AQA Level 2 Certificate in Further Mathematics Exam Practice Second Edition: Full Worked Solutions

1 Number and algebra I

Exercise 1.1 Numbers and the number system

1 $0.037 \times 54 = 1.998\,\text{kg}$ [2 marks]

2 136% of £17 $= 1.36 \times 17 =$ £23.12 [2 marks]

3 **(i)** 414 : 270 (or any equivalent ratio of integers, e.g. 23 : 15) [1 mark]

(ii) $1 : \frac{15}{23}$ (or any equivalent fraction) [1 mark]

[Total: 2 marks]

4 $4\frac{1}{6} : 1\frac{1}{4} = \frac{25}{6} : \frac{5}{4} = \frac{50}{12} : \frac{15}{12} = 50 : 15 = 10 : 3$ [2 marks]

5 94.4% of £23.59 $= 0.944 \times 23.59 = 22.268\,96$, which rounds to £22.27 [2 marks]

6 $\frac{3}{4} - \frac{2}{5} \div \frac{7}{8} = \frac{3}{4} - \frac{2}{5} \times \frac{8}{7} = \frac{3}{4} - \frac{16}{35} = \frac{105}{140} - \frac{64}{140} = \frac{41}{140}$ [3 marks]

7 $9\frac{3}{4} + 3\frac{1}{5} \times 2\frac{3}{4} = 9\frac{3}{4} + \frac{16}{5} \times \frac{11}{4} = 9\frac{3}{4} + \frac{4}{5} \times \frac{11}{1} = 9\frac{3}{4} + \frac{44}{5} = 9\frac{3}{4} + 8\frac{4}{5} = 17 + \frac{15}{20} + \frac{16}{20}$

$= 17 + \frac{31}{20} = 17 + 1\frac{11}{20} = 18\frac{11}{20}$ [3 marks]

8 **(i)** $\frac{13}{13 + 5} = \frac{13}{18}$ [1 mark]

(ii) $\frac{13}{18} \times 100 = 72.2\%$ [2 marks]

[Total: 3 marks]

9 **(i)** $\frac{3}{5} = 60\%$ [1 mark]

(ii) LCM of 6 and 5 is 30, so lowest possible number of arrows fired is 30 each.
Linda's hits to misses are 25 to 5, and Alan's hits to misses are 18 to 12.
Least possible difference $= 25 - 18 = 7$ [2 marks]

[Total: 3 marks]

Exercise 1.2 Simplifying expressions

1 **(i)** $x^2 + 2xy - xy + x^2 = 2x^2 + xy$ [1 mark]

(ii) $6p^2 - 8pq - 7pq - 35q^2 = 6p^2 - 15pq - 35q^2$ [1 mark]

[Total: 2 marks]

2 **(i)** $xy(x - y)$ [1 mark]

(ii) $2pq^2(4p^2 - 3q^3)$ [1 mark]

[Total: 2 marks]

3 **(i)** $a^2 + 6a - 4a + a^2 = 2a^2 + 2a = 2a(a + 1)$ [1 mark]

(ii) $p^2 + pq - 3p - pq + 7p = p^2 + 4p = p(p + 4)$ [1 mark]

[Total: 2 marks]

4 **(i)** $2 \times 3 \times a^2 \times a \times b \times b^3 = 6a^3b^4$ [1 mark]

(ii) $3 \times 2 \times 5 \times x^3 \times x \times y \times y^3 \times y^2 \times z = 30x^4y^6z$ [1 mark]

[Total: 2 marks]

5 **(i)** $\dfrac{3}{4mn^3}$ [1 mark]

(ii) $\dfrac{5xy}{8x^3} \times \dfrac{6xy^3}{25x^4y^2} = \dfrac{xy}{4x^3} \times \dfrac{3xy^3}{5x^4y^2} = \dfrac{3x^2y^4}{20x^7y^2} = \dfrac{3y^2}{20x^5}$ [1 mark]

[Total: 2 marks]

6 **(i)** $\dfrac{2q}{pq} + \dfrac{5p}{pq} = \dfrac{2q + 5p}{pq}$ [1 mark]

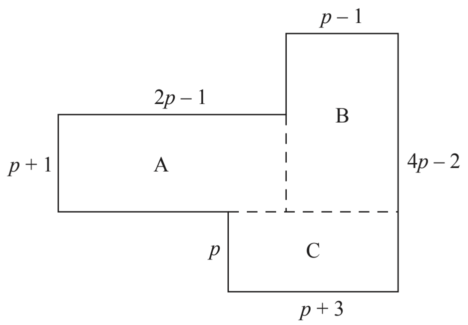
(ii) $\dfrac{x^2}{3x} + \dfrac{6}{3x} - \dfrac{5}{3x} = \dfrac{x^2 + 1}{3x}$ [1 mark]

[Total: 2 marks]

7 $\dfrac{x}{y} + \dfrac{2x}{3} \times \dfrac{5}{y} = \dfrac{x}{y} + \dfrac{10x}{3y} = \dfrac{3x}{3y} + \dfrac{10x}{3y} = \dfrac{13x}{3y}$ [2 marks]

8 **(i)** Perimeter $= 2 \times \big((4p-2) + (p-1) + (2p-1)\big) = 2(7p-4) = 14p - 8$ [1 mark]

(ii)



Area A $= (2p-1)(p+1) = 2p^2 + p - 1$

Area B $= (p-1)(3p-2) = 3p^2 - 5p + 2$

Area C $= p(p+3) = p^2 + 3p$

Total area $= 6p^2 - p + 1$ [2 marks]

[Total: 3 marks]

9 **(i)** $\dfrac{1}{2} \times 5x \times 12x = 30x^2$ [1 mark]

(ii) $5x + 12x + \sqrt{(5x)^2 + (12x)^2} = 13x + 17x = 30x$ [2 marks]

[Total: 3 marks]

Exercise 1.3 Solving linear equations

1 **(i)** $2x - x = -3 - 5 \quad \Rightarrow \quad x = -8$ [1 mark]

(ii) $5y - 3y = 7 + 3 \quad \Rightarrow \quad 2y = 10 \quad \Rightarrow \quad y = 5$ [1 mark]

[Total: 2 marks]

2 **(i)** $6x - 12 = x + 7 \quad \Rightarrow \quad 5x = 19 \quad \Rightarrow \quad x = \dfrac{19}{5}$ (or any equivalent value, e.g. 3.8) [2 marks]

(ii) $20x + 12 = 6x - 23 \quad \Rightarrow \quad 14x = -35 \quad \Rightarrow \quad x = -\dfrac{35}{14} = -2.5$ [2 marks]

[Total: 4 marks]

3 **(i)** $2x - 5 + x - 11 + 4x = 180 \quad \Rightarrow \quad 7x = 196 \quad \Rightarrow \quad x = 28$

$2x - 5 = 51, \quad x - 11 = 17, \quad 4x = 112$

smallest angle $= 17°$ [3 marks]

(ii) The three angles are different, so the triangle is scalene (and obtuse) [1 mark]

[Total: 4 marks]

4 $3p + 2p = 42 \quad \Rightarrow \quad 5p = 42 \quad \Rightarrow \quad p = \dfrac{42}{5}$ (or any equivalent value, e.g. 8.4) [2 marks]

5 **(i)** $w + w + (w+20) + (w+20) = 320$ (or any equivalent equation) [1 mark]

(ii) $4w + 40 = 320 \quad \Rightarrow \quad 4w = 280 \quad \Rightarrow \quad w = 70 \quad \therefore \text{area} = 70 \times 90 = 6300\,\text{m}^2$ [2 marks]

[Total: 3 marks]

6 $2(x+3) + 2x - 5 = 28 \quad \Rightarrow \quad 4x + 1 = 28 \quad \Rightarrow \quad x = \dfrac{27}{4}$ [2 marks]

7 **(i)** $3 \times 5 \times 4 = 60$ [1 mark]

(ii) $\dfrac{20(x+2)}{60} + \dfrac{12x}{60} - \dfrac{45x}{60} = 7 \quad \Rightarrow \quad 20(x+2) + 12x - 45x = 420$

$\Rightarrow \quad -13x = 380 \quad \Rightarrow \quad x = -\dfrac{380}{13}$ [2 marks]

[Total: 3 marks]

8 $x + (x+2) + (x+4) + (x+6) = 156 \quad \Rightarrow \quad 4x = 144 \quad \Rightarrow \quad x = 36$ [2 marks]

9 $2ax + 6 + 3ax - 15ab \equiv 7x - b \quad \Rightarrow \quad 2a + 3a = 7$ and $6 - 15ab = -b$

$\Rightarrow \quad a = 1.4$ so $6 - 21b = -b \quad \Rightarrow \quad 6 = 20b \quad \Rightarrow \quad b = 0.3$ [3 marks]

Exercise 1.4 Algebra and number

1 **(i)** $0.3m = 0.8n$, or any equivalent equation, e.g. $3m = 8n$ [1 mark]

(ii) $\dfrac{n}{m} = \dfrac{0.3}{0.8} = 0.375 = 37.5\%$ [1 mark]

[Total: 2 marks]

2 **(i)** $24 \times \dfrac{100+a}{100} = b \times \dfrac{100-24}{100}$ [2 marks]

(ii) $24(100+a) = 76b \quad \Rightarrow \quad 2400 + 24a = 76b \quad \Rightarrow \quad 19b - 6a = 600$ [2 marks]

[Total: 4 marks]

3 $\dfrac{2}{3} \times 12\% = 8\%$ of the population are left-handed men, and 4% are left-handed women.

$8\% \times 2 = 16\%$ of men are left-handed, and $4\% \times 2 = 8\%$ of women are left-handed.

$60 \times 0.16 = 9.6 \approx 10$ men $\qquad 25 \times 0.08 = 2$ women

Total ≈ 12 [3 marks]

4 $b + 15 : \dfrac{9}{10}b - 4 = 5 : 2 \quad \Rightarrow \quad \dfrac{b+15}{\frac{9}{10}b - 4} = \dfrac{5}{2} \quad \Rightarrow \quad 2(b+15) = 5\left(\dfrac{9}{10}b - 4\right)$

$\Rightarrow \quad 20(b+15) = 5(9b - 40) \quad \Rightarrow \quad 20b + 300 = 45b - 200 \quad \Rightarrow \quad b = 20$ [3 marks]

5 $1.25l \times W = lw$ where l is the original length, and w is the original width

$W = 0.8w = 80\%$ of w so there is a 20% reduction $\therefore x = 20$ [2 marks]

6 **(i)** $0.8h$ [1 mark]

(ii) $h : 1.2h = 1 : 1.2 = 10 : 12$ (or any equivalent ratio of integers, e.g. 5 : 6) [1 mark]

(iii) $1.2h - h = £35 \Rightarrow 0.2h = £35 \Rightarrow h = 5 \times £35 = £175$ [2 marks]

[Total: 4 marks]

7 **(i)** $\dfrac{x}{y} = \dfrac{3}{5} \quad \Rightarrow \quad 5x = 3y \quad \Rightarrow \quad x = \dfrac{3y}{5}$

$2x - y : 3x + y = \dfrac{6y}{5} - y : \dfrac{9y}{5} + y = \dfrac{y}{5} : \dfrac{14y}{5} = 1 : 14$ [2 marks]

(ii) $x : y = 21 : 35$ and $y : z = 35 : 10$ so $x : z = 21 : 10$

$\dfrac{z}{x} \times 100 = \dfrac{10}{21} \times 100 = 47.6\%$ so z is 47.6% of x [3 marks]

[Total: 5 marks]

8 $m = 1.2n$ and $\frac{p}{n} = \frac{6}{7}$

$5m = 6n$ and $7p = 6n$

$5m = 7p \quad \Rightarrow \quad p = \frac{5}{7}m$ so p is 71.4% of m [3 marks]

9 $0.6(2m-1) = 3 - m \quad \Rightarrow \quad 1.2m - 0.6 = 3 - m \quad \Rightarrow \quad 2.2m = 3.6 \quad \Rightarrow \quad m = \frac{18}{11}$ [2 marks]

Exercise 1.5 Expanding brackets

1 **(i)** $x^2 + 2x + x + 2 = x^2 + 3x + 2$ [1 mark]

(ii) $2x^2 - 6x + 5x - 15 = 2x^2 - x - 15$ [1 mark]

[Total: 2 marks]

2 **(i)** $m^3 - m^2 + m + m^2 - m + 1 = m^3 + 1$ [2 marks]

(ii) $x^5 + x^4 + x^3 + x^2 + x - x^4 - x^3 - x^2 - x - 1 = x^5 - 1$ [2 marks]

[Total: 4 marks]

3 **(i)** $(x+3)(x^2-1) = x^3 + 3x^2 - x - 3$ [2 marks]

(ii) $(6y^2 - 7y - 3)(y - 5) = 6y^3 - 37y^2 + 32y + 15$ [2 marks]

[Total: 4 marks]

4 **(i)** $x^2 - x - 12 - (x^2 + x - 2) = -2x - 10$ [2 marks]

(ii) $2m^2 - 7m - 15 - (m^2 + m - 12) = m^2 - 8m - 3$ [2 marks]

[Total: 4 marks]

5 $(x^2 - 3x + a)(x^2 + bx - 1) = x^4 + bx^3 - x^2 - 3x^3 - 3bx^2 + 3x + ax^2 + abx - a$

$\therefore \quad x^4 + (b-3)x^3 + (a - 3b - 1)x^2 + (3 + ab)x - a \equiv x^4 - 2x^3 + 7x - 4$

comparing x^3 terms: $b - 3 = -2 \quad \Rightarrow \quad b = 1$

comparing constant terms: $-a = -4 \quad \Rightarrow \quad a = 4$

check the x^2 terms: $a - 3b - 1 = 4 - 3 \times 1 - 1 = 0$ ✓

check the x terms: $3 + ab = 3 + 4 \times 1 = 7$ ✓ [3 marks]

6 **(i)** $(3m-1)(3m-1) = 9m^2 - 3m - 3m + 1 = 9m^2 - 6m + 1$ [1 mark]

(ii) $(3m-1)(9m^2 - 6m + 1) = 27m^3 - 27m^2 + 9m - 1$ [2 marks]

[Total: 3 marks]

7 **(i)** **(a)** $x^5 + 5x^3 - 2x^2 - 10$ [1 mark]

(b) $3x - 21 + x^4 - 7x^3 = x^4 - 7x^3 + 3x - 21$ [1 mark]

(ii) $x^5 + 5x^3 - 2x^2 - 10 - (x^4 - 7x^3 + 3x - 21) = x^5 - x^4 + 12x^3 - 2x^2 - 3x + 11$ [2 marks]

[Total: 4 marks]

8 **(i)** $(x+1)(x-2)(2x+1) = (x^2 - x - 2)(2x+1) = 2x^3 - x^2 - 5x - 2$ [2 marks]

(ii) $2(x+1)(x-2) + 2(x+1)(2x+1) + 2(x-2)(2x+1)$

$= 2(x^2 - x - 2) + 2(2x^2 + 3x + 1) + 2(2x^2 - 3x - 2)$

$= 2(5x^2 - x - 3) = 10x^2 - 2x - 6$ [2 marks]

[Total: 4 marks]

9 $(x+a)(bx+1)(3x-4) \equiv (bx^2+x+abx+a)(3x-4)$

$\equiv 3bx^3 - 4bx^2 + 3x^2 - 4x + 3abx^2 - 4abx + 3ax - 4a$

Comparing constant terms: $-4a = 8 \quad \Rightarrow \quad a = -2$

Comparing x^2 terms: $-4b + 3 + 3ab = -17$ so $-4b + 3 - 6b = -17 \quad \Rightarrow \quad b = 2$

Comparing x^3 terms: $3b = c$ so $c = 6$ [4 marks]

Exercise 1.6 Binomial expansions – using Pascal's triangle only

1 $x^3 + 3x^2 2 + 3x 2^2 + 2^3 = x^3 + 6x^2 + 12x + 8$ [2 marks]

2 $1(2x)^4 y^0 + 4(2x)^3 y^1 + 6(2x)^2 y^2 + 4(2x)^1 y^3 + 1(2x)^0 y^4$

$= 16x^4 + 32x^3 y + 24x^2 y^2 + 8xy^3 + y^4$ [3 marks]

3 **(i)** $a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5$ [2 marks]

(ii) $(2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)(-3y)^4 + (-3y)^5$

$= 32x^5 - 240x^4 y + 720x^3 y^2 - 1080x^2 y^3 + 810xy^4 - 243y^5$ [2 marks]

[Total: 4 marks]

4 **(i)** $1, \quad 1+12=13, \quad 12+66=78$ [1 mark]

(ii) 1st term $= 1 \times 2^{13} \times x^0 = 8192$

2nd term $= 13 \times 2^{12} \times x^1 = 53\,248x$

3rd term $= 78 \times 2^{11} \times x^2 = 159\,744x^2$ [2 marks]

[Total: 3 marks]

5 The 3rd term (in ascending powers of x) is $15 \times (2x)^2 \times 3^4 = 4860x^2$

$\therefore$ the coefficient of x^2 is 4860 [2 marks]

