therefore, and Since it follows that .

1. Considering the completed-square forms gives the equations and for the equations of the parabolas. Setting them equal at gives . Setting their derivatives equal at gives . Solving simultaneously gives and . So the parabolas have equations and . Their common tangent has gradient 2 and contains the point , so its equation is .
2. The lines have gradient 3 so the line is perpendicular to both and intersects the lines at the points and . The distance between these points is .

The radius of the circle is . The centre of the circle will lie on the line .

so let . Using the cosine rule .

so . Since passes through and passes through , it follows that is the point or the point . There are two possible solutions. One has the centre of the circle at the intersection of the lines and and the other has the centre at the intersection of the lines and . For and : , . For and : , .

The possible centres are at and . The equations of the two possible circles are

and

1. Let (so etc). First consider triangle , by the cosine rule . Triangles and are similar, so and hence . By symmetry, triangle is equilateral so triangle is similar to triangle with scale factor and hence the ratios of the areas is .

**Exercise 2 Admissions Test Multiple Choice Questions**

**TMUA style questions**

1. Neither circle contains the centre of the other circle and so the circles only touch when the centres are distance apart (the sum of their radii). It simplifies the algebra to work with the square of the distance between the centres and . Solving gives . Since , the correct answer is D.
2. A quick sketch shows that the line and parabola do not intersect. Let the closest points on the line and parabola be and respectively. The line is perpendicular to both the line and the parabola, hence the tangent to the parabola at is parallel to the line. The line has gradient and the gradient of the parabola is given by , so . The normal to the parabola at Q has equation which intersects the line at . The length . The correct answer is A.
3. The centre-radius forms for the circles are and . Neither circle contains the centre of the other and so the points that are maximum distance apart lie on the line passing through the centres. The distance between the centres is , and since the circles have radius 3 and 2 respectively, the furthest points are distance apart. The correct answer is C.
4. so the turning point has coordinates Letting denote the distance from the turning point to the origin gives . So the required values of are . The correct answer is D.
5. The tangent has equation . Substituting this into the equation of the circle gives a quadratic. The tangent meets the circle at a repeated root so you can find the value of by setting its discriminant equal to zero and solving.

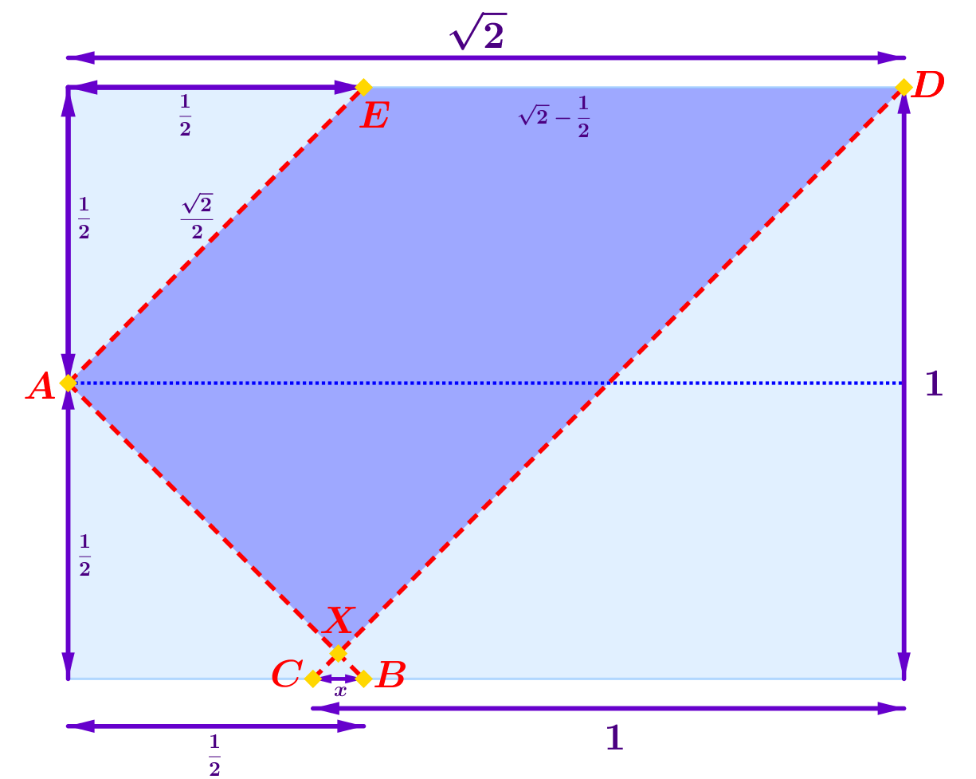
Repeated root when .

Since the tangent passes through the point and the positive axis, and hence the tangent intersects the positive axis when , so . The correct answer is C.

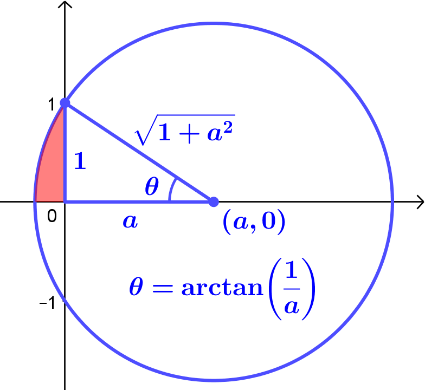
1. The centre-radius form of the equation of the circle is , so the circle has radius 4. The regular octagon consists of 8 isosceles triangles each formed by joining adjacent vertices to the centre of the circle. The area of each triangle is , so the area of the octagon is . The correct answer is E.
2. The line and the circle intersect when giving , so (note that there are 2 points of intersection provided that , and hence ). By considering the means of the coordinates of and you obtain (note that when computing the mean of the -coordinates of and the discriminant cancels, and the mean of the -coordinates of and is given by with ). By inspection, as varies, the coordinates of satisfy . The correct answer is B.
3. The centre-radius form for the circle is , so the circle’s centre, , lies vertically below the point . By symmetry, the other tangent meets the circle at the point . The correct answer is D.

**MAT style questions**

1. The graphs of and are reflections in the line . Hence they intersect on this line. Solving gives Setting the discriminant equal to zero (for a repeated root) gives , and hence . The correct answer is (c).
2. If the paper is unfolded, the following fold lines can be seen. The perimeter of the trapezium is required.



There are several isosceles right-angled triangles that can be seen. The length since it is the hypotenuse of a right-angled triangle with both of the other sides equal to . The length . The remaining lengths and require the side lengths of triangle to be known. This is a isosceles triangle with a right angle at . Let . Using the side that includes , giving . . From this . . The perimeter of the trapezium . The correct answer is (b)

1. The point to lies within the circle if and only if . Rearranging gives . The correct answer is (a).
2. A quick sketch rules out (b) and (d) since the vertical and horizontal tangents are and respectively. The point lies inside the circle (since ) and hence (c) can be eliminated. There are no obvious interior points for (a) and (e) so use direct substitution and check for a zero discriminant. Considering (a) first, you obtain which rearranges to with discriminant , so the line does not intersect the circle. Considering (e) first, you obtain which rearranges to give with discriminant 0. The correct answer is (e).
3. so the turning point of is . Hence the turning point of is , of is , and of is . The correct answer is (d).
4. When you need the image of the point when reflected in the line (which is ). Only option (b) gives this point so the rest can be eliminated. The correct answer is (b).
5. The chord formed by joining the points of intersection of the square with the circle is a diameter (as it subtends a right-angle). Hence is equal to the area of the semi-circle () plus the area of the triangle formed by the diameter with the point . This is 4 times the perpendicular height of the triangle which varies from 0 (when one side of the square is a tangent) to 4 (when the radius at is perpendicular to the diameter). Hence the maximum value for is . The correct answer is (d).
6. 