6 $35 \times (x^2)^3 \times \left(\frac{2}{x}\right)^4 = 35 \times x^6 \times 16x^{-4} = 560x^2$

coefficient = 560 [3 marks]

7 **(i)** $n = 5$ [1 mark]

(ii) $10 \times (ax)^3 \times y^2 = 80x^3 y^2 \quad \Rightarrow \quad 10a^3 = 80 \quad \Rightarrow \quad a = 2$ [2 marks]

(iii) $10 \times (2x)^2 \times y^3 = 40x^2 y^3$ so coefficient is 40 [2 marks]

[Total: 5 marks]

8 $252 \times (3x)^5 \times \left(-\frac{2}{x}\right)^5 = -1\,959\,552$ [2 marks]

9 **(i)** $1 \times 1^7 \times (ax)^0 + 7 \times 1^6 \times (ax)^1 + 21 \times 1^5 \times (ax)^2 = 1 + 7ax + 21a^2 x^2$ [2 marks]

(ii) $21a^2 = 9 \times 7a \quad \Rightarrow \quad a = 3$ [2 marks]

[Total: 4 marks]

Exercise 1.7 Surds: simplifying expressions containing square roots

1 **(i)** $\sqrt{72} = \sqrt{36\times2} = 6\sqrt{2}$ and $\sqrt{50} = \sqrt{25\times2} = 5\sqrt{2}$ [2 marks]

(ii) $6\sqrt{2}+5\sqrt{2} = 11\sqrt{2}$ [1 mark]

(iii) $11\sqrt{2} = \sqrt{121}\times\sqrt{2} = \sqrt{242}$ [1 mark]

[Total: 4 marks]

2 **(i)** $\dfrac{1}{\sqrt{2}} = \dfrac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ [2 marks]

(ii) $\dfrac{6}{\sqrt{2}} = \dfrac{6\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \dfrac{6\sqrt{2}}{2} = 3\sqrt{2}$ [2 marks]

(iii) $\dfrac{5}{\sqrt{3}} = \dfrac{5\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \dfrac{5\sqrt{3}}{3}$ [2 marks]

(iv) $\dfrac{\sqrt{7}}{\sqrt{2}} = \dfrac{\sqrt{7}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \dfrac{\sqrt{14}}{2}$ [2 marks]

(v) $\dfrac{6}{2\sqrt{3}} = \dfrac{6\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}} = \dfrac{6\sqrt{3}}{6} = \sqrt{3}$ [2 marks]

(vi) $\sqrt{\dfrac{8}{5}} = \sqrt{\dfrac{8\times5}{5\times5}} = \dfrac{\sqrt{40}}{5} = \dfrac{2\sqrt{10}}{5}$ [2 marks]

[Total: 12 marks]

3 $\dfrac{3}{\sqrt{2}} + \dfrac{5}{\sqrt{8}} = \dfrac{6}{\sqrt{8}} + \dfrac{5}{\sqrt{8}} = \dfrac{11}{\sqrt{8}} = \dfrac{11}{2\sqrt{2}} = \dfrac{11\sqrt{2}}{4}$ [2 marks]

4 $\dfrac{x\sqrt{32}}{\sqrt{2}} + \dfrac{2x\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{98}}{\sqrt{2}} \quad\Rightarrow\quad 4x + 2x = 7 \quad\Rightarrow\quad x = \dfrac{7}{6}$ [2 marks]

5 **(i)** $\dfrac{1}{\sqrt{2}-1} + \dfrac{1}{\sqrt{2}+1} = \dfrac{\sqrt{2}+1+\sqrt{2}-1}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)} = \dfrac{2\sqrt{2}}{2-1} = 2\sqrt{2} = 0 + \sqrt{8}$ [3 marks]

(ii) $\dfrac{3}{2+\sqrt{3}} + \dfrac{1}{2-\sqrt{3}} = \dfrac{3\left(2-\sqrt{3}\right)+1\left(2+\sqrt{3}\right)}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)} = \dfrac{6-3\sqrt{3}+2+\sqrt{3}}{4-3}$

$= 8 - 2\sqrt{3} = 8 - \sqrt{12}$ [3 marks]

(iii) $\dfrac{7}{\sqrt{8}-2} - \dfrac{3}{\sqrt{8}+2} = \dfrac{7\left(\sqrt{8}+2\right)-3\left(\sqrt{8}-2\right)}{\left(\sqrt{8}-2\right)\left(\sqrt{8}+2\right)} = \dfrac{7\sqrt{8}+14-3\sqrt{8}+6}{8-4}$

$= \dfrac{20+4\sqrt{8}}{4} = 5 + \sqrt{8}$ [3 marks]

(iv) $\dfrac{5}{4-\sqrt{2}} - \dfrac{3}{4+\sqrt{2}} = \dfrac{5\left(4+\sqrt{2}\right)-3\left(4-\sqrt{2}\right)}{\left(4-\sqrt{2}\right)\left(4+\sqrt{2}\right)} = \dfrac{20+5\sqrt{2}-12+3\sqrt{2}}{16-2}$ [3 marks]

$= \dfrac{8+8\sqrt{2}}{14} = \dfrac{4+4\sqrt{2}}{7} = \dfrac{4}{7} + \sqrt{\dfrac{32}{49}}$

[Total: 12 marks]

6 **(i)** 16 cm [1 mark]

(ii) $\left(4+\sqrt{2}\right)\left(4-\sqrt{2}\right)=16-2=14\,\text{cm}^2$ [2 marks]

(iii) $\sqrt{\left(4+\sqrt{2}\right)^2+\left(4-\sqrt{2}\right)^2}=\sqrt{16+8\sqrt{2}+2+16-8\sqrt{2}+2}=\sqrt{36}=6$ cm [2 marks]

[Total: 5 marks]

7 **(i)** $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ [2 marks]

(ii) $3^4+4\times 3^3\sqrt{2}+6\times 3^2\sqrt{2}^2+4\times 3\sqrt{2}^3+\sqrt{2}^4$

$=81+108\sqrt{2}+108+24\sqrt{2}+4$

$=193+132\sqrt{2}$ [2 marks]

[Total: 4 marks]

8 **(i)** $3x+1=\sqrt{8}\times\sqrt{2} \quad\Rightarrow\quad 3x+1=4 \quad\Rightarrow\quad x=1$ [2 marks]

(ii) $\dfrac{2w\sqrt{12}}{\sqrt{3}}-\dfrac{\sqrt{12}}{\sqrt{12}}=5\times\sqrt{12} \quad\Rightarrow\quad 4w-1=10\sqrt{3} \quad\Rightarrow\quad w=\dfrac{1}{4}\left(1+10\sqrt{3}\right)$ [3 marks]

(iii) $\dfrac{5y\sqrt{8}}{\sqrt{8}}=\sqrt{18}\times\sqrt{8}+\dfrac{y\sqrt{8}}{\sqrt{2}} \quad\Rightarrow\quad 5y=12+2y \quad\Rightarrow\quad y=4$ [3 marks]

(iv) $\dfrac{14}{\sqrt{20}}-\dfrac{2m}{\sqrt{5}}=\dfrac{1}{\sqrt{45}}+\dfrac{2m}{\sqrt{80}} \quad\Rightarrow\quad \dfrac{14\sqrt{80}}{\sqrt{20}}-\dfrac{2m\sqrt{80}}{\sqrt{5}}=\dfrac{\sqrt{80}}{\sqrt{45}}+\dfrac{2m\sqrt{80}}{\sqrt{80}}$

$\Rightarrow\quad 28-8m=\dfrac{4}{3}+2m \quad\Rightarrow\quad 10m=\dfrac{80}{3} \quad\Rightarrow\quad m=\dfrac{8}{3}$ [3 marks]

[Total: 11 marks]

9 $\sqrt{3}^3-3\times\sqrt{3}^2x+3\sqrt{3}x^2-x^3=9\sqrt{3}-9x-x^3 \quad\Rightarrow\quad 3\sqrt{3}-9x+3\sqrt{3}x^2=9\sqrt{3}-9x$

$\Rightarrow\quad 3\sqrt{3}x^2=6\sqrt{3} \quad\Rightarrow\quad x^2=2 \quad\Rightarrow\quad x=\pm\sqrt{2}$ [5 marks]

10 $\sqrt{2}^2+\sqrt{5}^2=\sqrt{a}^2 \quad\Rightarrow\quad 2+5=a \quad\Rightarrow\quad a=7$

$\sqrt{a}^2+\sqrt{2}^2=\sqrt{5}^2 \quad\Rightarrow\quad a+2=5 \quad\Rightarrow\quad a=3$ [3 marks]

Exercise 1.8 Surds: rationalising denominators with two terms

1 **(i)** $\dfrac{2}{\sqrt{2}-1}=\dfrac{2\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}=\dfrac{2\sqrt{2}+2}{2-1}=2\sqrt{2}+2$ [2 marks]

(ii) $\dfrac{5}{7+\sqrt{3}}=\dfrac{5\left(7-\sqrt{3}\right)}{\left(7+\sqrt{3}\right)\left(7-\sqrt{3}\right)}=\dfrac{35-5\sqrt{3}}{49-3}=\dfrac{35-5\sqrt{3}}{46}$ [2 marks]

(iii) $\dfrac{\sqrt{5}}{3+\sqrt{5}}=\dfrac{\sqrt{5}\left(3-\sqrt{5}\right)}{\left(3+\sqrt{5}\right)\left(3-\sqrt{5}\right)}=\dfrac{3\sqrt{5}-5}{9-5}=\dfrac{3\sqrt{5}-5}{4}$ [2 marks]

(iv) $\dfrac{2\sqrt{6}}{5-\sqrt{6}}=\dfrac{2\sqrt{6}\left(5+\sqrt{6}\right)}{\left(5-\sqrt{6}\right)\left(5+\sqrt{6}\right)}=\dfrac{10\sqrt{6}+12}{25-6}=\dfrac{10\sqrt{6}+12}{19}$ [2 marks]

(v) $\dfrac{\sqrt{7}}{9-2\sqrt{7}}=\dfrac{\sqrt{7}\left(9+2\sqrt{7}\right)}{\left(9-2\sqrt{7}\right)\left(9+2\sqrt{7}\right)}=\dfrac{9\sqrt{7}+14}{81-28}=\dfrac{9\sqrt{7}+14}{53}$ [2 marks]

(vi) $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}=\dfrac{\left(3+\sqrt{2}\right)\left(3+\sqrt{2}\right)}{\left(3-\sqrt{2}\right)\left(3+\sqrt{2}\right)}=\dfrac{9+6\sqrt{2}+2}{9-2}=\dfrac{11+6\sqrt{2}}{7}$ [3 marks]

(vii) $\dfrac{5-\sqrt{3}}{4+\sqrt{3}} = \dfrac{\left(5-\sqrt{3}\right)\left(4-\sqrt{3}\right)}{\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)} = \dfrac{20-9\sqrt{3}+3}{16-3} = \dfrac{23-9\sqrt{3}}{13}$ [3 marks]

(viii) $\dfrac{6+2\sqrt{10}}{3\sqrt{10}-7} = \dfrac{\left(6+2\sqrt{10}\right)\left(3\sqrt{10}+7\right)}{\left(3\sqrt{10}-7\right)\left(3\sqrt{10}+7\right)} = \dfrac{18\sqrt{10}+42+60+14\sqrt{10}}{90-49} = \dfrac{102+32\sqrt{10}}{41}$ [3 marks]

[Total: 19 marks]

2 **(i)** $\dfrac{\sqrt{2}}{5-\sqrt{2}} = \dfrac{\sqrt{2}\left(5+\sqrt{2}\right)}{\left(5-\sqrt{2}\right)\left(5+\sqrt{2}\right)} = \dfrac{5\sqrt{2}+2}{25-2} = \dfrac{2}{23}+\dfrac{5}{23}\sqrt{2}$ [3 marks]

(ii) $\dfrac{7+\sqrt{2}}{7-\sqrt{2}} = \dfrac{\left(7+\sqrt{2}\right)\left(7+\sqrt{2}\right)}{\left(7-\sqrt{2}\right)\left(7+\sqrt{2}\right)} = \dfrac{49+14\sqrt{2}+2}{49-2} = \dfrac{51}{47}+\dfrac{14}{47}\sqrt{2}$ [3 marks]

(iii) $\dfrac{6+\sqrt{2}}{\sqrt{2}-1} = \dfrac{\left(6+\sqrt{2}\right)\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)} = \dfrac{6\sqrt{2}+6+2+\sqrt{2}}{2-1} = 8+7\sqrt{2}$ [3 marks]

(iv) $\dfrac{5-\sqrt{8}}{\sqrt{18}+4} = \dfrac{\left(5-\sqrt{8}\right)\left(\sqrt{18}-4\right)}{\left(\sqrt{18}+4\right)\left(\sqrt{18}-4\right)} = \dfrac{5\sqrt{18}-20-12+4\sqrt{8}}{18-16}$

$= \dfrac{15\sqrt{2}-32+8\sqrt{2}}{2} = -16+\dfrac{23}{2}\sqrt{2}$ [3 marks]

[Total: 12 marks]

3 **(i)** $\dfrac{2}{\sqrt{3}-1} = \dfrac{2\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)} = \dfrac{2\sqrt{3}+2}{3-1} = 1+\sqrt{3}$ [3 marks]

(ii) $\dfrac{5}{\sqrt{2}-1} = \dfrac{5\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)} = \dfrac{5\sqrt{2}+5}{2-1} = 5+5\sqrt{2} = 5+\sqrt{50}$ [3 marks]

(iii) $\dfrac{3}{6-\sqrt{3}} = \dfrac{3\left(6+\sqrt{3}\right)}{\left(6-\sqrt{3}\right)\left(6+\sqrt{3}\right)} = \dfrac{18+3\sqrt{3}}{36-3} = \dfrac{6}{11}+\dfrac{1}{11}\sqrt{3} = \dfrac{6}{11}+\sqrt{\dfrac{3}{121}}$ [3 marks]

(iv) $\dfrac{2+\sqrt{5}}{4-\sqrt{5}} = \dfrac{\left(2+\sqrt{5}\right)\left(4+\sqrt{5}\right)}{\left(4-\sqrt{5}\right)\left(4+\sqrt{5}\right)} = \dfrac{8+6\sqrt{5}+5}{16-5} = \dfrac{13}{11}+\dfrac{6}{11}\sqrt{5} = \dfrac{13}{11}+\sqrt{\dfrac{180}{121}}$ [3 marks]

[Total: 12 marks]

4 **(i)** $5+\sqrt{10}-\sqrt{10}-2 = 3$ [1 mark]

(ii) $\dfrac{\left(\sqrt{5}+\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)}{\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)} = \dfrac{5+\sqrt{10}+\sqrt{10}+2}{3} = \dfrac{7+2\sqrt{10}}{3}$ [2 marks]

[Total: 3 marks]

5 Length $= \dfrac{18+\sqrt{2}}{5-\sqrt{2}} = \dfrac{\left(18+\sqrt{2}\right)\left(5+\sqrt{2}\right)}{\left(5-\sqrt{2}\right)\left(5+\sqrt{2}\right)} = \dfrac{90+23\sqrt{2}+2}{25-2} = \dfrac{92+23\sqrt{2}}{23} = \left(4+\sqrt{2}\right)$cm

[3 marks]

6 $\left(\sqrt{2}-1\right)^2+\left(2-\sqrt{2}\right)^2 = 2-2\sqrt{2}+1+4-4\sqrt{2}+2 = 9-6\sqrt{2}$

$\left(3-\sqrt{2}\right)^2 = 9-6\sqrt{2}+2 = 11-6\sqrt{2}$

$9-6\sqrt{2} \neq 11-6\sqrt{2}$ so the triangle is not right-angled [4 marks]

7 **(i)** Sum $= 7 + 2\sqrt{2} + 6 - 3\sqrt{2} = \left(13 - \sqrt{2}\right)\text{cm}$ [2 marks]

(ii) Area $= \dfrac{1}{2}\left(a+b\right)h \quad \therefore \quad 50 + \sqrt{162} = \dfrac{1}{2} \times \left(13 - \sqrt{2}\right)h \quad \Rightarrow \quad h = \dfrac{2\left(50 + 9\sqrt{2}\right)}{13 - \sqrt{2}}$

$= \dfrac{\left(100 + 18\sqrt{2}\right)\left(13 + \sqrt{2}\right)}{\left(13 - \sqrt{2}\right)\left(13 + \sqrt{2}\right)} = \dfrac{1300 + 334\sqrt{2} + 36}{169 - 2} = \dfrac{1336 + 334\sqrt{2}}{167} = \left(8 + 2\sqrt{2}\right)\text{cm}$ [4 marks]

[Total: 6 marks]

8 **(i)** $3 - \sqrt{6} + \sqrt{3} + \sqrt{6} - 2 + \sqrt{2} - \sqrt{3} + \sqrt{2} - 1 = 2\sqrt{2}$ [2 marks]

(ii) $\dfrac{5 - \sqrt{2}}{\sqrt{3} + \sqrt{2} - 1} = \dfrac{\left(5 - \sqrt{2}\right)\left(\sqrt{3} - \sqrt{2} + 1\right)}{\left(\sqrt{3} + \sqrt{2} - 1\right)\left(\sqrt{3} - \sqrt{2} + 1\right)} = \dfrac{5\sqrt{3} - 5\sqrt{2} + 5 - \sqrt{6} + 2 - \sqrt{2}}{2\sqrt{2}}$

$= \dfrac{\left(5\sqrt{3} - 6\sqrt{2} - \sqrt{6} + 7\right) \times \sqrt{2}}{\left(2\sqrt{2}\right) \times \sqrt{2}} = \dfrac{5\sqrt{6} - 12 - 2\sqrt{3} + 7\sqrt{2}}{4}$ [3 marks]

(iii) $\dfrac{3\left(\sqrt{5} + \sqrt{3} - 2\right)}{\left(\sqrt{5} - \sqrt{3} + 2\right)\left(\sqrt{5} + \sqrt{3} - 2\right)} = \dfrac{3\sqrt{5} + 3\sqrt{3} - 6}{5 + \sqrt{15} - 2\sqrt{5} - \sqrt{15} - 3 + 2\sqrt{3} + 2\sqrt{5} + 2\sqrt{3} - 4}$

$= \dfrac{3\sqrt{5} + 3\sqrt{3} - 6}{-2 + 4\sqrt{3}} = \dfrac{\left(3\sqrt{5} + 3\sqrt{3} - 6\right)\left(2 + 4\sqrt{3}\right)}{\left(-2 + 4\sqrt{3}\right)\left(2 + 4\sqrt{3}\right)}$

$= \dfrac{6\sqrt{5} + 12\sqrt{15} + 6\sqrt{3} + 36 - 12 - 24\sqrt{3}}{-4 - 8\sqrt{3} + 8\sqrt{3} + 48}$

$= \dfrac{6\sqrt{5} + 12\sqrt{15} - 18\sqrt{3} + 24}{44} = \dfrac{3\sqrt{5} + 6\sqrt{15} - 9\sqrt{3} + 12}{22}$ [4 marks]

[Total: 9 marks]

9 $\dfrac{\sqrt{5} + \sqrt{2} - 3}{\sqrt{5} - \sqrt{2} + 1} = \dfrac{\left(\sqrt{5} + \sqrt{2} - 3\right)\left(\sqrt{5} + \sqrt{2} - 1\right)}{\left(\sqrt{5} - \sqrt{2} + 1\right)\left(\sqrt{5} + \sqrt{2} - 1\right)} = \dfrac{5 + \sqrt{10} - \sqrt{5} + \sqrt{10} + 2 - \sqrt{2} - 3\sqrt{5} - 3\sqrt{2} + 3}{5 + \sqrt{10} - \sqrt{5} - \sqrt{10} - 2 + \sqrt{2} + \sqrt{5} + \sqrt{2} - 1}$

$= \dfrac{10 + 2\sqrt{10} - 4\sqrt{5} - 4\sqrt{2}}{2 + 2\sqrt{2}} = \dfrac{5 + \sqrt{10} - 2\sqrt{5} - 2\sqrt{2}}{1 + \sqrt{2}} = \dfrac{\left(5 + \sqrt{10} - 2\sqrt{5} - 2\sqrt{2}\right)\left(1 - \sqrt{2}\right)}{\left(1 + \sqrt{2}\right)\left(1 - \sqrt{2}\right)}$

$= \dfrac{5 - 5\sqrt{2} + \sqrt{10} - \sqrt{20} - 2\sqrt{5} + 2\sqrt{10} - 2\sqrt{2} + 4}{1 - 2} = \dfrac{9 + 3\sqrt{10} - 4\sqrt{5} - 7\sqrt{2}}{-1}$

$= 7\sqrt{2} + 4\sqrt{5} - 3\sqrt{10} - 9$ [5 marks]

Exercise 1.9 The product rule for counting

1 $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ [1 mark]

2 **(i)** The last digit must be 5. The first three digits can be written in any order.

$\therefore$ the number of arrangements of three different digits is $3! = 6$ [1 mark]

(ii) Altogether $4! = 24$ different four-digit numbers can be formed.

$\therefore$ the number of non-multiples of 5 are $24 - 6 = 18$ [2 marks]

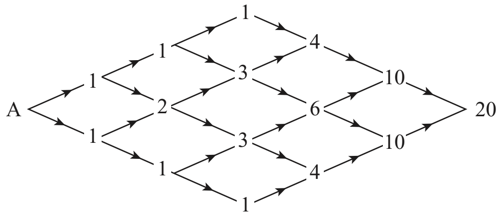
[Total: 3 marks]

3 **(i)** $10 \times 10 \times 10 \times 10 = 10\,000$ [1 mark]

(ii) $10 \times 9 \times 8 \times 7 = 5040$ [1 mark]

[Total: 2 marks]

4 The numbers at each stage give the number of different ways of reaching that point from A.



So there are 20 ways of reaching B from A. [2 marks]

5 **(i)** $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$ [1 mark]

(ii) $10! \div 2 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 1\,814\,400$ [2 marks]

[Total: 3 marks]

6 **(i)** Each block of 10 000 has $9 \times 9 \times 9 \times 4 = 2916$ possible numbers.

Total $= 6 \times 2916 = 17\,496$ [2 marks]

(ii) Numbers with no 3s $= 9 \times 9 \times 9 \times 9 \times 6 = 39\,366$
Numbers with at least one 3 $= 60\,000 - 39\,366 = 20\,634$ [2 marks]

[Total: 4 marks]

7 **(i)** D must be in the middle. There are then 3! arrangements of the first three letters, which must be mirrored at the end of the arrangement.
Number of arrangements = 6 [1 mark]

(ii) D cannot be used. The first two cards could be AB or AC or BC or these reversed.

Number of arrangements = 6 [1 mark]

(iii) Two cards = 3 palindromes

Three cards = (two cards) × 3 = 9 palindromes

Four cards = 6 palindromes

Five cards = (four cards) × 2 = 12 palindromes

Six cards = same as seven cards = 6 palindromes

Seven cards = 6 palindromes

Total = 42 palindromes [3 marks]

[Total: 5 marks]

8 **(i)** $8 \times 7 \times 6 \times 5 \times 4 = 6720$ [2 marks]

(ii) $8 \times 8 \times 8 \times 8 \times 8 = 32\,768$ [1 mark]

[Total: 3 marks]

9 $6 \times 7 \times 7 \times 4 = 1176$ [3 marks]

2 Algebra II

Exercise 2.1 Factorising

1 **(i)** $2wx^2(4x - 3w)$ [1 mark]

(ii) $3a^2b(3ab + 4)$ [1 mark]

(iii) $3p^2q^3(5p^2 - 7)$ [1 mark]

[Total: 3 marks]

2 **(i)** $ab + bc + ac + c^2 = b(a + c) + c(a + c)$

$= (a + c)(b + c)$ [1 mark]

(ii) $p^2 + pq + pr + qr = p(p + q) + r(p + q)$

$= (p + q)(p + r)$ [1 mark]

(iii) $a^2 + ab + ac + bc = a(a + b) + c(a + b)$

$= (a + c)(a + b)$ [1 mark]

[Total: 3 marks]

3 **(i)** $2ab + 3ac - 4b^2 - 6bc = a(2b + 3c) - 2b(2b + 3c)$

$= (a - 2b)(2b + 3c)$ [1 mark]

(ii) $2r^2 + 6rs - 3rt - 9st = 2r(r + 3s) - 3t(r + 3s)$

$= (2r - 3t)(r + 3s)$ [1 mark]

(iii) $3g^2 - gh - 6gk + 2hk = g(3g - h) - 2k(3g - h)$

$= (g - 2k)(3g - h)$ [1 mark]

[Total: 3 marks]

4 **(i)** $x^2 + 5x + 6 = (x + 2)(x + 3)$ [1 mark]

(ii) $x^2 - 7x + 10 = (x - 2)(x - 5)$ [1 mark]

(iii) $x^2 - 3x - 10 = (x - 5)(x + 2)$ [1 mark]

(iv) $p^2 + 9p + 14 = (p + 2)(p + 7)$ [1 mark]

(v) $r^2 - 15r + 36 = (r - 3)(r - 12)$ [1 mark]

(vi) $t^2 - 10t - 75 = (t - 15)(t + 5)$ [1 mark]

[Total: 6 marks]

5 **(i)** $2a^2 + 5a + 2 = (2a + 1)(a + 2)$ [1 mark]

(ii) $2x^2 - x - 6 = (2x + 3)(x - 2)$ [1 mark]

(iii) $3p^2 - 8p - 3 = (3p + 1)(p - 3)$ [1 mark]

(iv) $2a^2 - 13a + 20 = (2a - 5)(a - 4)$ [1 mark]

(v) $4c^2 + c - 18 = (4c + 9)(c - 2)$ [1 mark]

(vi) $3x^2 + 10x - 8 = (3x - 2)(x + 4)$ [1 mark]

[Total: 6 marks]

6 **(i)** $x^2 + 6xy + 8y^2 = (x + 2y)(x + 4y)$ [1 mark]

(ii) $r^2 + 2ar - 15a^2 = (r + 5a)(r - 3a)$ [1 mark]

(iii) $y^2 + 9yz + 20z^2 = (y + 4z)(y + 5z)$ [1 mark]

(iv) $a^2 - 5ab - 6b^2 = (a - 6b)(a + b)$ [1 mark]

(v) $p^2 + 7pq + 12q^2 = (p + 3q)(p + 4q)$ [1 mark]

(vi) $s^2 - 4st + 4t^2 = (s - 2t)(s - 2t) = (s - 2t)^2$ [1 mark]

[Total: 6 marks]

7 **(i)** $2x^2 + 5xy + 3y^2 = (2x + 3y)(x + y)$ [1 mark]

(ii) $2a^2 - 5ab + 2b^2 = (2a - b)(a - 2b)$ [1 mark]

(iii) $9p^2 - 6pq + q^2 = (3p - q)(3p - q) = (3p - q)^2$ [1 mark]

(iv) $2x^2 - 11xy + 5y^2 = (2x - y)(x - 5y)$ [1 mark]

(v) $3a^2 - 4ab + b^2 = (3a - b)(a - b)$ [1 mark]

(vi) $6p^2 + 5pq - 6q^2 = (2p + 3q)(3p - 2q)$ [1 mark]

[Total: 6 marks]

8 **(i)** $x^2 - 4y^2 = (x + 2y)(x - 2y)$ [1 mark]

(ii) $a^2 - (b + 2)^2 = (a + (b + 2))(a - (b + 2))$
$= (a + b + 2)(a - b - 2)$ [2 marks]

(iii) $(p + 3)^2 - 4q^2 = (p + 3 + 2q)(p + 3 - 2q)$ [1 mark]

(iv) $s^2 - 9(t + 2)^2 = (s + 3(t + 2))(s - 3(t + 2))$
$= (s + 3t + 6)(s - 3t - 6)$ [2 marks]

(v) $(x - 1)^2 - 16y^2 = (x - 1 + 4y)(x - 1 - 4y)$ [1 mark]

(vi) $(2x - 1)^2 - y^2 = (2x - 1 + y)(2x - 1 - y)$ [1 mark]

[Total: 8 marks]

9 **(i)** $x^3 - 9x = x(x^2 - 9)$
$= x(x + 3)(x - 3)$ [2 marks]

(ii) $a^4 - 4a^2 = a^2(a^2 - 4)$
$= a^2(a + 2)(a - 2)$ [2 marks]

(iii) $4p^3 - 16p = 4p(p^2 - 4)$
$= 4p(p + 2)(p - 2)$ [2 marks]

(iv) $4(a + b)^2 - (a - b)^2 = (2(a + b) + (a - b))(2(a + b) - (a - b))$
$= (2a + 2b + a - b)(2a + 2b - a + b)$
$= (3a + b)(a + 3b)$ [2 marks]

(v) $(p + 2q)^2 - (p - 2q)^2 = ((p + 2q) + (p - 2q))((p + 2q) - (p - 2q))$
$= (p + 2q + p - 2q)(p + 2q - p + 2q)$
$= (2p)(4q)$
$= 8pq$ [2 marks]

(vi) $4(x + y)^2 - 9(x - y)^2 = (2(x + y) + 3(x - y))(2(x + y) - 3(x - y))$
$= (2x + 2y + 3x - 3y)(2x + 2y - 3x + 3y)$
$= (5x - y)(5y - x)$ [2 marks]

[Total: 12 marks]

Exercise 2.2 Rearranging mathematical formulae

1 $x^2 + y^2 = r^2 \quad \Rightarrow \quad x^2 = r^2 - y^2$

$\Rightarrow \quad x = \pm\sqrt{r^2 - y^2}$ [1 mark]

2 $A = \frac{1}{2}bh \quad \Rightarrow \quad h = \frac{2A}{b}$ [1 mark]

3 $P = 2(l + b) \quad \Rightarrow \quad \frac{P}{2} = l + b$

$\Rightarrow \quad b = \frac{P}{2} - l \quad \left(\text{or } \frac{P - 2l}{2}\right)$ [1 mark]

4 $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad r^3 = \frac{3V}{4\pi}$

$\Rightarrow \quad r = \sqrt[3]{\frac{3V}{4\pi}}$ [1 mark]

5 $A = \pi r^2 + \pi rl \quad \Rightarrow \quad \pi rl = A - \pi r^2$

$\Rightarrow \quad l = \frac{A - \pi r^2}{\pi r} \quad \left(\text{or } \frac{A}{\pi r} - r\right)$ [1 mark]

6 **(i)** $A = \frac{1}{2}(b + c)h \quad \Rightarrow \quad h = \frac{2A}{b + c}$ [1 mark]

(ii) $A = \frac{1}{2}(b + c)h \quad \Rightarrow \quad 2A = bh + ch$

$\Rightarrow \quad 2A - ch = bh$

$\Rightarrow \quad b = \frac{2A - ch}{h} \quad \left(\text{or } \frac{2A}{h} - c\right)$ [2 marks]

[Total: 3 marks]

7 **(i)** $V = \pi r^2 h \quad \Rightarrow \quad h = \frac{V}{\pi r^2}$ [1 mark]

(ii) $V = \pi r^2 h \quad \Rightarrow \quad r^2 = \frac{V}{\pi h}$

$\Rightarrow \quad r = \sqrt{\frac{V}{\pi h}}$ (a length must be positive) [1 mark]

[Total: 2 marks]

8 **(i)** $V = \frac{1}{3}x^2 y \quad \Rightarrow \quad y = \frac{3V}{x^2}$ [1 mark]

(ii) $V = \frac{1}{3}x^2 y \quad \Rightarrow \quad x^2 = \frac{3V}{y}$

$\Rightarrow \quad x = \sqrt{\frac{3V}{y}}$ (a length must be positive) [1 mark]

[Total: 2 marks]

9 **(i)** $v^2 - u^2 = 2as \quad \Rightarrow \quad a = \frac{v^2 - u^2}{2s}$ [1 mark]

(ii) $v^2 - u^2 = 2as \quad \Rightarrow \quad v^2 - 2as = u^2$

$\Rightarrow \quad u = \pm\sqrt{v^2 - 2as}$ (can be in positive or negative direction) [1 mark]

[Total: 2 marks]

10 (i) $A = \frac{(a+b)h}{2} \quad \Rightarrow \quad h = \frac{2A}{(a+b)}$ [1 mark]

(ii) $A = \frac{(a+b)h}{2} \quad \Rightarrow \quad 2A = ah + bh$

$\Rightarrow \quad ah = 2A - bh$

$\Rightarrow \quad a = \frac{2A - bh}{h} \quad \left(\text{or } \frac{2A}{h} - b\right)$ [2 marks]

[Total: 3 marks]

11 (i) $S = \frac{n}{2}[2a + (n-1)d] \quad \Rightarrow \quad 2S = 2an + n(n-1)d \quad \Rightarrow \quad 2S - n(n-1)d = 2an$

$\Rightarrow \quad a = \frac{1}{2n}[2S - n(n-1)d] \quad \Rightarrow \quad a = \frac{S}{n} - \frac{n-1}{2}d$ [2 marks]

(ii) $S = \frac{n}{2}[2a + (n-1)d] \quad \Rightarrow \quad 2S = 2an + n(n-1)d \quad \Rightarrow \quad 2S - 2an = n(n-1)d$

$\Rightarrow \quad d = \frac{2(S - an)}{n(n-1)}$ [2 marks]

[Total: 4 marks]

Exercise 2.3 Rearranging more general formulae and equations

1 $2p + 3q = 4pq \quad \Rightarrow \quad 3q = 4pq - 2p$

$\Rightarrow \quad 3q = p(4q - 2)$

$\Rightarrow \quad p = \frac{3q}{4q-2}$ [2 marks]