The area of the shaded region is . Setting this equal to (so that the area enclosed by the overlapping circles is ) gives . The correct answer is (b).

**Chapter 5: Functions**

**Two questions to think about (page 42)**

Completing the square gives

for all real so the maximum value of this is required.

This is when giving

There could be a local minimum between and

gives

Since , (i.e. also between and ) and (i.e. not between and )

From the shape of the graph of (since the coefficient of is positive), where there is a local minimum.

and

This is the minimum value for the given domain.

**Try it out (page 46)**

If then and if then and

and are calculated from so

and are calculated from and and are calculated from . As ,

The pattern will therefore be 1 positive 1 followed by 2 negative 1s followed by 4 positive 1s followed by 8 negative 1s and so on. The changes take place at terms where is a positive integer.

Since and , the final change will take place at .

so the correct answer is (a)

**Exercise 1**



For these three terms to be in a geometric sequence,

The simple counter example gives and so the terms are not necessarily in a geometric sequence

For these three terms to be in a geometric sequence,

For this case, the simplifies to and the simplifies to .

Assuming the arithmetic sequence for and the geometric sequence for are not the trivial sequences then the three terms do form a geometric sequence.

The correct answer is (b)

may give a local minimum in . is not in

so gives a local minimum

The minimum value in is

The correct answer is (b)

* + - 1. Most of the work for this has been done in the previous question. Since there are no other turning points, the maximum value is .

The correct answer is (d)

completing the square gives

This is maximum when

The maximum value is

* + - 1. is at its minimum value when and at its maximum when

is at its minimum value when and at its maximum value when .

The value of must lie between and (these are bounds – we do not know yet if the function can achieve these values, just that it can’t lie outside of these values).

There are only solutions for if and for the same values of

For , when . Since , there is only one solution in and (b) is the correct answer.

This is maximum when .

The maximum value is

The correct answer is (e)

Any term calculated as or will be . Terms calculated as will be if and if . The only values that will be will stem from and based on this. Hence, the only values that will be equal to are where is a non-negative integer.

In the numbers to , the powers of are so the product . Since is raised to a negative power, the product will be .

* + - 1. For to , the values of are and

Most of these are prime so their gpf will be simply the value itself. The gpf of 25 is . The gpf of 49 is , the gpf of is , the gpf of is and the gpf of is .

The prime factorisation of is

so the correct answer is (b)

* + - 1. This could be done by writing an exhaustive list but that would not be particularly efficient

A better way is to consider the possible totals that can be found using the two rules and use this to reduce the number of possibilities that need to be considered.

The rules lead to either a multiplication by or a division by .

Let be the number of multiplications by and the number of multiplications by . Both and are non-negative integer values.

Every possible total can therefore be calculated by

Since the total needs to be less than , since

The maximum number of steps overall is (otherwise values of for are ) so . The maximum value of is since would mean applying five times from which would give .

The possible values are and

These are the possibilities to focus on. Most can be quickly assessed using mental arithmetic.

corresponds to , corresponds to , corresponds to , corresponds to , corresponds to and corresponds to .

corresponds to but this is

For each of the others there is more than one value that can be reached for example represents two cases of and three cases of but these could be in any order e.g. .

gives either or

gives , or

gives , or

gives , , , , or

gives , , or

, and are slightly more awkward as some combinations go beyond the values being used. They also all lie on the line so their totals should be calculated as they may be higher than .

For example

gives , , , or but for only one of these, which is equal to

So

gives

gives , , , or and five others with

but , , , and so none of these are valid.

gives , , or and six others with

so all are valid

This gives a total of

The correct answer is (c)

* + - 1. Multiplication is commutative so the order of and operations is not important in terms of the result e.g. anything with two and one in any order will give required.

If function is applied times and function is applied times then the result will be

So

This can be written as i.e. so

The only possibilities are and since and must be positive integers and from the conditions in the question.

The arrangements that give must therefore have either

two applications of and one application of

or three applications of each

For the first of these there are possibilities (you are choosing two of the 3 positions to place the and the has to go in whatever place is left). You could even list these as , and .

For three applications of each, there are possibilities (you are choosing three of the six positions to place the ’s and the remaining positions must be occupied by ‘s.

Hence there are 23 different ways to generate and the answer is (b)

**Try it Out (page 50)**

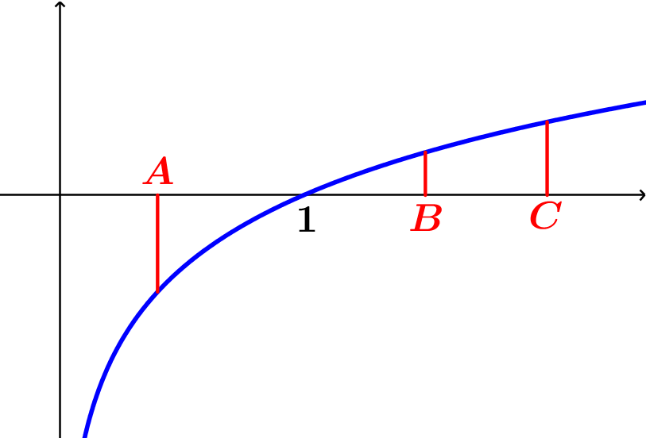
If then is not necessarily true for all and .

A simple counter example with , and demonstrates that this is the case

definitely does not imply that .

It is true for as it is an increasing function for all (and it doesn’t exist for any other values of ).

The graph of looks like this:



for would be an example where

**Exercise 2**

Let

so , so giving

so (since ) giving

so . This will need a comparison to

so . This will need a comparison to

Both and are between and . A number between and can be compared to both. This may take more than one trial if the number selected is either greater tan both or less than both and .

Comparing to : Assume so

this is true so

Comparing to : Assume so

this is not true so hence .

Note: if this had been inconclusive (e.g. both were ) then a different value between and would need to be used.

To compare and : Assume i.e. then i.e. this is true since hence

The correct order is i.e.

This can be done by estimating the values using the approximation

The correct order is , , ,

since

Since ,

since

Let

This means that so there are two equal values and the correct order is

, (since ), (since ),

The correct order is , , ,

So

So

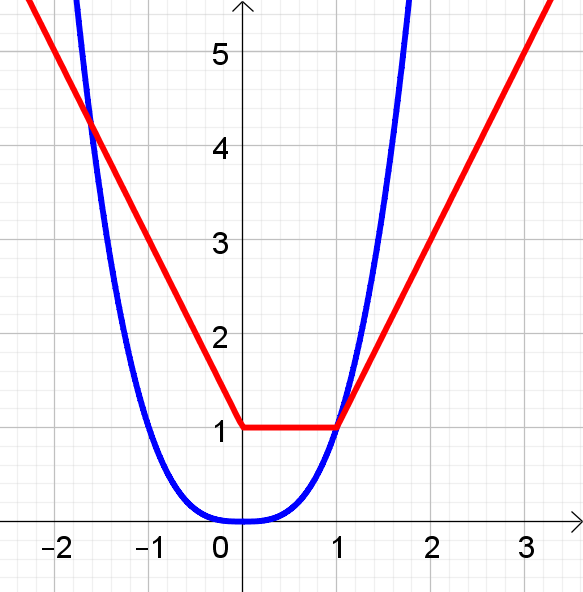
So

A sketch of the graphs of and will help.

For there are two critical values, where changes from being calculated as to being calculates as and where changes from being calculated as to being calculates as

Hence, for , can be sketched using , for it can be sketched using and for it can be sketched using

For , when , it can be sketched using



Both pass through the point .

Since the gradient of at is greater than , there are no further crossing points for

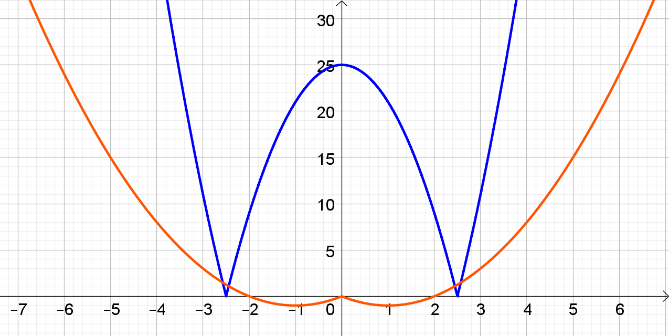
The graphs cross in two places so there are real solutions and the correct answer is (c)

This also requires a sketch.

so has two critical points at and .

For the graph can be sketched as

The graph of can be sketched as for and as for



The graphs cross at points so there will be real solutions.

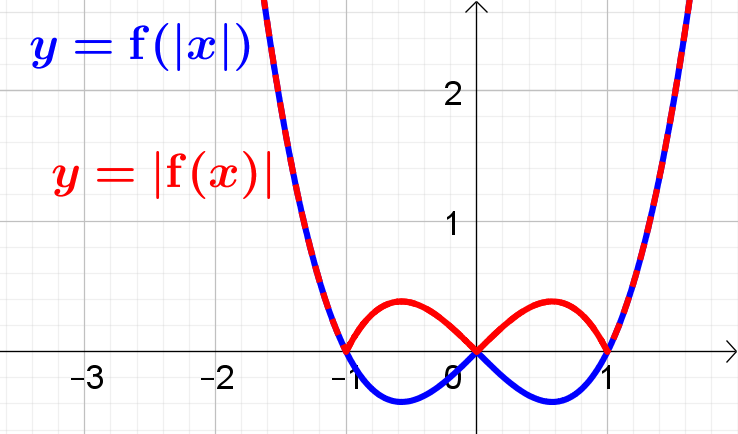
The correct answer is (a)

.

For , sketch

For sketch

This can be sketched by reflecting every point on where in the axis.

The two graphs will be the same for and

Area =

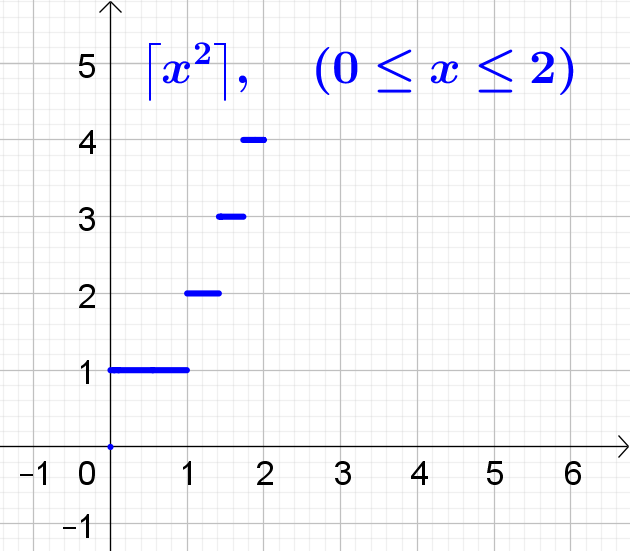
1. When , so

so

so

so

so



The integral will be the sum of the areas of the rectangles between each line section and the axis.

The correct answer is (b)

* + - 1. The integral can be separated into two parts

when

when so for ,

when so for ,

when so for ,

so for ,

so for ,

At

So the integral is equal to

The correct answer is (a)

**Exercise 3 Admissions Test Multiple Choice Questions**

**TMUA style questions**

Since , the values will repeat in sets of the seven values

so there will be 14 complete sets followed by

The answer is E

all the values will be so

For odd, the terms are

This is an arithmetic sequence with and .

is the th term in the sequence so

Hence .

The correct answer is B

, , ,

Geometric series with and so terms

The correct answer is C.

Left

The correct answer is C.

,

so possible values for are , and

If , which satisfies all of the required conditions

If , making untrue so reject this

If , making untrue so reject this

If , making untrue so reject this

If , making and untrue so reject this

If , making untrue so reject this

The only value that fits all criteria is so the correct answer is D

Let and so from the first equation A

From , B

From A:

Substituting this into B:

or

If ,

If ,

From : and

From : and

The maximum value of .

The correct answer is D

Let

but non-zero solution required

The correct answer is A

Let and

A

B

Substitute into A:

In B:

The correct answer is E

A

B

C

3A + B:

D

D – C:

The answer is D

Using : A

Using : B

Substituting A into B:

Using the factor theorem with :

so is a factor

Equating coefficients of :

For the quadratic factor, the discriminant shows that there are no further real solutions and the only real solution is .