2 $2(a + t) = 3(2t - b) \quad \Rightarrow \quad 2a + 2t = 6t - 3b$

$\Rightarrow \quad 2a + 3b = 4t$

$\Rightarrow \quad t = \frac{2a + 3b}{4}$ [2 marks]

3 $pr = 2(p - r) \quad \Rightarrow \quad pr = 2p - 2r$

$\Rightarrow \quad pr + 2r = 2p$

$\Rightarrow \quad r(p + 2) = 2p$

$\Rightarrow \quad r = \frac{2p}{p+2}$ [2 marks]

4 $6b - 2c = 3bc \quad \Rightarrow \quad 6b = 3bc + 2c$

$\Rightarrow \quad 6b = c(3b + 2)$

$\Rightarrow \quad c = \frac{6b}{3b+2}$ [2 marks]

5 $a(d - 2) = 2b(3 - 2d) \quad \Rightarrow \quad ad - 2a = 6b - 4bd$

$\Rightarrow \quad ad + 4bd = 2a + 6b$

$\Rightarrow \quad d(a + 4b) = 2(a + 3b)$

$\Rightarrow \quad d = \frac{2(a+3b)}{(a+4b)}$ [2 marks]

6 $b = \frac{2a - 3}{3 + 2a} \quad \Rightarrow \quad b(3 + 2a) = 2a - 3$

$\Rightarrow \quad 3b + 2ab = 2a - 3$

$\Rightarrow \quad 3b + 3 = 2a - 2ab$

$\Rightarrow \quad 3(b + 1) = 2a(1 - b)$

$\Rightarrow \quad a = \frac{3(b+1)}{2(1-b)}$ [2 marks]

7 $y = \dfrac{x-1}{2x+1}$ $\Rightarrow$ $y(2x+1) = x-1$

$\Rightarrow$ $2xy + y = x - 1$

$\Rightarrow$ $1 + y = x - 2xy$

$\Rightarrow$ $1 + y = x(1-2y)$

$\Rightarrow$ $x = \dfrac{1+y}{1-2y}$ [2 marks]

8 $p = \dfrac{2s+3a}{s}$ $\Rightarrow$ $ps = 2s + 3a$

$\Rightarrow$ $ps - 2s = 3a$

$\Rightarrow$ $s(p-2) = 3a$

$\Rightarrow$ $s = \dfrac{3a}{p-2}$ [2 marks]

9 $x = \dfrac{w+2}{w-3}$ $\Rightarrow$ $x(w-3) = w+2$

$\Rightarrow$ $wx - 3x = w + 2$

$\Rightarrow$ $wx - w = 3x + 2$

$\Rightarrow$ $w(x-1) = 3x + 2$

$\Rightarrow$ $w = \dfrac{3x+2}{x-1}$ [2 marks]

10 $3p + 2a = \dfrac{2p}{3-a}$ $\Rightarrow$ $(3p+2a)(3-a) = 2p$

$\Rightarrow$ $9p - 3ap + 6a - 2a^2 = 2p$

$\Rightarrow$ $7p - 3ap = 2a^2 - 6a$

$\Rightarrow$ $p(7-3a) = 2a(a-3)$

$\Rightarrow$ $p = \dfrac{2a(a-3)}{(7-3a)}$ [2 marks]

11 $w = \sqrt{\dfrac{x+3}{4-x}}$ $\Rightarrow$ $w^2 = \dfrac{x+3}{4-x}$ $\Rightarrow$ $w^2(4-x) = x+3$ $\Rightarrow$ $4w^2 - w^2x = x + 3$

$\Rightarrow$ $4w^2 - 3 = w^2x + x$ $\Rightarrow$ $4w^2 - 3 = x\left(w^2+1\right)$ $\Rightarrow$ $x = \dfrac{4w^2-3}{w^2+1}$ [4 marks]

Exercise 2.4 Simplifying algebraic fractions

1 **(i)** $\dfrac{3ab^2}{6a^2b} = \dfrac{b}{2a}$ [1 mark]

(ii) $\dfrac{12a^2b^3}{9a^3c^2} = \dfrac{4b^3}{3ac^2}$ [1 mark]

(iii) $\dfrac{6x^2y}{4xy^2} = \dfrac{3x}{2y}$ [1 mark]

[Total: 3 marks]

2 **(i)** $\dfrac{2x+4}{4x+2} = \dfrac{2(x+2)}{2(2x+1)}$

$= \dfrac{x+2}{2x+1}$ [1 mark]

(ii) $\dfrac{3x-9}{x^2-x-6} = \dfrac{3(x-3)}{(x-3)(x+2)}$

$= \dfrac{3}{x+2}$ [1 mark]

(iii) $\dfrac{x^2+3x-18}{2x-6} = \dfrac{(x+6)(x-3)}{2(x-3)}$

$= \dfrac{x+6}{2}$ [1 mark]

[Total: 3 marks]

3 (i) $\dfrac{4x^2-9}{6x^2-13x+6} = \dfrac{(2x+3)(2x-3)}{(3x-2)(2x-3)}$

$= \dfrac{2x+3}{3x-2}$ [2 marks]

(ii) $\dfrac{x^2+2x-8}{x^2-6x+8} = \dfrac{(x+4)(x-2)}{(x+2)(x+4)}$

$= \dfrac{x-2}{x+2}$ [2 marks]

(iii) $\dfrac{6x^2-13x+6}{9x^2-4} = \dfrac{(3x-2)(2x-3)}{(3x+2)(3x-2)}$

$= \dfrac{2x-3}{3x+2}$ [2 marks]

[Total: 6 marks]

4 (i) $\dfrac{4x^2-1}{3x} \times \dfrac{2x}{2x^2+3x+1} = \dfrac{\cancel{(2x+1)}(2x-1)}{3\cancel{x}} \times \dfrac{2\cancel{x}}{\cancel{(2x+1)}(x+1)}$

$= \dfrac{2(2x-1)}{3(x+1)}$ [2 marks]

(ii) $\dfrac{3x^2+5x-2}{6x+3} \times \dfrac{2x+1}{2x^2+3x-2} = \dfrac{(3x-1)\cancel{(x+2)}}{3\cancel{(2x+1)}} \times \dfrac{\cancel{(2x+1)}}{(2x-1)\cancel{(x+2)}}$

$= \dfrac{3x-1}{3(2x-1)}$ [2 marks]

(iii) $\dfrac{x+3}{x^2-9} \times \dfrac{x^2+2x-15}{x^2+3x-10} = \dfrac{\cancel{(x+3)}}{\cancel{(x+3)}\cancel{(x-3)}} \times \dfrac{\cancel{(x+5)}\cancel{(x-3)}}{\cancel{(x+5)}(x-2)}$

$= \dfrac{1}{x-2}$ [2 marks]

(iv) $\dfrac{x-2}{x^2-x-6} \times \dfrac{x^2+4x+4}{x^2+x-6} = \dfrac{\cancel{(x-2)}}{(x-3)\cancel{(x+2)}} \times \dfrac{\cancel{(x+2)}(x+2)}{(x+3)\cancel{(x-2)}}$

$= \dfrac{x+2}{(x-3)(x+3)}$

[2 marks]

[Total: 8 marks]

5 (i) $\dfrac{3x}{10x-4} \div \dfrac{4x^2}{5x-2} = \dfrac{3\cancel{x}}{2\cancel{(5x-2)}} \times \dfrac{\cancel{(5x-2)}}{4\cancel{x^2}\,x}$

$= \dfrac{3}{8x}$ [1 mark]

(ii) $\dfrac{2a-3}{3a-2} \div \dfrac{4a-6}{6a-4} = \dfrac{(2a-3)}{(3a-2)} \times \dfrac{2(3a-2)}{2(2a-3)}$

$= 1$ [1 mark]

(iii) $\dfrac{x+3}{x^2-9} \div \dfrac{2x^2+5x+3}{3x^2+x-2} = \dfrac{\cancel{(x+3)}}{\cancel{(x+3)}(x-3)} \times \dfrac{(3x-2)\cancel{(x+1)}}{(2x+3)\cancel{(x+1)}}$

$= \dfrac{3x-2}{(x-3)(2x+3)}$ [2 marks]

(iv) $\dfrac{x^2-4}{x^2-2x} \div \dfrac{x^2-4x+4}{x+2} = \dfrac{(x+2)\cancel{(x-2)}}{x\cancel{(x-2)}} \times \dfrac{(x+2)}{(x-2)(x-2)}$

$= \dfrac{(x+2)^2}{x(x-2)^2}$ [2 marks]

[Total: 6 marks]

6 **(i)** $\dfrac{4p}{3} + \dfrac{3p}{4} = \dfrac{4(4p)+3(3p)}{12}$

$= \dfrac{25p}{12}$ [1 mark]

(ii) $\dfrac{4}{3p} + \dfrac{3}{4p} = \dfrac{16}{12p} + \dfrac{9}{12p}$

$= \dfrac{25}{12p}$ [1 mark]

(iii) $\dfrac{4p}{3} + \dfrac{3}{4p} = \dfrac{16p^2+9}{12p}$ [1 mark]

[Total: 3 marks]

7 **(i)** $\dfrac{3}{4a+3} + \dfrac{4}{3a+2} = \dfrac{3(3a+2)+4(4a+3)}{(4a+3)(3a+2)}$

$= \dfrac{9a+6+16a+12}{(4a+3)(3a+2)}$

$= \dfrac{25a+18}{(4a+3)(3a+2)}$ [2 marks]

(ii) $\dfrac{a}{2a+3} + \dfrac{a}{2a-3} = \dfrac{a(2a-3)+a(2a+3)}{(2a+3)(2a-3)}$

$= \dfrac{2a^2-3a+2a^2+3a}{(2a+3)(2a-3)}$

$= \dfrac{4a^2}{(2a+3)(2a-3)}$ [2 marks]

(iii) $\dfrac{a}{a+b} + \dfrac{b}{a-b} = \dfrac{a(a-b)+b(a+b)}{(a+b)(a-b)}$

$= \dfrac{a^2-ab+ab+b^2}{(a+b)(a-b)}$

$= \dfrac{a^2+b^2}{(a+b)(a-b)}$ [2 marks]

[Total: 6 marks]

8 **(i)** $\dfrac{3x}{2} - \dfrac{2x}{3} = \dfrac{9x}{6} - \dfrac{4x}{6}$

$= \dfrac{5x}{6}$ [1 mark]

(ii) $\dfrac{4}{3x}-\dfrac{3}{4x}=\dfrac{16}{12x}-\dfrac{9}{12x}$

$=\dfrac{7}{12x}$ [1 mark]

(iii) $\dfrac{3x}{2}-\dfrac{2}{3x}=\dfrac{9x^2}{6x}-\dfrac{4}{6x}$

$=\dfrac{9x^2-4}{6x}$ [1 mark]

[Total: 3 marks]

9 **(i)** $\dfrac{3}{p^2+2p}-\dfrac{2}{p^2-2p}=\dfrac{3}{p(p+2)}-\dfrac{2}{p(p-2)}$

$=\dfrac{3(p-2)-2(p+2)}{p(p+2)(p-2)}$

$=\dfrac{3p-6-2p-4}{p(p+2)(p-2)}$

$=\dfrac{p-10}{p(p+2)(p-2)}$ [2 marks]

(ii) $\dfrac{4x}{x^2-4}-\dfrac{2}{x+2}=\dfrac{4x}{(x+2)(x-2)}-\dfrac{2}{(x+2)}$

$=\dfrac{4x-2(x-2)}{(x+2)(x-2)}$

$=\dfrac{4x-2x+4}{(x+2)(x-2)}$

$=\dfrac{2x+4}{(x+2)(x-2)}$

$=\dfrac{2(x+2)}{(x+2)(x-2)}=\dfrac{2}{x-2}$ [3 marks]

(iii) $\dfrac{2x}{(x+1)(x+2)}-\dfrac{3x}{(x+2)(x+3)}=\dfrac{2x(x+3)-3x(x+1)}{(x+1)(x+2)(x+3)}$

$=\dfrac{2x^2+6x-3x^2-3x}{(x+1)(x+2)(x+3)}$

$=\dfrac{3x-x^2}{(x+1)(x+2)(x+3)}\quad\left(=\dfrac{x(3-x)}{(x+1)(x+2)(x+3)}\right)$ [2 marks]

[Total: 7 marks]

10 (i) $\dfrac{a+1}{a}+\dfrac{a^2+1}{a^2}+\dfrac{a^3+1}{a^3}=\dfrac{a^2(a+1)+a(a^2+1)+(a^3+1)}{a^3}$

$=\dfrac{a^3+a^2+a^3+a+a^3+1}{a^3}$

$=\dfrac{3a^3+a^2+a+1}{a^3}$ [2 marks]

(ii) $\dfrac{3x+y}{3} - \dfrac{2x+y}{2} - \dfrac{2x-y}{6} = \dfrac{2(3x+y) - 3(2x+y) - (2x-y)}{6}$

$= \dfrac{6x + 2y - 6x - 3y - 2x + y}{6}$

$= \dfrac{-2x}{6}$

$= -\dfrac{x}{3}$ [2 marks]

[Total: 4 marks]

11 $\dfrac{x(x+1)}{3x} - \dfrac{3 \times 2}{3x} + \dfrac{4 \times 3x}{3x} = \dfrac{x^2 + 13x - 6}{3x}$ [3 marks]

Exercise 2.5 Solving linear equations involving fractions

1 **(i)** $6(3w-4) = 8(5w+1) \Rightarrow 18w - 24 = 40w + 8 \Rightarrow -32 = 22w \Rightarrow w = -\dfrac{16}{11}$ [2 marks]

(ii) $7(2x+3) = 5(3x-4) \Rightarrow 14x + 21 = 15x - 20 \Rightarrow x = 41$ [2 marks]

(iii) $9(5y-2) = 4(8y+1) \Rightarrow 45y - 18 = 32y + 4 \Rightarrow 13y = 22 \Rightarrow y = \dfrac{22}{13}$ [2 marks]

[Total: 6 marks]

2 **(i)** $x + \dfrac{x}{3} = \dfrac{3}{4} \Rightarrow \dfrac{4x}{3} = \dfrac{3}{4}$

$\Rightarrow x = \dfrac{9}{16}$ [1 mark]

(ii) $\dfrac{x}{4} + \dfrac{2x}{5} = \dfrac{13}{10} \Rightarrow \dfrac{5x}{20} + \dfrac{8x}{20} = \dfrac{26}{20}$

$\Rightarrow 13x = 26$

$\Rightarrow x = 2$ [2 marks]

(iii) $\dfrac{3x}{2} + \dfrac{2x}{3} = 5 \Rightarrow \dfrac{9x}{6} + \dfrac{4x}{6} = 5$

$\Rightarrow \dfrac{13x}{6} = 5$

$\Rightarrow x = \dfrac{30}{13} = 2\dfrac{4}{13}$ [2 marks]

[Total: 5 marks]

3 **(i)** $x - \dfrac{x}{3} = \dfrac{3}{4} \Rightarrow \dfrac{2x}{3} = \dfrac{3}{4}$

$\Rightarrow x = \dfrac{9}{8} = 1\dfrac{1}{8}$ [1 mark]

(ii) $\dfrac{2x}{3} - \dfrac{x}{5} = 7 \Rightarrow \dfrac{10x}{15} - \dfrac{3x}{15} = 7$

$\Rightarrow 7x = 7 \times 15$

$\Rightarrow x = 15$ [2 marks]

(iii) $\dfrac{3x}{2} - \dfrac{2x}{3} = 5 \Rightarrow \dfrac{9x}{6} - \dfrac{4x}{6} = 5$

$\Rightarrow 5x = 5 \times 6$

$\Rightarrow x = 6$ [2 marks]

[Total: 5 marks]

4 **(i)** $\dfrac{x+1}{4}+\dfrac{x-1}{2}=4 \quad \Rightarrow \quad \dfrac{x+1}{4}+\dfrac{2(x-1)}{4}=4$

$\Rightarrow \quad x+1+2x-2=16$

$\Rightarrow \quad 3x=17$

$\Rightarrow \quad x=5\dfrac{2}{3}$ [2 marks]

(ii) $\dfrac{2x}{3}-\dfrac{x-1}{5}=3 \quad \Rightarrow \quad \dfrac{10x}{15}-\dfrac{3x-3}{15}=3$

$\Rightarrow \quad 10x-3x+3=45$

$\Rightarrow \quad 7x=42$

$\Rightarrow \quad x=6$ [2 marks]

(iii) $\dfrac{x+1}{2}+\dfrac{3x-1}{4}=9 \quad \Rightarrow \quad \dfrac{2x+2}{4}+\dfrac{3x-1}{4}=9$

$\Rightarrow \quad 2x+2+3x-1=36$

$\Rightarrow \quad 5x+1=36$

$\Rightarrow \quad x=7$ [2 marks]

[Total: 6 marks]

5 **(i)** $\dfrac{x+1}{5}-\dfrac{x}{6}=1 \quad \Rightarrow \quad \dfrac{6x+6}{30}-\dfrac{5x}{30}=1$

$\Rightarrow \quad 6x+6-5x=30$

$\Rightarrow \quad x=24$ [2 marks]