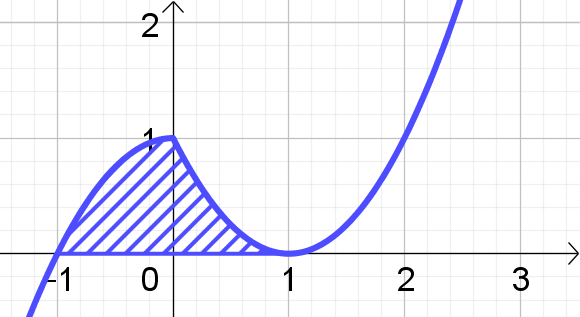
Substituting this into A: so

The correct answer is G

Let

For ,

For ,



The correct answer is D

* + - 1. can be rewritten as

This can be sketched using

for

for

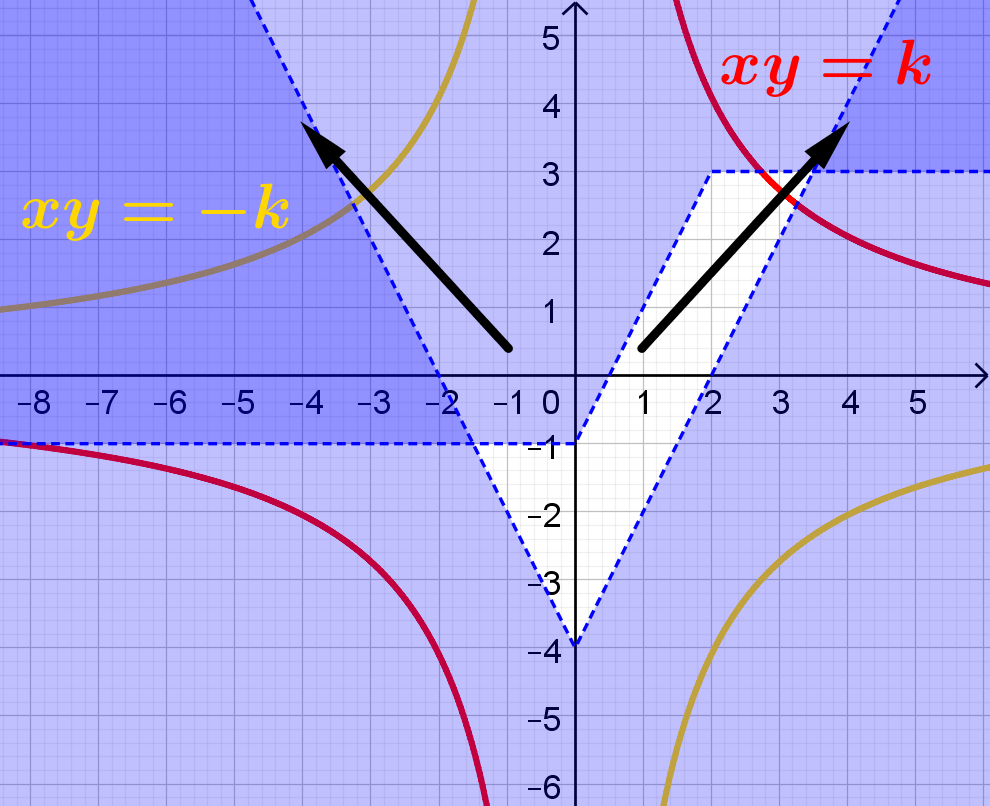
for

can be rewritten as

This can be sketched using

for

for

Let so either or

Consider the graphs of and as increases

The maximum value of occurs when the line crosses the line

At that point

The maximum value of is

The correct answer is B

**MAT style questions**

* + - 1. Completing the square for gives

This is maximum when

The correct answer is (d)

* + - 1. Using

Completing the square for

This gives its **least** value when

The correct answer is (b)

* + - 1. A mixture of approximation and calculation can be used for this question

and

So (a)

For (b)

As ,

Since (b) is greater than (a)

For (c) using gives

This shows that (c) is greater than (a)

For (d)

so (d) is greater than (a), (b) and (c)

For (e) as , so

The correct answer is (d)

* + - 1. For (a) so

For (b)

So (a) is less than (b)

For (c) . As , both are lower than (a)

For (d)

. As ( and ) this gives

this is lower than (a)

For (e) this is greater than (d)

To compare (c) and (d) some more precision can be used for (c)

Since so this is greater than (d).

The **least** value is (d).

* + - 1. This can be done by considering simple counter examples

For (a) and this is not true

For (b) and this is not true

For (c) and this is not true

For (d) and this is true as no square number ends in

For (e) and this is not true

The correct answer is (d)

* + - 1. Let so A

Another equation in is required

For so

Let so B

B :

A :

Subtracting:

So

The correct answer is (a)

* + - 1. The digits from to inclusive will sum to

The digits from to will be the sum of ten ’s and the digits to i.e.

For to this will be , for to this will be and so on

The correct answer is (b)

The correct answer is (d)

* + - 1. All of the values will be or . The only values that will be will stem from . They have to be odd powers of .

These are the only possible values in to since the next time occurs is for which is not a part of the sum.

There are six values that give and values that give .

Hence the sum is

The correct answer is (a)

1. Let

By the factor theorem:

This is true if is odd.

The correct answer is (a)

1. divided by remainder

divided by remainder

Using the remainder theorem

in gives A

Using the factor theorem

in gives B

From A

Completing the square for gives

From the options, if (options (a) and (e)) then

but this means that is not an integer as specified in the question so (a) and (e) can be eliminated.

If then and again, is not an integer so (b) can be eliminated

If then giving since , must be otherwise . So it is possible that (c) is correct with and

If then and again, is not an integer.

(c) is the correct answer.

A final substituting and into the LHS of B: gives

which is consistent.

The correct answer is (c)

1. The answer can be found by using a simple value for

If then

Using the remainder theorem with

It looks like (c) is the correct answer. To confirm this the other values can be checked

(a) , (b) , (d) , (e)

The correct answer is (c)

1. From A

From B

From C

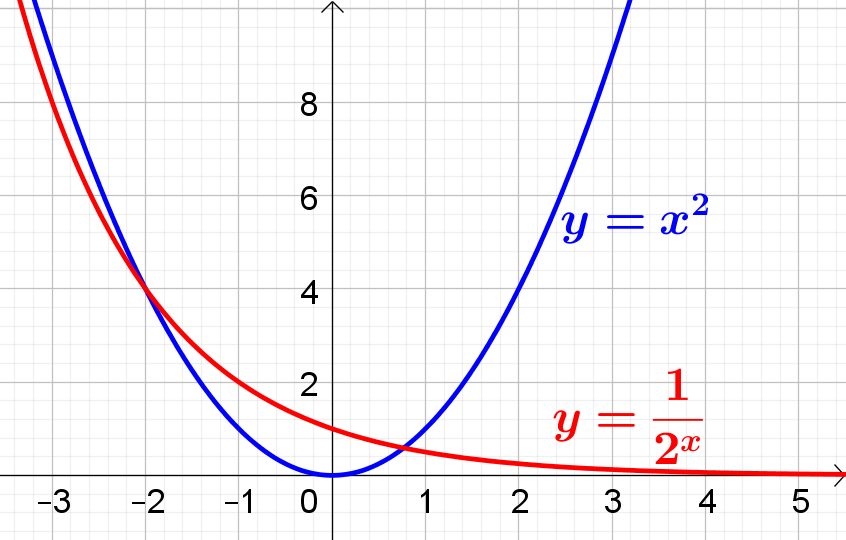
Since , and all feature as the base of a logarithm, all three must be positive.

Substituting B into A gives so and so since

A becomes , B becomes and C becomes i.e.

Substituting into gives

To see if this gives unique values for , the graphs of and can be sketched



There is one solution for (negative values can be ignored since you know that )

As there is a unique value for , since , there is also a unique value for .

and all have unique solutions.

The correct answer is (e)

1. The first thing to note is that as is the base of a logarithm, it must be greater than .

The equation can be rewritten as

(\*)

There are three solutions to the equation (\*)

When , which is not greater than .