(ii) $\dfrac{x}{6}-\dfrac{x+1}{5}=\dfrac{1}{10} \quad \Rightarrow \quad \dfrac{5x}{30}-\dfrac{6x+6}{30}=\dfrac{3}{30}$

$\Rightarrow \quad -x-6=3$

$\Rightarrow \quad x=-9$ [2 marks]

(iii) $\dfrac{2x-1}{3}+\dfrac{x+2}{4}=2 \quad \Rightarrow \quad \dfrac{8x-4}{12}+\dfrac{3x+6}{12}=2$

$\Rightarrow \quad 11x+2=24$

$\Rightarrow \quad x=2$ [2 marks]

[Total: 6 marks]

6 **(i)** $\dfrac{3}{x}-\dfrac{2}{x}=10 \quad \Rightarrow \quad 3-2=10x$

$\Rightarrow \quad x=0.1$ [1 mark]

(ii) $\dfrac{3}{x}+\dfrac{2}{x}=10 \quad \Rightarrow \quad 3+2=10x$

$\Rightarrow \quad x=0.5$ [1 mark]

(iii) $\dfrac{3}{x}-\dfrac{4}{3x}=10 \quad \Rightarrow \quad 9-4=30x$

$\Rightarrow \quad x=\dfrac{1}{6}$ [1 mark]

[Total: 3 marks]

7 **(i)** $\dfrac{3}{2x}-\dfrac{2}{3x}=5 \quad \Rightarrow \quad \dfrac{9}{6x}-\dfrac{4}{6x}=5$

$\Rightarrow \quad 5=30x$

$\Rightarrow \quad x=\dfrac{1}{6}$ [1 mark]

(ii) $\dfrac{3+x}{2x} - \dfrac{2+x}{3x} = 1$

$$\Rightarrow \quad 3(3+x) - 2(2+x) = 6x$$
$$\Rightarrow \quad 9 + 3x - 4 - 2x = 6x$$
$$\Rightarrow \quad 5 = 5x$$
$$\Rightarrow \quad x = 1$$

[2 marks]

(iii) $\dfrac{3-x}{2x} + \dfrac{2-x}{3x} + 1 = 0$

$$\Rightarrow \quad 3(3-x) + 2(2-x) + 6x = 0$$
$$\Rightarrow \quad 9 - 3x + 4 - 2x + 6x = 0$$
$$\Rightarrow \quad 13 + x = 0$$
$$\Rightarrow \quad x = -13$$

[2 marks]

[Total: 5 marks]

8 **(i)** $\dfrac{3}{x} - \dfrac{4}{2x+1} = 1$

$$\Rightarrow \quad 3(2x+1) - 4x = x(2x+1)$$
$$\Rightarrow \quad 6x + 3 - 4x = 2x^2 + x$$
$$\Rightarrow \quad 2x^2 - x - 3 = 0$$
$$\Rightarrow \quad (2x-3)(x+1) = 0$$
$$\Rightarrow \quad x = 1.5 \text{ or } x = -1$$

[2 marks]

(ii) $1 + \dfrac{1}{x+1} = \dfrac{2x}{x+1}$

$$\Rightarrow \quad (x+1) + 1 = 2x$$
$$\Rightarrow \quad x + 2 = 2x$$
$$\Rightarrow \quad x = 2$$

[1 mark]

(iii) $\dfrac{2}{2x+1} - \dfrac{3}{3x+1} + \dfrac{1}{2} = 0$

$$\Rightarrow \quad 4(3x+1) - 6(2x+1) + (2x+1)(3x+1) = 0$$
$$\Rightarrow \quad 12x + 4 - 12x - 6 + 6x^2 + 5x + 1 = 0$$
$$\Rightarrow \quad 6x^2 + 5x - 1 = 0$$
$$\Rightarrow \quad (6x-1)(x+1) = 0$$
$$\Rightarrow \quad x = \frac{1}{6} \text{ or } x = -1$$

[3 marks]

[Total: 6 marks]

9 **(i)** $\dfrac{3(p+2)}{5} + 2 = p$

$$\Rightarrow \quad 3(p+2) + 10 = 5p$$
$$\Rightarrow \quad 3p + 6 + 10 = 5p$$
$$\Rightarrow \quad 16 = 2p$$
$$\Rightarrow \quad p = 8$$

[2 marks]

(ii) $q - \dfrac{3(q+1)}{4} = 1$

$$\Rightarrow \quad 4q - 3(q+1) = 4$$
$$\Rightarrow \quad 4q - 3q - 3 = 4$$
$$\Rightarrow \quad q = 7$$

[2 marks]

(iii) $\dfrac{2(r+1)}{3} + \dfrac{(r+2)}{5} = 1$

$$\Rightarrow \quad 10(r+1) + 3(r+2) = 15$$
$$\Rightarrow \quad 10r + 10 + 3r + 6 = 15$$
$$\Rightarrow \quad 13r = -1$$
$$\Rightarrow \quad r = -\frac{1}{13}$$

[2 marks]

[Total: 6 marks]

Exercise 2.6 Completing the square

1 $x^2 + 6x + 12 \equiv (x + a)^2 + b$

$\equiv x^2 + 2ax + a^2 + b$

$\Rightarrow \quad 6 = 2a \quad$ and $\quad 12 = a^2 + b$

$6 = 2a \quad \Rightarrow \quad a = 3$

$12 = a^2 + b \quad \Rightarrow \quad 12 = 9 + b$

$\Rightarrow \quad b = 3$ [2 marks]

2 $x^2 - cx + 9 \equiv (x - 2)^2 + d$

$\equiv x^2 - 4x + (4 + d)$

$\Rightarrow -c = -4 \quad$ and $\quad 9 = 4 + d$

$\Rightarrow c = 4 \quad$ and $\quad d = 5$ [2 marks]

3 $4 - 2x - x^2 \equiv p - (x + q)^2$

$\equiv p - x^2 - 2qx - q^2$

$\Rightarrow 4 = p - q^2 \quad$ and $\quad -2 = -2q$

$\Rightarrow \quad q = 1$

$4 = p - 1 \Rightarrow p = 5$ [2 marks]

4 $6 + 4x - x^2 \equiv r - (x + s)^2$

$\equiv r - x^2 - 2sx - s^2$

$\Rightarrow 6 = r - s^2 \quad$ and $\quad 4 = -2s$

$\Rightarrow s = -2$

$6 = r - (-2)^2 \Rightarrow r = 10$ [2 marks]

5 $3x^2 + ax + 7 \equiv b(x + 2)^2 - c$

$\equiv b(x^2 + 4x + 4) - c$

$\equiv bx^2 + 4bx + (4b - c)$

Coeff $x^2 \Rightarrow b = 3$

Coeff $x \Rightarrow a = 4b = 12$

Constant term $\Rightarrow 7 = 4b - c$

$= 12 - c$

$\Rightarrow c = 5$ [3 marks]

6 $2x^2 - 12x + 23 \equiv p(x - q)^2 + r$

$\equiv p(x^2 - 2qx + q^2) + r$

$\equiv px^2 - 2pqx + (pq^2 + r)$

Coeff $x^2 \Rightarrow p = 2$

Coeff $x \Rightarrow -2pq = -12$

$\Rightarrow -4q = -12$

$\Rightarrow q = 3$

Constant term: $\quad 23 = pq^2 + r$

$= 2(3)^2 + r$

$\Rightarrow r = 5$ [3 marks]

7 $1 - ax - bx^2 \equiv 3 - c(x + 1)^2$

$\equiv 3 - c(x^2 + 2x + 1)$

$\equiv -cx^2 - 2cx + (3 - c)$

Constant term $\Rightarrow 1 = 3 - c$

$\Rightarrow \quad c = 2$

Coeff $x \quad \Rightarrow -a = -2c$ [3 marks]

$\Rightarrow a = 4$

Coeff $x^2 \quad \Rightarrow -b = -c$

$\Rightarrow b = 2$

8 $p + q(x + r)^2 = 5x^2 - 20x + 16$

$p + q(x^2 + 2rx + r^2) = 5x^2 - 20x + 16$

$qx^2 + 2qrx + (p + qr^2) = 5x^2 - 20x + 16$

Coeff x^2 $\Rightarrow q = 5$

Coeff x $\Rightarrow 2qr = -20$

$\Rightarrow 10r = -20$ so $r = -2$

Constant term $\Rightarrow p + qr^2 = 16$

$\Rightarrow p + 5(-2)^2 = 16$ so $p = -4$ [3 marks]

9 (i) $x^2 + 6x + 20 \equiv (x + a)^2 + b$

$\equiv x^2 + 2ax + a^2 + b$

Coeff x $\Rightarrow 2a = 6$

$\Rightarrow a = 3$ [2 marks]

Constant term $\Rightarrow 20 = a^2 + b$

$\Rightarrow b = 11$

(ii) Using these values $y = (x + 3)^2 + 11$

$\Rightarrow y - 11 = (x + 3)^2$

$\Rightarrow x + 3 = \pm\sqrt{y - 11}$

$\Rightarrow x = -3 \pm \sqrt{y - 11}$ [2 marks]

[Total: 4 marks]

10 (i) $4x^2 - 16x + 11 = p(x + q)^2 + r$

$= p(x^2 + 2qx + q^2) + r$

$= px^2 + 2pqx + (pq^2 + r)$

Coeff x^2 $\Rightarrow p = 4$

Coeff x $\Rightarrow 8q = -16$

$\Rightarrow q = -2$

Constant term $\Rightarrow 11 = 4\,(-2)^2 + r$

$\Rightarrow r = -5$ [3 marks]

(ii) Using these values $y = 4(x - 2)^2 - 5$

$\Rightarrow \dfrac{y + 5}{4} = (x - 2)^2$

$\Rightarrow x - 2 = \pm\dfrac{\sqrt{y + 5}}{2}$

$\Rightarrow x = 2 \pm \dfrac{\sqrt{y + 5}}{2}$ [2 marks]

[Total: 5 marks]

11 $b - 3(x + c)^2 = b - 3x^2 - 6cx - 3c^2 \quad \Rightarrow \quad a = -3$ and $-6c = -12 \quad \Rightarrow \quad c = 2$

$b - 3 \times 2^2 = 5 \quad \Rightarrow \quad b = 17$ [3 marks]

3 Algebra III

Exercise 3.1 Function notation

1 **(i)** $f(-2) = (-2) + 2$
$= 0$ [1 mark]

(ii) $f(0) = 0 + 2$
$= 2$ [1 mark]

(iii) $f(2) = 2 + 2$
$= 4$ [1 mark]

(iv) $g(-2) = (-2)^2 + 2$
$= 6$ [1 mark]

(v) $g(0) = 0 + 2$
$= 2$ [1 mark]

(vi) $g(2) = 2^2 + 2$
$= 6$ [1 mark]

[Total: 6 marks]

2 **(i)** $f(-3) = 2(-3)^2$
$= 18$ [1 mark]

(ii) $f(3) = 2(3)^2$
$= 18$ [1 mark]

(iii) $g(-3) = \frac{2}{-3}$
$= -\frac{2}{3}$ [1 mark]

(iv) $g(3) = \frac{2}{3}$ [1 mark]

[Total: 4 marks]

3 $f(x) = 2x - 3;\ g(x) = 3x - 2$

(i) $f(x) = 0 \Rightarrow x = 1.5$ [1 mark]

(ii) $g(x) = 0 \Rightarrow x = \frac{2}{3}$ [1 mark]

(iii) $f(x) = g(x) \Rightarrow 2x - 3 = 3x - 2$
$\Rightarrow -x = 1$
$\Rightarrow x = -1$ [1 mark]

[Total: 3 marks]

4 $f(x) = 3x + 2;\ g(x) = 2x - 3$

(i) $f(x) - g(x) = 0 \Rightarrow (3x + 2) - (2x - 3) = 0$
$\Rightarrow 3x + 2 - 2x + 3 = 0$
$\Rightarrow x = -5$ [2 marks]

(ii) $f(x) + g(x) = 0 \Rightarrow (3x + 2) + (2x - 3) = 0$
$\Rightarrow 5x - 1 = 0$
$\Rightarrow x = \frac{1}{5}$ (or 0.2) [2 marks]

(iii) $f(x) = 2g(x) \Rightarrow (3x + 2) = 2(2x - 3)$
$\Rightarrow 3x + 2 = 4x - 6$
$\Rightarrow x = 8$ [2 marks]

[Total: 6 marks]

5 $f(x) = x^2 - 1$; $g(x) = 2x - 2$

(i) $f(x) - g(x) = 0 \Rightarrow (x^2 - 1) - (2x - 2) = 0$
$\Rightarrow x^2 - 1 - 2x + 2 = 0$
$\Rightarrow x^2 - 2x + 1 = 0$
$\Rightarrow (x - 1)^2 = 0$
$\Rightarrow x = 1$ [2 marks]

(ii) $f(x) + g(x) = 0 \Rightarrow (x^2 - 1) + (2x - 2) = 0$
$\Rightarrow x^2 + 2x - 3 = 0$
$\Rightarrow (x + 3)(x - 1) = 0$
$\Rightarrow x = -3$ or $x = 1$ [2 marks]

(iii) $f(x) = [g(x)]^2 \Rightarrow x^2 - 1 = (2x - 2)^2$
$\Rightarrow x^2 - 1 = 4x^2 - 8x + 4$
$\Rightarrow 0 = 3x^2 - 8x + 5$
$\Rightarrow (3x - 5)(x - 1) = 0$
$\Rightarrow x = 1\frac{2}{3}$ or $x = 1$ [2 marks]

[Total: 6 marks]

6 $f(x) = 2x + 3$

(i) $f(2x + 1) = 2(2x + 1) + 3$
$= 4x + 5$ [1 mark]

(ii) $f((x - 1)^2) = 2(x - 1)^2 + 3$
$= 2(x^2 - 2x + 1) + 3$
$= 2x^2 - 4x + 5$ [1 mark]

(iii) $[f(x - 1)]^2 = [2(x - 1) + 3]^2$
$= [2x + 1]^2$
$= 4x^2 + 4x + 1$ [1 mark]

[Total: 3 marks]

7 $f(x) = 2x^2 + 3x + 1$

(i) $f(2x) = 2(2x)^2 + 3(2x) + 1$
$= 8x^2 + 6x + 1$ [1 mark]

(ii) $f(-2x) = 2(-2x)^2 + 3(-2x) + 1$
$= 8x^2 - 6x + 1$ [1 mark]

(iii) $f(2x)^2 = f(4x^2)$
$= 2(4x^2)^2 + 3(4x^2) + 1$
$= 32x^4 + 12x^2 + 1$ [1 mark]

(iv) $f(2x^2) = 2(2x^2)^2 + 3(2x^2) + 1$
$= 8x^4 + 6x^2 + 1$ [1 mark]

[Total: 4 marks]

8 $f(x) = \dfrac{x^2 - 4}{3}$

(i) $f(4) = \dfrac{4^2 - 4}{3}$

$= 4$ [1 mark]

(ii) **(a)** $f(x) = 0 \Rightarrow \dfrac{x^2 - 4}{3} = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$ [1 mark]

(b) $f(x) = 4 \Rightarrow \dfrac{x^2 - 4}{3} = 4$

$\Rightarrow x^2 - 4 = 12$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$ [2 marks]

[Total: 4 marks]

9 $f(x) = \dfrac{x - 4}{5}$

(i) $f(0) = \dfrac{0 - 4}{5}$

$= -\dfrac{4}{5}$ (or -0.8) [1 mark]

(ii) **(a)** $f(x) = 0 \Rightarrow \dfrac{x - 4}{5} = 0$

$\Rightarrow x = 4$ [1 mark]

(b) $f(x) = f(2x) \Rightarrow \dfrac{x - 4}{5} = \dfrac{2x - 4}{5}$

$\Rightarrow x - 4 = 2x - 4$

$\Rightarrow x = 0$ [1 mark]

[Total: 3 marks]

10 $f(x) = \dfrac{2x - 3}{2x + 3}$

(i) **(a)** $f(x) = 0 \Rightarrow \dfrac{2x - 3}{2x + 3} = 0$

$\Rightarrow 2x = 3$

$\Rightarrow x = 1.5$ [1 mark]

(b) $f(x) = 2 \Rightarrow \dfrac{2x - 3}{2x + 3} = 2$

$\Rightarrow 2x - 3 = 4x + 6$

$\Rightarrow -9 = 2x$

$\Rightarrow x = -4.5$ [2 marks]

(ii) $f(x) = 1 \Rightarrow \dfrac{2x - 3}{2x + 3} = 1$

$\Rightarrow 2x - 3 = 2x + 3$

$\Rightarrow -6 = 0$ not possible [1 mark]

[Total: 4 marks]

11 (i) $f(2x) = (3(2x) + 2)^2$
$= (6x + 2)^2$
$= [2(3x + 1)]^2$
$= 4(3x + 1)^2$ [2 marks]

(ii) $f(x - 2) = (3(x - 2) + 2)^2$
$= (3x - 6 + 2)^2$
$= (3x - 4)^2$ [1 mark]

(iii) $(f(x) - 4)^2 = (f(x))^2 - 8f(x) + 16$
$= (3x + 2)^4 - 8(3x + 2)^2 + 16$

then either:

Let $(3x + 2) = p$

$(f(x) - 4)^2 = p^4 - 8p^2 + 16$
$= (p^2 - 4)^2$
$= ((3x + 2)^2 - 4)^2$
$= (9x^2 + 12x + 4 - 4)^2$
$= (9x^2 + 12x)^2$
$= (3x)^2(3x + 4)^2$
$= 9x^2(3x + 4)^2$

Or, expand $(3x + 2)^4 - 8(3x + 2)^2 + 16$ and simplify. [3 marks]

[Total: 6 marks]

12 Let $f(x) = ax + b$ where a and b are constants.

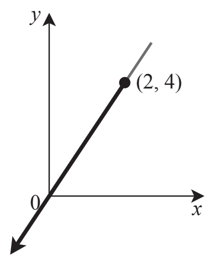
$ax + b + 2 \equiv a(x + 1) + b \quad \Rightarrow \quad ax + b + 2 = ax + a + b \quad \Rightarrow \quad b + 2 = a + b \quad \Rightarrow \quad a = 2$

So b can take any value.