When , which is not greater than .

When , which is greater than .

So x = 3 is the only solution to the equation.

The correct answer is (b)

1. The expression can be written

and from this

For this to have a repeated root, the discriminant must be equal to .

Let

The correct answer is (d)

1. for

for

When ,

The correct answer is (c)

1. The integral can be written as a difference of two integrals

The first integral can be considered to be the area of a rectangle of base units and height units

When ,

For , so

For , so

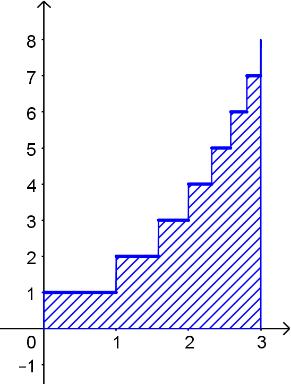
For , so

For , so

Since is a natural number, it follows that the value of at the upper limit is

For , so

The integral will be the sum of several rectangular areas



It is a good idea to express the base length of each rectangle entirely in terms of logarithms to base 2.

Rectangle 1 has an area of

Rectangle 2 has an area of

Rectangle 3 has an area of

The final rectangle has an area of

The correct answer is (c)

There is a quick way to answer the question by selecting simple values for . Two of the options are written with this in mind. Using gives options (a), (b) and (d) as well as (c).

Using gives option (d) as well as (c). Using would identify (c) as correct.

**Chapter 6: Calculus**

**Try it out (page 58)**

1. .
2. .
3. , so and .
4. , so . The gradient of the tangent to the curve where is . It follows that the gradient of the normal to the curve where is .
5. . If then , meaning that the gradient of the normal on the curve where is . The co-ordinates of are . Using the gradient of the normal, the co-ordinates of are and the co-ordinates of are .  
   Then, realising that, for a straight line, ratios of lengths along the line are the same as the ratio of the horizontal distances travelled or the ratio of the vertical distances travelled, you can conclude that (using the co-ordinate of each point). There is no need to waste time using Pythagoras’ Theorem!
6. , so the gradient of the tangent at is and the equation of the tangent is .  
   Since lies on both the line and the curve, and . Substituting one equation into the other gives . Since , is the only valid solution.
7. Let the co-ordinates of the point of contact of the two curves be .   
    so .  
   Also, if then .  
   Since the gradients of the two curves are equal at , it follows that .  
   The point also lies on both curves, so and , giving , so . This gives

**For the MAT only**

1. and .
2. and .
3. and .

**Try it out (page 61)**

1. Let . Then precisely when or . When , . When , . Knowing the shape of a cubic curve when the coefficient of is positive, it follows that , above the -axis, is a local maximum, and , below the -axis, is a local minimum. Since as and and . The curve therefore crosses the -axis at three distinct points, meaning that the equation has three distinct roots.
2. Let . Then (since . It follows that this cubic curve has no stationary points and, in particular, no turning points. It therefore crosses the -axis just once; the equation must have only one (real) root.
3. Let . Then (spotting that when and applying the Factor Theorem to find the first linear factor). This is a quartic curve. When , and, when , . Therefore, the only two stationary points are and . Also, since the coefficient of is positive, as . This information is enough to determine that is a local maximum and that is stationary point of inflection (just try sketching the curve; it may also help to first sketch the graph of ). It follows that the equation has two roots (one of which is repeated).

**Exercise 1**

Let . Then .

Let

or

,

,

is to the left of and above it. Using the shape of a cubic with a positive coefficient of indicates that gives a local maximum and a local minimum. To do this, the curve must pass through the axis once only. The equation therefore has one real root.

1. Let . Then . When , , and, when , . The only two stationary points are , above the -axis, and , below the -axis. The coefficient of is positive, meaning that as and and . Sketching the curve shows that the curve crosses the -axis at three distinct points. The equation therefore has three real roots.
2. Let . Then .

or

,

,

is to the left of and above it. Using the shape of a cubic with a positive coefficient of indicates that gives a local maximum and a local minimum. To do this, the curve must pass through the axis at only one point. The equation therefore has one real roots.

1. Let . Then . When , . Since , ; all that matters is that there is a stationary point at which is above the -axis. When , . There is a stationary point at that is below the -axis. Since the coefficient of is positive, as and and . Sketching the curve shows that when the stationary point is a local maximum and when the stationary point is a local minimum. In particular, the curve crosses the -axis at three distinct points. The equation therefore has three real roots.
2. Let . Then (spotting that when and applying the Factor Theorem to find the first linear factor). When , . When , . When , . Since the coefficient of is positive, as . This information is enough to determine that the graph crosses the -axis at four distinct points. It follows that the equation has four distinct real roots.
3. Let . Then

or

,

,

, this is possibly a point of inflexion

, is a local minimum.

From the general shape of a quartic, this can only cross the axis twice. It follows that the equation has four distinct real roots.

1. . The discriminant, , for this quadratic is

meaning that has no real roots; there are no stationary points.

1. , so it follows that there is just one stationary point. (In fact, computing shows the same thing).
2. . The discriminant of this is , meaning that has two distinct real roots. (That is not a square number means that the roots are not rational numbers, but the question does not ask for where the stationary points are but simply how many there are). There are therefore two stationary points.
3. (the discriminant of is , so this helps realise that the quadratic function is the square of a linear factor). It follows that there are two values of , and , at which . The function has two stationary points.
4. . However, since for all real values of , . In particular, has no real solutions and the function has no stationary points.

.

Possible linear factors are

and clearly won’t work.

Try :

so is a factor

Equation coefficients of : so

of : so

of : so

To factorise the quartic notice that the first three terms are , and

This is the difference of two squares so the equation becomes

1. In summary, which factorises to so gives

There is one stationary point when for , the discriminant is negative so there will be no stationary points that can be calculated from that factor. The discriminant of is positive giving two further real solutions (neither are equal to ) and therefore indicating two further stationary points.

There are three stationary points in total.

1. Find the value(s) of for which the function is a quadratic function with a maximum turning point at .  
     
   . When , , so . (That the value of is the same as the value of in the question is a coincidence; in solving the cubic equation in it was necessary to find that is a root in order to realise that is a factor, or perhaps the expansion of is something worth recognising?).  
   Note that the solution is not yet complete. It has been established is that is the only value for which there is a stationary point at . It could be a maximum or a minimum.  
   When , the quadratic function is , since the coefficient of is negative, it follows that the stationary point at is a maximum turning point.  
     
     
   An alternative approach would be to realise that any quadratic function that has a maximum turning point at can be expressed as , for real numbers and with . In particular, the coefficient of is four times greater than the coefficient of , so . From here, the solution continues as above.  
     
   Either way, the correct answer is .
2. . This has exactly one real stationary point when the discriminant of is zero:  
    precisely when .
3. . When , . Since this is independent of , for all values of . However, a stationary point need not be a minimum turning point. . If a stationary point satisfies then it is a minimum turning point. At , this happens if . The answer is (d).
4. . When , precisely when or . This is the condition for there to be a stationary point, but not necessarily for there to be a maximum.  
   When : , and . When , and . It follows that the stationary point at is a local maximum.  
   When : , and . When , and . It follows that the stationary point at is a local minimum.  
   When : , and . When , and . It follows that the stationary point at is a local maximum.  
   Therefore or is a necessary and sufficient condition for there to be a local maximum at . The answer is (f).
5. . Realising that does help suggest that the Factor Theorem may help, and it does! When , , so . Therefore the number of stationary points on the quartic curve given in the question is either , or depending on whether the discriminant of is negative, zero or positive respectively. The discriminant of is .  
   The curve has one stationary point (at ) when .  
   The curve has two stationary points (at and at the repeated root of the quadratic function) when or .  
   The curve has three stationary points (at and at the two distinct roots of the quadratic function) when or .

**Try it out (page 63)**

Factorising, . This curve crosses the -axis at , and . For , the curve is above the -axis (since negativepositivenegative is positive). It follows that, for , the curve is below the -axis and, for it is above (since the curve crosses and never touches the -axis).  
Therefore, the requested area is equal to square units.

**Try it out (page 64)**

Where the curve and the line intersect, or . Knowing the general shape of a cubic curve which has a positive coefficient of , it follows that , and the curve is below the line, when either or , and that , and the curve is above the line, when either or . The requested area is therefore square units.

**Try it out (page 68)**

1. , since the graph lies on the -axis between and .
2. For , the required area is a rectangle: .
3. For , the required area is a trapezium whose parallel sides, unit apart, have lengths and : .
4. Since is being substituted in the values returned by the function will depend on this

so and are not possible

when , when i.e. or

when , when i.e. or

when , when i.e. or

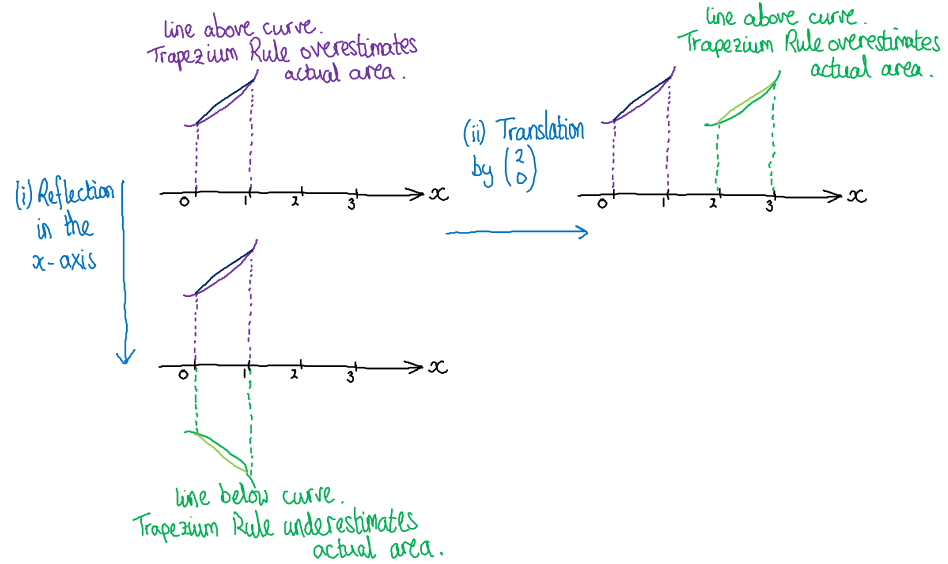
will be an even function so the integral can be found using the limits and and then doubling the result



Since , for all values of in

**Exercise 2**

1. 1. and   
        
      Where the graphs of the two functions intersect, or . For , , so the graph of the line is above the graph of the curve.  
      .
   2. and   
        
      Where the graphs of the two functions intersect, or . For , , so the graph of the first curve is below the graph of the second.  
      .
   3. and   
        
      Where the graphs of the two functions intersect, or . For , which, for positive values of , implies that , so the curve is above the line.  
      .
   4. and   
        
      Where the graphs of the two functions intersect, (twice) or . (Noticing that and , if you are lucky, would speed up the factorisation even more). Knowing the shape of a cubic curve with positive coefficient of and appreciating that the -axis is tangent to the curve at , one can deduce that , and therefore , when .  
      .
   5. and   
        
      Where the graphs of the two functions intersect, or . Knowing the general shape of a cubic curve with positive coefficient of , it is clear that , and therefore , for , and that , and therefore , for . The total requested area is therefore .
2. The Trapezium Rule with two equal width trapezia would use and . Performing the Trapezium Rule gives . (1)  
   With four trapezia of equal width, and are used, giving , so that . (2)  
   The Trapezium Rule with one trapezium of would be . However, subtracting equation (1) from equation (2) gives so .
3. Using the Trapezium Rule with five trapezia of equal width provides an overestimate for both and . Prove that using the Trapezium Rule with five trapezia of equal width also provides an overestimate for .  
     
   That the Trapezium Rule with five equal-width trapezia provides an overestimate of means that .  
   That the Trapezium Rule with five equal-width trapezia provides an overestimate of means that .  
   Then the Trapezium Rule, with five equal-width trapezia, of , is .  
     
   1. The graph of is a reflection in the -axis of the graph of . Therefore, any two diagrams showing the Trapezium Rule, each of the same number of trapezia and applied to the same interval of values for , applied to and to , must also be reflections in the -axis of each other. Since the Trapezium Rule with five equal width trapezia overestimates , overall, the (possibly) sloped sides of the trapezia must be above the graph more (in terms of area) than they are below. Reflecting this in the -axis, however, would result in the sloped sides of the trapezia now be below more than they are above. Therefore, the Trapezium Rule with five equal width trapezia must underestimate .  
      (See the diagram after the answer to a) for an illustration of this).

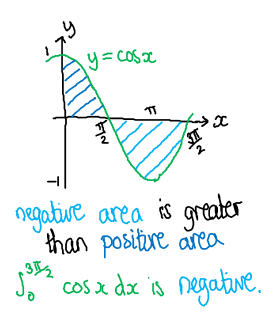
The graph of is a translation by of the graph of . You know that the Trapezium Rule with five equal width trapezia overestimates . The diagram illustrating the Trapezium Rule with five equal width trapezia applied to is just a translation by of the diagram illustrating the Trapezium Rule with five equal width trapezia ; it follows that this also gives (the same) overestimate as before.  
  