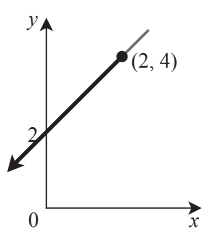
Therefore $f(x) = 2x + b$ [3 marks]

Exercise 3.2 Domain and range of a function

1 (i) $f(x) = 2x \qquad x \leqslant 2$
$\Rightarrow f(x) \leqslant 4$ [2 marks]

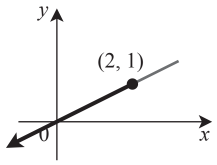


(ii) $f(x) = x + 2 \qquad x \leqslant 2$
$\Rightarrow f(x) \leqslant 4$ [2 marks]



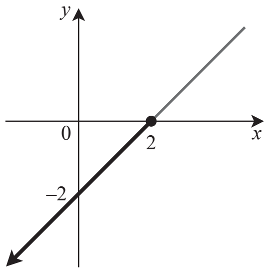
(iii) $f(x) = \frac{x}{2} \qquad x \leqslant 2$

$\Rightarrow f(x) \leqslant 1$ [2 marks]



(iv) $f(x) = x - 2 \qquad x \leqslant 2$

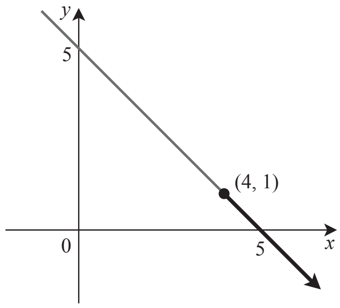
$\Rightarrow f(x) \leqslant 0$ [2 marks]



[Total: 8 marks]

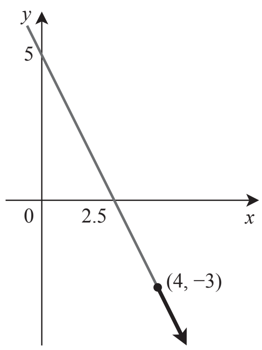
2 **(i)** $f(x) = 5 - x \qquad x \geqslant 4$

$\Rightarrow f(x) \leqslant 1$ [2 marks]



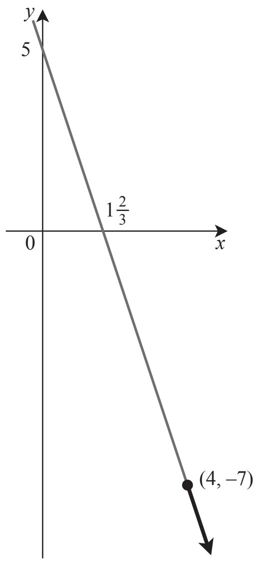
(ii) $f(x) = 5 - 2x \qquad x \geqslant 4$

$\Rightarrow f(x) \leqslant -3$ [2 marks]



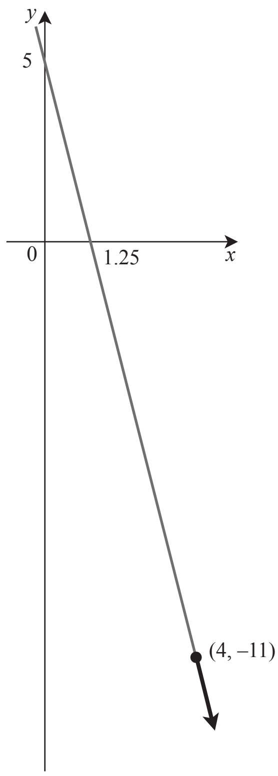
(iii) $f(x) = 5 - 3x \qquad x \geqslant 4$

$\Rightarrow f(x) \leqslant -7$ [2 marks]



(iv) $f(x) = 5 - 4x \qquad x \geqslant 4$

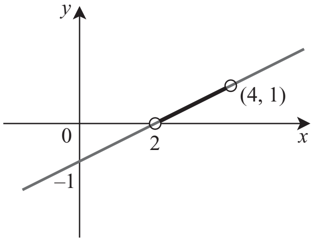
$\Rightarrow f(x) \leqslant -11$ [2 marks]



[Total: 8 marks]

3 **(i)** $f(x) = \frac{x}{2} - 1 \qquad 2 < x < 4$

$\Rightarrow 0 < f(x) < 1$ [2 marks]

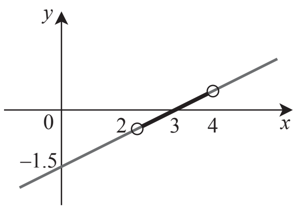


(ii) $f(x) = \dfrac{x-1}{2} - 1 \qquad 2 < x < 4$

$= \dfrac{x-1-2}{2}$

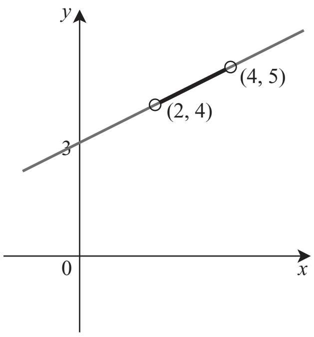
$= \dfrac{x-3}{2}$

$\Rightarrow -0.5 < f(x) < 0.5$ [3 marks]



(iii) $f(x) = \dfrac{x}{2} + 3 \qquad 2 < x < 4$

$\Rightarrow 4 < f(x) < 5$ [2 marks]

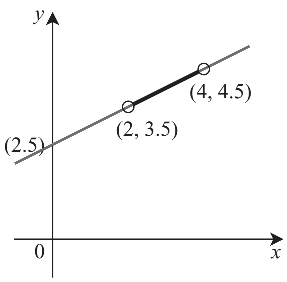


(iv) $f(x) = \dfrac{x-1}{2} + 3 \qquad 2 < x < 4$

$= \dfrac{x-1+6}{2}$

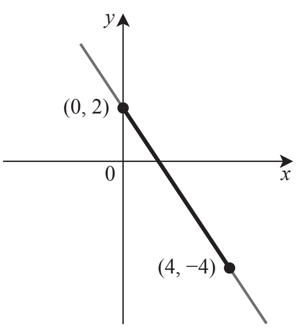
$= \dfrac{x+5}{2}$

$\Rightarrow 3.5 < f(x) < 4.5$ [3 marks]



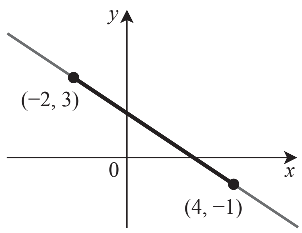
(v) $f(x) = \dfrac{4 - 3x}{2} \qquad 0 \leqslant x \leqslant 4$

$\Rightarrow -4 \leqslant f(x) \leqslant 2$ [3 marks]



(vi) $f(x) = \dfrac{5 - 2x}{3} \qquad -2 \leqslant x \leqslant 4$

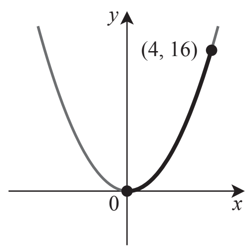
$\Rightarrow -1 \leqslant f(x) \leqslant 3$ [3 marks]



[Total: 16 marks]

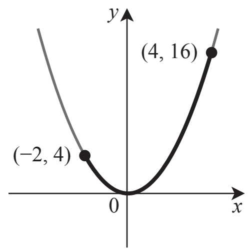
4 **(i)** $f(x) = x^2 \qquad 0 \leqslant x \leqslant 4$

$\Rightarrow 0 \leqslant f(x) \leqslant 16$ [2 marks]



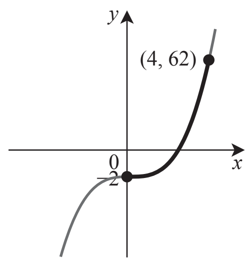
(ii) $f(x) = x^2 \qquad -2 \leqslant x \leqslant 4$

$\Rightarrow 0 \leqslant f(x) \leqslant 16$ [2 marks]



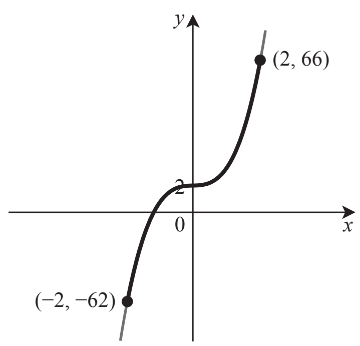
(iii) $f(x) = x^3 - 2 \qquad 0 \leqslant x \leqslant 4$

$\Rightarrow -2 \leqslant f(x) \leqslant 62$ [2 marks]



(iv) $f(x) = x^3 + 2 \qquad -4 \leqslant x \leqslant 4$

$\Rightarrow -62 \leqslant f(x) \leqslant 66$ [2 marks]



[Total: 8 marks]

5 **(i)** Domain $-2 \leqslant x \leqslant 5$
Range $-1 \leqslant f(x) \leqslant 13$ [2 marks]

(ii) Domain $-3 \leqslant x \leqslant 4$
Range $-5 \leqslant f(x) \leqslant 9$ [2 marks]

(iii) Domain $-2 \leqslant x \leqslant 3$
Range $0 \leqslant x \leqslant 16$ [2 marks]

(iv) Domain $-2 \leqslant x \leqslant 2$
Range $-7 \leqslant x \leqslant 9$ [2 marks]

[Total: 8 marks]

6 **(i)** $f(x) = x^2 + x$

Quadratic curve is symmetrical so minimum f(x) is when $x = -\frac{1}{2}$ $x = \frac{1}{2} \Rightarrow f(x) = -\frac{1}{4}$

Range is $-\frac{1}{4} \leqslant x < 6$ [2 marks]

(ii) $f(x) = x^2 - x$

Quadratic curve is symmetrical so minimum f(x) is when $x = \frac{1}{2}$ $x = \frac{1}{2} \Rightarrow f(x) = -\frac{1}{4}$

Range is $-\frac{1}{4} \leqslant x < 12$ [2 marks]

(iii) $f(x) = x^2 + x + 3$

Quadratic curve is symmetrical so minimum f(x) is when $x = \frac{-3+2}{2} = -\frac{1}{2}$ $x = -\frac{1}{2} \Rightarrow f(x) = 2\frac{3}{4}$

Range is $2\frac{3}{4} \leqslant x < 9$ [2 marks]

(iv) $f(x) = x^2 + x - 4$

Quadratic curve is symmetrical so minimum f(x) is when $x = \frac{-3+2}{2} = -\frac{1}{2}$ $x = -\frac{1}{2} \Rightarrow f(x) = -4\frac{1}{4}$

Range is $-4\frac{1}{4} \leqslant x < 2$ [2 marks]

[Total: 8 marks]

7 **(i)** $-8 < f(x) < 12$ [1 mark]

(ii) $-16 < f(x) < 20$ [1 mark]

(iii) $0 \leqslant f(x) < 9$ [1 mark]

(iv) $-9 < f(x) < 3$ [1 mark]

[Total: 4 marks]

8 **(i)** $\left.\begin{array}{l} x = -2 \Rightarrow f(x) = 64 \\ x = 3 \Rightarrow f(x) = 9 \end{array}\right\}$ From graph $0 \leqslant f(x) \leqslant 64$ [1 mark]

(ii) $\left.\begin{array}{l} x = -3 \Rightarrow f(x) = -9 \\ x = 2 \Rightarrow f(x) = 16 \end{array}\right\}$ From graph $-9 \leqslant f(x) \leqslant 16$ [1 mark]

(iii) From graph, lowest point is $(-1, -5)$

$\left.\begin{array}{l} x = -5 \Rightarrow f(x) = 11 \\ x = 2 \Rightarrow f(x) = 4 \end{array}\right\}$ From graph $-5 \leqslant f(x) \leqslant 11$ [1 mark]

(iv) By symmetry, lowest point is when $x = \left(\dfrac{-2+4}{2}\right) = 1$

$$\left.\begin{aligned} x = 1 &\Rightarrow f(x) = -9 \\ x = -3 &\Rightarrow f(x) = 7 \\ x = 5 &\Rightarrow f(x) = 7 \end{aligned}\right\} \text{ Range is } -9 \leqslant f(x) \leqslant 7$$

[2 marks]

[Total: 5 marks]

9 $x^2 - 7 = 2 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$ but $-5 \leqslant x \leqslant 1$ so $x = -3$ [3 marks]

Exercise 3.3 Composite functions

1 $(2 - x)^3$: subtract x and then cube
$(2 - x)^3 = fg(x)$ where $g(x) = 2 - x$, $f(x) = x^3$ [2 marks]

2 $f(x) = x - 2$; $g(x) = x^2 - 2$

(i) **(a)** $fg(x) = f(x^2 - 2)$
$= (x^2 - 2) - 2$
$= x^2 - 4$ [1 mark]

(b) $gf(x) = g(x - 2)$
$= (x - 2)^2 - 2$
$= x^2 - 4x + 2$ [1 mark]

(ii) **(a)** $fg(2) = 2^2 - 4$
$= 0$ [1 mark]

(b) $gf(2) = 4 - 8 + 2$
$= -2$ [1 mark]

(iii) $fg(x) = gf(x) \Rightarrow x^2 - 4 = x^2 - 4x + 2$
$\Rightarrow 4x = 6$
$\Rightarrow x = 1.5$ [2 marks]

[Total: 6 marks]

3 $f(x) = 2x$ $g(x) = 2 + x$

(i) **(a)** $fg(x) = f(2 + x)$
$= 2(2 + x)$ [1 mark]

(b) $gf(x) = g(2x)$
$= 2 + 2x$ [1 mark]

(ii) **(a)** $fg(2) = 2(4)$
$= 8$ [1 mark]

(b) $gf(2) = 2 + 4$
$= 6$ [1 mark]

(iii) $f(x) = g(x) \Rightarrow 2x = 2 + x$
$\Rightarrow x = 2$ [1 mark]

[Total: 5 marks]

4 **(i)** $f(x) = \sqrt{1+x}$ i.e. $x \to 1 + x \to \sqrt{1+x}$

$gh(x) = \sqrt{1+x} \Rightarrow h(x) = 1 + x$

$g(x) = \sqrt{x}$ [2 marks]

(ii) $f(x) = (1 - x)^2$ i.e. $x \to (1 - x) \to (1 - x)^2$

$gh(x) = (1 - x)^2 \Rightarrow h(x) = 1 - x$

$g(x) = x^2$ [2 marks]

(iii) $f(x) = \dfrac{1}{1 + x}$ i.e. $x \to 1 + x \to \dfrac{1}{1 + x}$

$gh(x) = \dfrac{1}{1 + x} \Rightarrow h(x) = 1 + x$

$g(x) = \dfrac{1}{x}$ [2 marks]

[Total: 6 marks]

5 **(i)** $f(x) = \sqrt{1+3x}$ i.e. $x \to 3x \to 1 + 3x \to \sqrt{1+3x}$

$ghk(x) = \sqrt{1+3x} \Rightarrow k(x) = 3x; h(x) = 1 + x; g(x) = \sqrt{x}$ [3 marks]

(ii) $f(x) = (1 - 2x)^3$ i.e. $x \to 2x \to 1 - 2x \to (1 - 2x)^3$

$ghk(x) = (1 - 2x)^3 \Rightarrow k(x) = 2x; h(x) = 1 - x; g(x) = x^3$ [3 marks]

(iii) $f(x) = \dfrac{1}{1 - 4x}$ i.e. $x \to 4x \to 1 - 4x \to \dfrac{1}{1 - 4x}$

$ghk(x) = \dfrac{1}{1 - 4x} \Rightarrow k(x) = 4x; h(x) = 1 - x; g(x) = \dfrac{1}{x}$ [3 marks]

[Total: 9 marks]

6 **(i)** $f(x) = \sin 4x$ i.e. $x \to 4x \to \sin 4x$

$gh(x) = \sin 4x \Rightarrow h(x) = 4x; g(x) = \sin x$ [2 marks]

(ii) $f(x) = \cos(x - 45°)$ i.e. $x \to x - 45° \to \cos(x - 45°)$

$gh(x) = \cos(x - 45°) \Rightarrow h(x) = x - 45°; g(x) = \cos x$ [2 marks]

(iii) $f(x) = \dfrac{1}{2}\tan x$ i.e. $x \to \tan x \to \dfrac{1}{2}\tan x$

$gh(x) = \dfrac{1}{2}\tan x \Rightarrow h(x) = \tan x;\ g(x) = \dfrac{1}{2}x$ [2 marks]

[Total: 6 marks]

7 **(i)** $f(x) = 3\tan 4x$ i.e. $x \to 4x \to \tan 4x \to 3\tan 4x$

$ghk(x) = 3\tan 4x \Rightarrow k(x) = 4x; h(x) = \tan x; g(x) = 3x$ [3 marks]

(ii) $f(x) = \dfrac{1}{2}\sin(x + 30°)$ i.e. $x \to x + 30° \to \sin(x + 30°) \to \dfrac{1}{2}\sin(x + 30°)$

$ghk(x) = \dfrac{1}{2}\sin(x + 30°) \Rightarrow k(x) = x + 30°; h(x) = \sin x; g(x) = \dfrac{1}{2}x$ [3 marks]

(iii) $f(x) = \dfrac{2}{\cos 3x}$ i.e. $x \to 3x \to \cos 3x \to \dfrac{2}{\cos 3x}$

$ghk(x) = \dfrac{2}{\cos 3x} \Rightarrow k(x) = 3x;\ h(x) = \cos x;\ g(x) = \dfrac{2}{x}$ [3 marks]

[Total: 9 marks]

8 $f(x) = x^2$ $g(x) = ax - 2$

(i) $fg(x) = f(ax - 2)$
$= (ax - 2)^2$

$gf(x) = g(x^2)$
$= ax^2 - 2$ [2 marks]

(ii) $a = 2 \Rightarrow fg(x) = (2x - 2)^2$ and $gf(x) = 2x^2 - 2$

$fg(x) = gf(x) \Rightarrow 4x^2 - 8x + 4 = 2x^2 - 2$

$\Rightarrow 2x^2 - 8x + 6 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x - 1)(x - 3) = 0$

$\Rightarrow x = 1$ or $x = 3$

$fg(x) < gf(x) \Rightarrow 1 < x < 3$ [3 marks]

(iii) $a = 1 \Rightarrow fg(x) = (x - 2)^2$ and $gf(x) = x^2 - 2$

$fg(x) = gf(x) \Rightarrow x^2 - 4x + 4 = x^2 - 2$

$\Rightarrow 6 - 4x = 0$

$\Rightarrow x = 1.5$ [2 marks]

(iv) $a = 0 \Rightarrow f(x) = x^2$ and $g(x) = -2$

$f(x) = g(x) \Rightarrow x^2 = -2$

No real solution [1 mark]

[Total: 8 marks]

9 d e b a c [2 marks]

10 (i) $2 \times (-4) + 1 \leqslant f(x) \leqslant 2 \times 4 + 1 \Rightarrow -7 \leqslant f(x) \leqslant 9$ [2 marks]

(ii) $f^{-1}(x) = \dfrac{x-1}{2}$ $-7 \leqslant x \leqslant 9$ [2 marks]

(iii) $f^2(x) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$ [2 marks]

(iv) $f^2(3) = f(7)$ but 7 is not in the domain of f [1 mark]

(v) $fgh(x) = fg\left(\dfrac{1}{x}\right) = f\left(\dfrac{1}{x^2} - 3\right) = 2\left(\dfrac{1}{x^2} - 3\right) + 1 = \dfrac{2}{x^2} - 5$ $1 < x \leqslant 2$ [3 marks]

(vi) $2\left(\left(\dfrac{1}{2}\right)^2 - 3\right) + 1 \leqslant fgh(x) < 2\left(\left(\dfrac{1}{1}\right)^2 - 3\right) + 1 \Rightarrow -4.5 \leqslant fgh(x) < -3$ [2 marks]

[Total: 12 marks]

Exercise 3.4 Graphs of linear functions

1 **(i)** A(2, 4), B(4, 8), gradient $= \dfrac{8-4}{4-2} = 2$ [1 mark]

(ii) A(3, 5), B(5, 5), gradient $= \dfrac{5-5}{5-3} = 0$ [1 mark]

(iii) A(2, 4), B(7, 6), gradient $= \dfrac{6-4}{7-2} = 0.4$ [1 mark]

(iv) A(3, 6), B(6, 0), gradient $= \dfrac{0-6}{6-3} = -2$ [1 mark]

(v) A(2, 9), B(5, 3), gradient $= \dfrac{3-9}{5-2} = -2$ [1 mark]

(vi) A(6, 7), B(2, 2), gradient $= \dfrac{2-7}{2-6} = 1.25$ [1 mark]

(vii) A(−2, 3), B(2, 6), gradient $= \dfrac{6-3}{2-(-2)} = 0.75$ [1 mark]

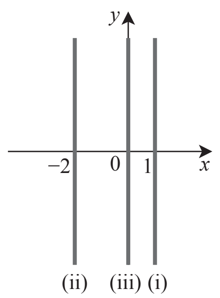
(viii) A(3, −4), B(−7, 6), gradient $= \dfrac{6-(-4)}{-7-3} = -1$ [1 mark]

(ix) A(−2, −1), B(−1, −2), gradient $= \dfrac{(-2)-(-1)}{(-1)-(-2)} = -1$ [1 mark]

(x) A(−2, 0), B(0, −4), gradient $= \dfrac{(-4-0)}{0-(-2)} = -2$ [1 mark]

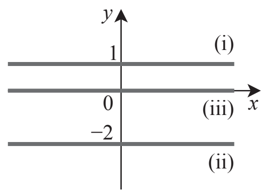
[Total: 10 marks]

2



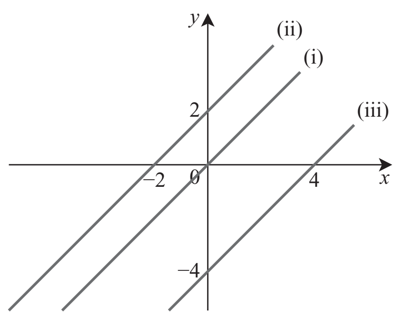
[2 marks]

3



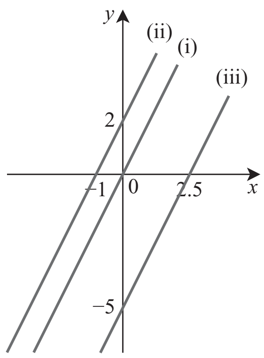
[2 marks]

4



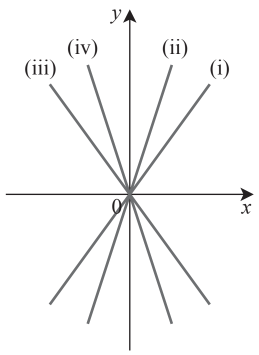
[2 marks]

5



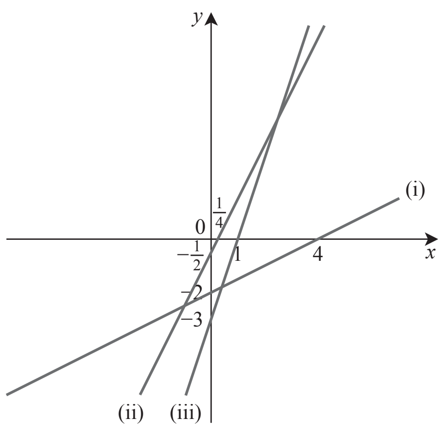
[2 marks]

6



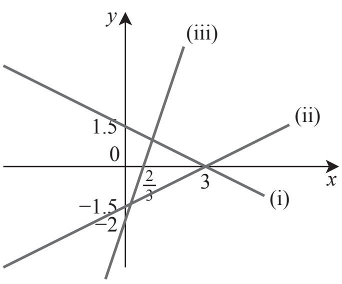
[2 marks]

7



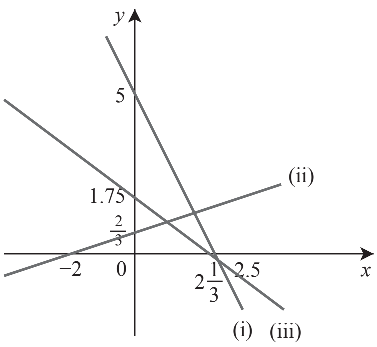
[3 marks]

8



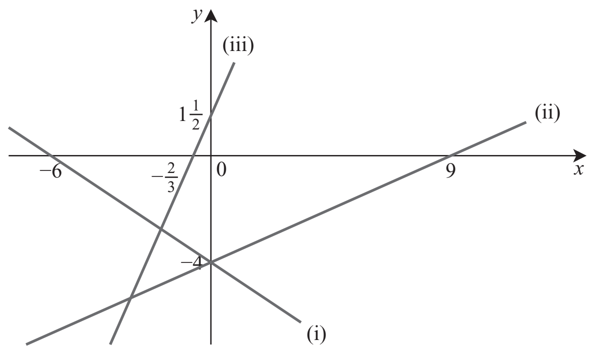
[3 marks]

9



[3 marks]

10



[3 marks]

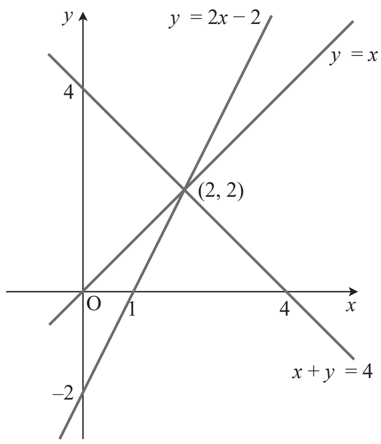
11 (i) t hours work costs £$25t$
$\Rightarrow$ total cost $C = 50 + 25t$ [1 mark]

(ii) $t = 3 \Rightarrow C = 50 + 25(3)$
$\Rightarrow$ cost of a 3 hour job = £125 [1 mark]

(iii) $C = 250 \Rightarrow 50 + 25t = 250$
$\Rightarrow 25t = 200$
$\Rightarrow t = 8$ hours [2 marks]

[Total: 4 marks]

12



The y-intercepts are $2(0) - 2 = -2$ and $4 - 0 = 4$ and 0

The x-intercepts are $\dfrac{0 + 2}{2} = 1$ and $4 - 0 = 4$ and 0

The graphs intersect each other when $2x - 2 = x \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad y = 2$

Also $2 + 2 = 4$

So all three graphs meet at $(2, 2)$ [4 marks]

Exercise 3.5 Finding the equation of a line

1 **(i)** $y = 3$ [1 mark]

(ii) $y = -2x$ or $2x + y = 0$ [1 mark]

(iii) $y = -3x$ or $3x + y = 0$ [1 mark]

(iv) $x = -2$ [1 mark]

[Total: 4 marks]

2 **(i)** $y = 2 - x$ [1 mark]

(ii) $y = 2 - 2x$ [1 mark]

(iii) $y = 2 - 3x$ [1 mark]

(iv) $y = 3x + 2$ [1 mark]

(v) $y = 2x + 2$ [1 mark]

(vi) $y = x + 2$ [1 mark]

[Total: 6 marks]

3 **(i)** $x + 2y = 3$ [1 mark]

(ii) $x + y = 3$ [1 mark]

(iii) $2x + y = 3$ [1 mark]

(iv) $2x - y = 3$ [1 mark]

(v) $x - 2y = 3$ [1 mark]

[Total: 5 marks]

4 **(i)** Gradient = 1 $\Rightarrow y = x + c$
Through (2, 5) $\Rightarrow 5 = 2 + c \Rightarrow c = 3$
Line $y = x + 3$ [2 marks]

(ii) Gradient = 2 $\Rightarrow y = 2x + c$
Through (−1, 3) $\Rightarrow 3 = -2 + c \Rightarrow c = 5$
Line $y = 2x + 5$ [2 marks]

(iii) Gradient = 3 $\Rightarrow y = 3x + c$
Through (3, −1) $\Rightarrow -1 = 9 + c \Rightarrow c = -10$
Line $y = 3x - 10$ [2 marks]

(iv) Gradient = 4 $\Rightarrow y = 4x + c$
Through (−2, −5) $\Rightarrow -5 = -8 + c \Rightarrow c = 3$
Line $y = 4x + 3$ [2 marks]

[Total: 8 marks]

5 **(i)** Gradient −1 $\Rightarrow y = -x + c$
Through (−1, 2) $\Rightarrow 2 = 1 + c \Rightarrow c = 1$
Line $y = 1 - x$ [2 marks]

(ii) Gradient $-2 \Rightarrow y = -2x + c$
Through $(-2, -1) \Rightarrow -1 = 4 + c \Rightarrow c = -5$
Line $y = -2x - 5$ or $2x + y + 5 = 0$ [2 marks]

(iii) Gradient $-3 \Rightarrow y = -3x + c$
Through $(-1, -4) \Rightarrow -4 = 3 + c \Rightarrow c = -7$
Line $y = -3x - 7$ or $3x + y + 7 = 0$ [2 marks]

(iv) Gradient $-4 \Rightarrow y = -4x + c$
Through $(5, 0) \Rightarrow 0 = -20 + c \Rightarrow c = 20$
Line $y = -4x + 20$ or $4x + y - 20 = 0$ [2 marks]
[Total: 8 marks]

6 **(i)** A(2, 3), B(3, 2)

Gradient $= \dfrac{2-3}{3-2} = -1$

Line $y - 3 = -1(x - 2) = -x + 2$
$\Rightarrow x + y = 5$ [2 marks]

(ii) A(2, 4), B(6, 4)

Gradient $= \dfrac{4-4}{6-2} = 0$

Line $y = 4$ [2 marks]

(iii) A(3, 5), B(5, 1)

Gradient $= \dfrac{1-5}{5-3} = -2$

Line $y - 5 = -2(x - 3)$
$\Rightarrow y - 5 = -2x + 6$
$\Rightarrow 2x + y = 11$ [3 marks]

(iv) A(1, 6), B(4, 5)

Gradient $= \dfrac{5-6}{4-1} = -\dfrac{1}{3}$

Line $y - 6 = -\dfrac{1}{3}(x - 1)$

$\Rightarrow 3y - 18 = -x + 1$

$\Rightarrow x + 3y = 19$ [3 marks]
[Total: 10 marks]

7 **(i)** A(−1, 2), B(2, −1)

Gradient $= \dfrac{-1-2}{2-(-1)} = -1$

Line $y - 2 = -1(x + 1)$
$\Rightarrow y - 2 = -x - 1$
$\Rightarrow x + y = 1$ [2 marks]

(ii) A(−2, 4), B(2, −4)

Gradient $= \dfrac{-4-4}{2-(-2)} = -2$

Line $y - 4 = -2(x + 2)$

$\Rightarrow y - 4 = -2x - 4$

$\Rightarrow y = -2x$ [2 marks]

(iii) A(−1, −2), B(−2, −1)

Gradient $= \dfrac{-1-(-2)}{-2-(-1)} = -1$

Line $y + 2 = -1(x + 1)$

$\Rightarrow y + 2 = -x - 1$

$\Rightarrow x + y + 3 = 0$ [2 marks]

(iv) A(−2, −4), B(−4, −2)

Gradient $= \dfrac{-2-(-4)}{-4-(-2)} = -1$

Line $y + 4 = -1(x + 2)$

$\Rightarrow y + 4 = -x - 2$

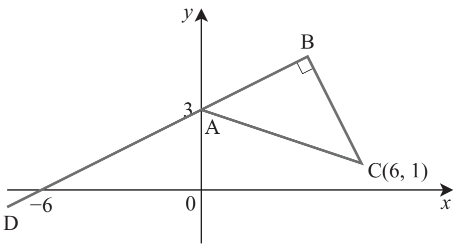
$\Rightarrow x + y + 6 = 0$ [2 marks]

[Total: 8 marks]

8 **(i)** Line $x - 2y + 6 = 0$

Meets x-axis when $y = 0 \Rightarrow x = -6$

Meets y-axis when $x = 0 \Rightarrow y = 3$



[2 marks]

(ii) AC is line through (0, 3) and (6, 1)

Gradient $= \dfrac{1-3}{6-0} = -\dfrac{1}{3}$

Equation $y - 3 = -\dfrac{1}{3}(x - 0) \Rightarrow y = -\dfrac{1}{3}x + 3$

BC is line through (6, 1) perpendicular to AB

Gradient AB $= \dfrac{3-0}{0-(-6)} = \dfrac{1}{2} \Rightarrow$ Gradient BC $= -2$

Equation $y - 1 = -2(x - 6)$

$\Rightarrow y - 1 = -2x + 12$

$\Rightarrow 2x + y - 13 = 0$ [4 marks]

(iii) At B, AB meets BC: AB $\equiv x - 2y + 6 = 0$

$\Rightarrow 2x - 4y + 12 = 0 \quad (1)$

$\text{BC} \Rightarrow 2x + y - 13 = 0 \quad (2)$

$(2) - (1) \Rightarrow 5y - 25 = 0$

$\Rightarrow y = 5$

Sub into eq^n AB: $x - 10 + 6 = 0 \Rightarrow x = 4$

B is point (4, 5)

A(0, 3), B(4, 5), C(6, 1)

$\text{AB} = \sqrt{(4-0)^2 + (5-3)^2} = 2\sqrt{5}$

$\text{BC} = \sqrt{(6-4)^2 + (1-5)^2} = 2\sqrt{5}$

Area $\Delta\text{ABC} = \frac{1}{2} \times 2\sqrt{5} \times 2\sqrt{5} = 10$ units^2 [4 marks]

[Total: 10 marks]

9 (i) Let £m be the cost of a meal and £w be the cost of a bottle of wine.