Here is a sketch that demonstrates a) and b). It is not a proof, but it illustrates the ideas behind this question when just one trapezia is used in the Trapezium Rule:  
  


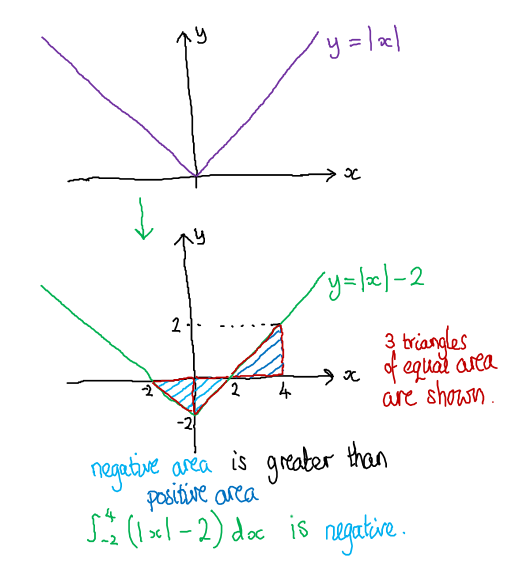
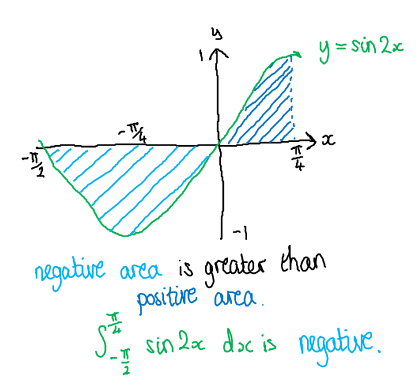
b)

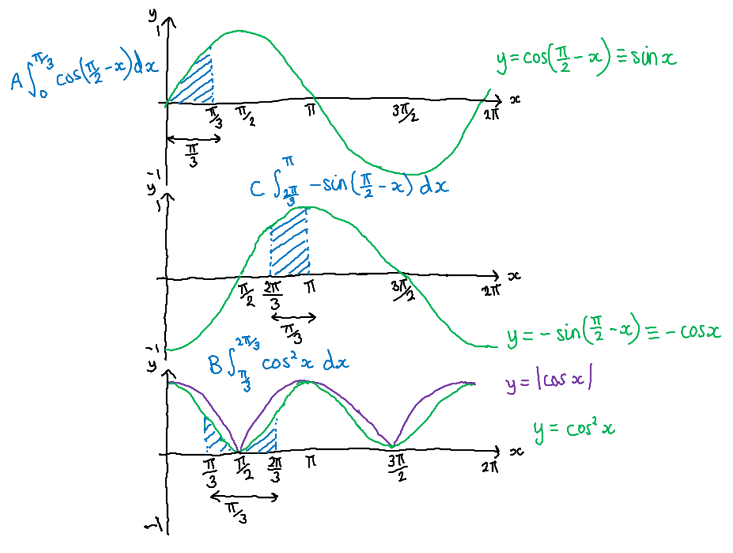
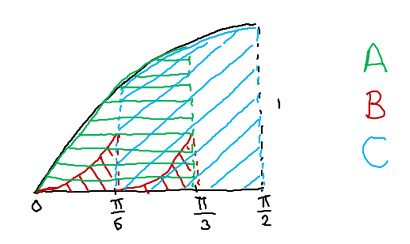
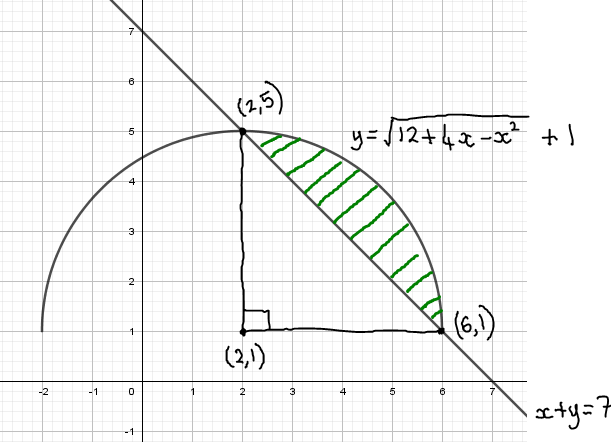
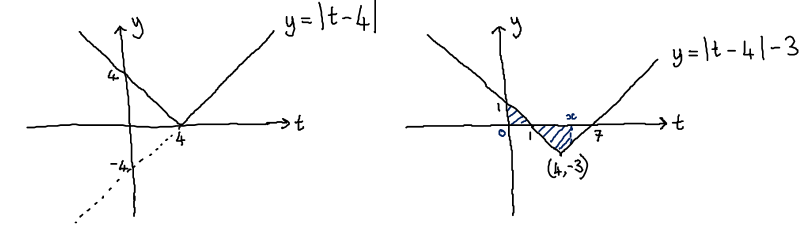
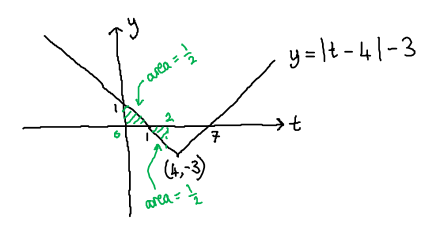
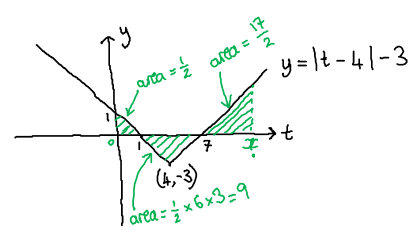
a)

. Since the Trapezium Rule with five equal width trapezia gives an overestimate when applied to either or , by a), it must give an underestimate when applied to . Applied to , it could, therefore, give either an overestimate or an underestimate depending on which is bigger, the area by which is overestimated or the area by which is underestimated. It could even happen by chance to calculate the exact area!  
  
An alternative approach could be to apply b) and realise that if gives an overestimate then gives an underestimate. Since you could relabel as and vice versa, it follows that both an underestimate and an overestimate are possible.

**Exercise 3**

1. All that is needed here is to determine whether each of the definite integrals are zero, positive or negative. Simply sketching the curve of each integrand should show this.  
   :   
   

:  
  
:  
  
Therefore is negative negative negative, which is negative.

1. The term in that has the highest power of is . Differentiating this twice gives .  
   The term in that has the highest power of is . Differentiating this once gives .  
   Since (notice how it was not necessary to spend time computing the value of or ), it follows that the highest power of is and the coefficient of this power of is .
2. If , and . Substituting these into the differential equation gives . Since , it follows that or .
3. For A and C, the identity helps.  
    and .  
   A quick sketch of the graph of each of the integrands indicates that C has the largest value.  
     
   C has the largest area. From a sketch this is less obvious than from a drawing that is to scale. The following diagram may help, in which each of three areas have been placed inside the area represented by .  
     
     
   The correct option is C.
4. It is easy to go on cruise control and approach this as you would do normally, solving simultaneous equations to find where these graphs intersect and then integrating, but is a lot of work, unguided, to integrate and you have to be aware that you may be wasting a lot of time.  
     
   On closer inspection . So is the equation of the part of the circle, with centre and radius units, for which (notice the one-way implication sign in the working above, and realise that is equal to a positive square root). Sketching this semi-circle and the line , passing through with gradient , one quickly establishes the following picture:  
     
     
     
   It follows that the requested area is simply the area of the quarter circle, , with the area of the right-angled triangle, , removed. This is , and this has answer has been reached without the need to do any integration at all!
5. A sketch here is essential:  
     
   Taking area below the -axis as negative, the question requires you to find all values of such that the area from to is zero, in other words, the area above the -axis is the same as the area below. From the sketch you can see that this will happen for just two values of .  
     
   If , there is are two congruent triangles, one above the -axis and one below, so the value of the integral is zero.  
     
   As increases from here, the value of becomes negative and decreases to its lowest value when . From there, it increases, reaching zero one more time.  
     
   The question only asks for how many solutions the original equation has; you can already say that there are two solutions, answering the question.  
     
   (If you were asked to find what the value of is in the above diagram then you could do the following. Area above Area below gives since .  
   So . Since , it follows that .)
6. Integrating, with respect to , both sides of the equation between and gives the following:  
   .  
   However, realising that both (this is just relabelling the variable) and (since the graph of is a reflection in the -axis of the graph of , the area between and will be the same for both graphs), the above simplifies to what is shown below:  
    where .  
   So .

**Exercise 4 Admissions Test Multiple Choice Questions**

**TMUA style questions**

1. You could compute this directly, although this is fiddly and it is easy to make a mistake.  
   It is always worth making a quick check first of all to see if there is any other easier approach.  
     
   Let . Then . Any function, , such that , is called an odd function. Although this is not currently included on school syllabuses, it is helpful to know a bit about odd and even functions, and why they are useful.  
   In particular, the graph of any odd function, , since if is a point on the graph then so is , has rotational symmetry of order two about the origin. This means that for any odd function and any real value of , since the area to the right of the -axis is the negative of the area to the left.

As soon as you realise the above, you instantly know that the answer must be zero. The answer is A.

The integral is actually an improper integral (consider what happens when in ).

1. The first curve has equation . The maximum point on this curve is .  
   The second curve has equation . The maximum point on this curve is .  
   At this stage, you would normally be using Pythagoras’ Theorem to find expressions for the distance between these two points and finding the value of that minimises this expression.  
   However, pause and look at the pair of co-ordinates. Each ordinate is either always zero or a multiple of , so, in particular, setting would make all ordinates zero at the same time. Since when the two points are the same (namely, the origin), it follows that the shortest possible distance between them is zero. The answer is A.  
     
   This is another good example of a question from which you benefit if you pause before diving into the usual method!

You could even have spotted this by considering the original equations with as both being .

1. The last of the three equations can also be written as , since drawing the graph of from to is the same as drawing the graph of from to , so the areas are the same.

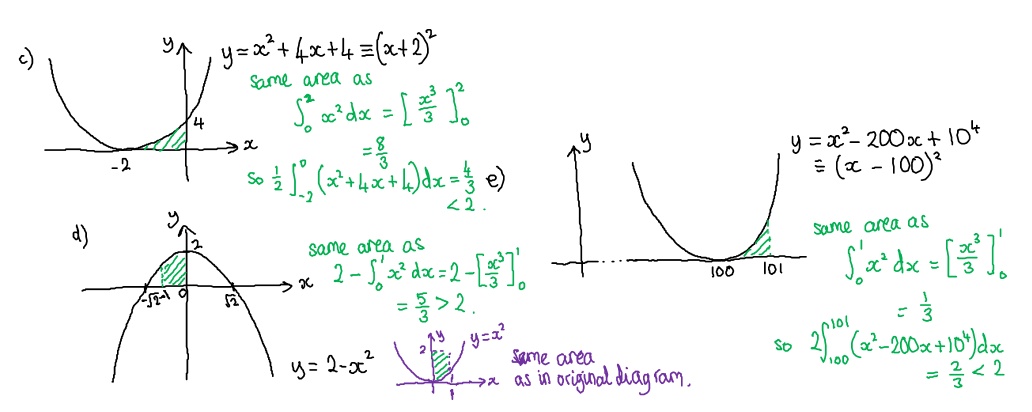
So

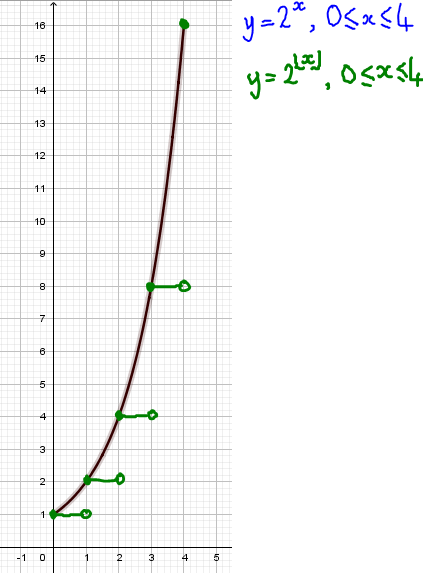
Then . The answer is D.

1. Let .  
   When , .  
   When , .  
   When , .  
   When , .  
   When , .  
   Applying the Trapezium Rule to , with four trapezia of equal width, gives:  
   .  
   The answer is A.
2. Consider the function . Then precisely when or .  
   If then .  
   If then . The two turning points are at and .  
   It follows that the equation has three distinct real solutions when . The answer is D.

**MAT style questions**

1. First consider the graph of . Then precisely when . When , . Since is positive when , is the only stationary point on the curve, and it is a minimum turning point. It follows that the equation has: no real solutions if ; one (repeated) real solution if ; two real solutions if . In particular, has exactly two real solutions. The answer is c).
2. If The co-ordinates of the minimum turning point on the graph of are . Then the co-ordinates of the minimum point on must be and the co-ordinates of the minimum point on must be . The answer is b).
3. The distance of any point, , from the origin is , and this is greatest when is greatest. If the point also lies on the above ellipse then . This is greatest when , giving , so the greatest distance from the origin is . The answer is c).
4. A quick sketch will shows that and

c), d) and e) are quadratic curves. All quadratic curves are a transformation of the graph of : is a translation by of ; is a reflection in the -axis of followed by a translation by ; is a translation by of . It may be easier to find an equivalent area below or above the curve.  
  
d) has the greatest value (in fact, the work above establishes that e) c) a) b) d)).

1. 
2. Use your graph to find when
3. This is just a sum of the areas of all of the rectangles in the above diagram. Each rectangle has width one unit, so it is a sum of the heights.  
   .  
     
   (Notice that the height of the last rectangle is if is an integer. This is worth bearing in mind for when you deal with bigger numbers in the later parts of the question.)
4. This is now the sum of the areas of the first two rectangles and half of the third.  
   .

Show that   
  
Realise that for all values of , with equality only at integer values, so for any positive value of . The diagram demonstrates this for .  
  
In a similar way, . You can obtain the value of by using the formula for the sum of terms in a geometric sequence, but it is useful to know that the sum of the first positive powers of two is always one less than the next power of two.

1. Find

If then , so applying a one-way stretch, scale factor , parallel to the -axis, to the graph of from to , gives the graph of from to .  
The quick way to answer this question is therefore to realise that .

Show that

if and are both positive integers

This is a generalisation of what was done above. Applying a one-way stretch, scale factor , parallel to the -axis, to the graph of from to , gives the graph of from to . This suggests that .

Find the exact value of

Give you answer in the form where and are positive integers.

Applying a one-way stretch, scale factor , parallel to the -axis, to the graph of from to , gives the graph of from to . So .