$m + w = 32.75 \quad (1)$

$2m + w = 53.75 \quad (2)$

$(2) - (1) \Rightarrow m = 21$ i.e. meal costs £21 [2 marks]

(ii) In (1) $21 + w = 32.75 \Rightarrow w = 11.75$

Bottle of wine costs £11.75 [1 mark]

[Total: 3 marks]

10 (i) $x + 3(0.9) = 5.7 \Rightarrow x = 3$

7 mile journey costs £$(3 + 7(0.9))$ = £9.30 [2 marks]

(ii) Let the journey be m miles.

$3 + m\,(0.9) = 12$

$\Rightarrow 0.9m = 9$

$\Rightarrow m = 10$ i.e. journey is 10 miles [2 marks]

[Total: 4 marks]

11 When $x = 2,\ y = 2^2 - \frac{1}{2} = \frac{7}{2}$ and when $x = 5,\ y = 5^2 - \frac{1}{5} = \frac{124}{5}$

So the gradient of the line is $\dfrac{\frac{124}{5} - \frac{7}{2}}{5-2} = \dfrac{71}{10}$ and the equation is $y - \frac{7}{2} = \frac{71}{10}(x-2)$

$\Rightarrow 10y - 35 = 71x - 142 \quad \Rightarrow \quad 71x - 10y - 107 = 0$ [4 marks]

Exercise 3.6 Graphs of quadratic functions

1 (a) $y = 4x - x^2$ (eq^n (ii))

(b) $y = x^2 - 2x + 3$ (eq^n (iv)) [2 marks]

2 (a) $y = x^2 - 3x + 6$ (eq^n (i))

(b) $y = x^2 - 6x + 6$ (eq^n (iv)) [2 marks]

3 **(a)** $y = 4 - x^2$ (eq^n (i))

(b) $y = -x^2 + 2x - 1$ (eq^n (iii)) [2 marks]

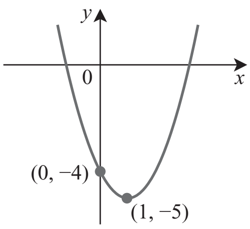
4 **(i)** $y = x^2 - 2x - 4 = (x - 1)^2 - 1 - 4$
$= (x - 1)^2 - 5$

(a) Vertex $(1, -5)$ [2 marks]

(b) Line of symmetry $x = 1$ [1 mark]

(c) Intersects y-axis at $(0, -4)$ [1 mark]

(ii) [1 mark]



[Total: 5 marks]

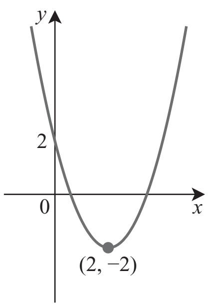
5 **(i)** $y = x^2 - 4x + 2 = (x - 2)^2 - 4 + 2$
$= (x - 2)^2 - 2$

(a) Vertex $(2, -2)$ [2 marks]

(b) Line of symmetry $x = 2$ [1 mark]

(c) Intersects y-axis at $(0, 2)$ [1 mark]

(ii) [1 mark]



[Total: 5 marks]

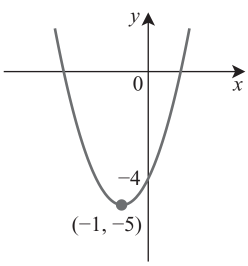
6 **(i)** $y = x^2 + 2x - 4 = (x + 1)^2 - 1 - 4$
$= (x + 1)^2 - 5$

(a) Vertex $(-1, -5)$ [2 marks]

(b) Line of symmetry $x = -1$ [1 mark]

(c) Intersects y-axis at $(0, -4)$ [1 mark]

(ii) [1 mark]



[Total: 5 marks]

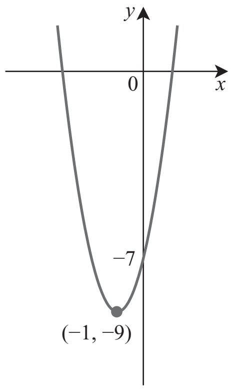
7 **(i)** $y = 2x^2 + 4x - 7 = 2(x^2 + 2x - 3.5)$
$= 2[(x + 1)^2 - 1 - 3.5]$
$= 2(x + 1)^2 - 9$

(a) Vertex $(-1, -9)$ [2 marks]

(b) Line of symmetry $x = -1$ [1 mark]

(c) Intersects y-axis at $(0, -7)$ [1 mark]

(ii) [1 mark]



[Total: 5 marks]

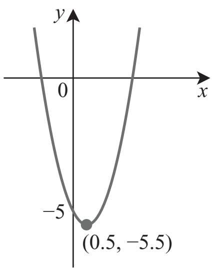
8 **(i)** $y = 2x^2 - 2x - 5 = 2(x^2 - x - 2.5)$
$= 2[(x - 0.5)^2 - 0.25 - 2.5]$
$= 2(x - 0.5)^2 - 5.5$ [2 marks]

(a) Vertex $(0.5, -5.5)$ [1 mark]

(b) Line of symmetry $x = 0.5$ [1 mark]

(c) Intersects y-axis at $(0, -5)$ [1 mark]

(ii)



[Total: 5 marks]

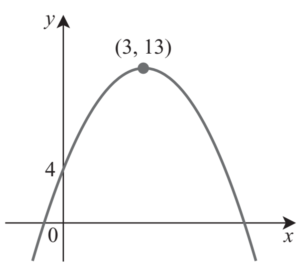
9 **(i)** $y = 4 + 6x - x^2$
$= 4 - (x^2 - 6x)$
$= 4 - [(x - 3)^2 - 9]$
$= 13 - (x - 3)^2$

(a) Vertex $(3, 13)$ [2 marks]

(b) Line of symmetry $x = 3$ [1 mark]

(c) Intersects y-axis at $(0, 4)$ [1 mark]

(ii) [1 mark]



[Total: 5 marks]

10 (i) ∪ shaped and intersects x-axis at (1, 0) and (3, 0)
Eq^n $y = c(x - 1)(x - 3)$
Through $(2, -1) \Rightarrow -1 = c(2 - 1)(2 - 3)$
(Or use through (0,3))
$\Rightarrow c = 1$
Eq^n $y = (x - 1)(x - 3)$ [3 marks]

(ii) ∪ shaped and intersects x-axis at (−1, 0) and (2, 0)
Eq^n $y = c(x + 1)(x - 2)$
Through $(0, -6) \Rightarrow -6 = c(1)(-2)$
$\Rightarrow c = 3$
Eq^n $y = 3(x + 1)(x - 2)$ [3 marks]

(iii) ∩ shaped and intersects x-axis at (0, 0) and (2, 0)
Eq^n $y = -cx(x - 2)$
Through $(1, 1) \Rightarrow 1 = -c(1)(-1) \Rightarrow c = 1$
Eq^n $y = -x(x - 2)$ [3 marks]

[Total: 9 marks]

11 $y = a(x-3)^2 + 7$ so $-11 = a(0-3)^2 + 7 \Rightarrow -18 = 9a \Rightarrow a = -2$

So the equation is $y = -2x^2 + 12x - 11$ [4 marks]

Exercise 3.7 Inverse functions

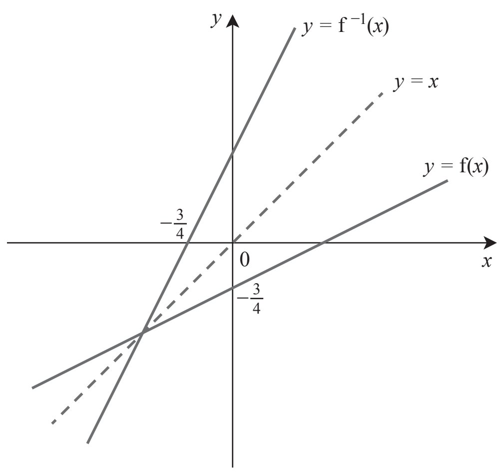
1 (i) $\mathrm{f}(x) = \dfrac{2x - 3}{4}$

$$x \xrightarrow{\times 2} 2x \xrightarrow{-3} 2x - 3 \xrightarrow{\div 4} \frac{2x - 3}{4} = \mathrm{f}(x)$$

$$\frac{4x + 3}{2} \xleftarrow{\div 2} 4x + 3 \xleftarrow{+3} 4x \xleftarrow{\times 4} x$$

$\mathrm{f}^{-1}(x) = \dfrac{4x + 3}{2}$ [2 marks]

(ii) [2 marks]



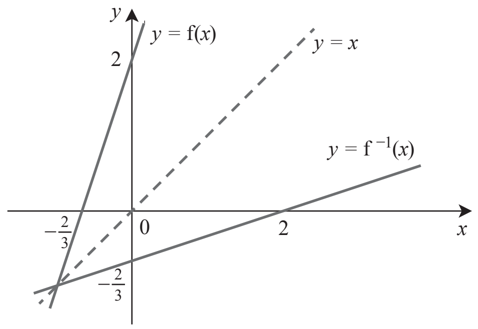
(iii) $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections in $y = x$ [1 mark]

[Total: 5 marks]

2 **(i)** $f(x) = 3x + 2$ i.e. $\times 3$ then $+ 2$
for inverse $- 2$ then $\div 3$

$f^{-1}(x) = \dfrac{x - 2}{3}$ [2 marks]

(ii) [2 marks]



(iii) $ff^{-1}(3) = f\left(\dfrac{1}{3}\right) = 3$

$f^{-1}f(3) = f^{-1}(11) = 3$ [2 marks]

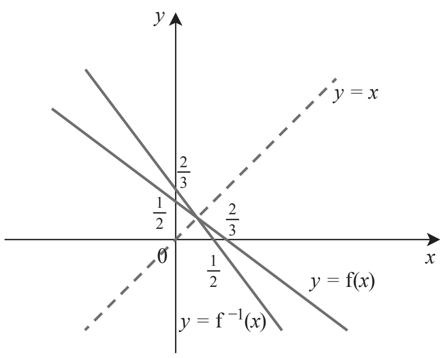
(iv) This suggests $ff^{-1}(x) = f^{-1}f(x)$ [1 mark]

[Total: 7 marks]

3 **(i)** $f(x) = \dfrac{2 - 3x}{4}$ i.e. $\times(-3)$ then $+2$ then $\div 4$
for inverse $\times 4$ then -2 then $\div -3$

$f^{-1}(x) = \dfrac{4x - 2}{-3} = \dfrac{2 - 4x}{3}$ [2 marks]

(ii) [2 marks]

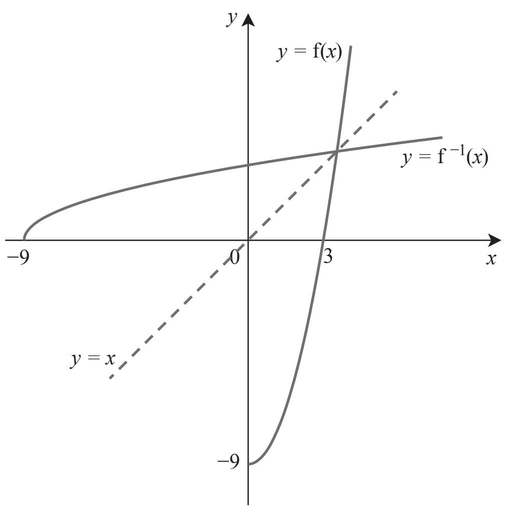


[Total: 4 marks]

4 **(i)** $f(x) = x^2 - 9$ $x \geqslant 0$ i.e. square then $- 9$
for inverse $+9$ then positive $\sqrt{\ }$

$f^{-1}(x) = \sqrt{x+9};\ x \geqslant -9$ [2 marks]

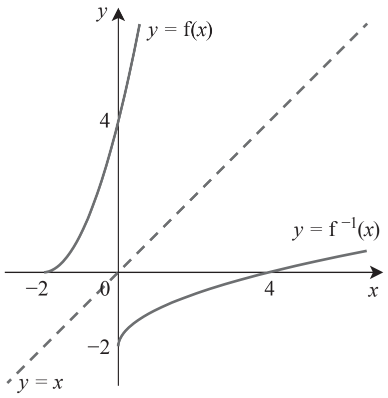
(ii) [2 marks]



[Total: 4 marks]

5 **(i)** $f(x) = (x + 2)^2 \quad x \geqslant -2$ i.e. $+ 2$ then square
for inverse square root then -2
$f^{-1}(x) = \sqrt{x} - 2; \ x \geqslant 0$ [2 marks]

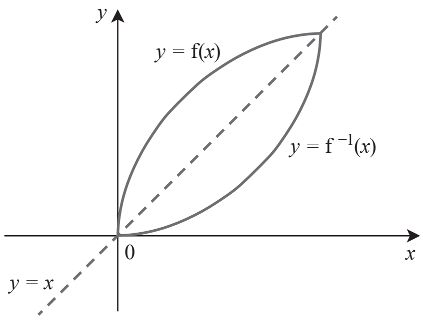
(ii) [2 marks]



[Total: 4 marks]

6 **(i)** $f(x) = 3\sqrt{x} \quad x \geqslant 0$ i.e. square root then $\times 3$
for inverse $\div 3$ then square
$f^{-1}(x) = \left(\frac{x}{3}\right)^2 = \frac{x^2}{9}; x \geqslant 0$ [2 marks]

(ii) [2 marks]

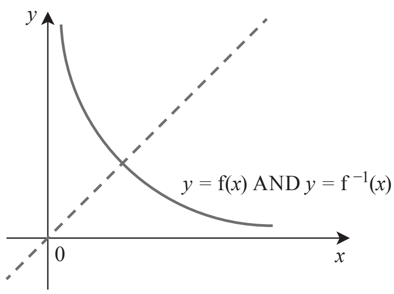


[Total: 4 marks]

7 **(i)** $f(x) = \frac{3}{x}$ for $x > 0$

Curves of the form $y = \frac{a}{x}$ are symmetrical about
the line $y = x \Rightarrow$ inverse function $f^{-1}(x) = \frac{3}{x}$ for $x > 0$ [2 marks]

(ii) [1 mark]



[Total: 3 marks]

8 **(i)** $f(x) = 3x - 2$ i.e. $\times 3$ then -2
For inverse $+2$ then $\div 3$
$f^{-1}(x) = \frac{x + 2}{3} \quad f^{-1}(-3) = -\frac{1}{3}$ [2 marks]

(ii) $f(x) = 2x - 3$ i.e. $\times 2$ then -3

For inverse $+3$ then $\div 2$

$f^{-1}(x) = \dfrac{x+3}{2}$ $f^{-1}(-3) = 0$ [2 marks]

(ii) $f(x) = \dfrac{3+4x}{2}$ i.e. $\times 4$ then $+3$ then $\div 2$

For inverse $\times 2$ then -3 then $\div 4$

$f^{-1}(x) = \dfrac{2x-3}{4}$ $f^{-1}(-3) = -2\dfrac{1}{4}$ [2 marks]

[Total: 6 marks]

9 **(i)** $f(x) = x^2 + 4$ for $x \geqslant 0$ i.e. square then $+4$

For inverse -4 then square root

$f^{-1}(x) = \sqrt{x-4}$ for $x \geqslant 0$ $f^{-1}(8) = \sqrt{8-4} = 2$ [3 marks]

(ii) $f^{-1}(x) = (x+4)^2$ for $x \geqslant -4$ i.e. $+4$ then square

For inverse square root then -4

$f^{-1}(x) = \sqrt{x} - 4$ for $x \geqslant 0$ $f^{-1}(8) = \sqrt{8} - 4 = 2\sqrt{2} - 4$ [3 marks]

(iii) $f(x) = 4x^2 - 3$ for $x \geqslant 0$ i.e. square then $\times 4$ then -3

For inverse $+3$ then $\div 4$ then square root

$f^{-1}(x) = \sqrt{\dfrac{x+3}{4}}$ for $x \geqslant -3$ $f^{-1}(8) = \sqrt{\dfrac{11}{4}} = \dfrac{\sqrt{11}}{2}$ [3 marks]

[Total: 9 marks]

10 **(i)** $y = 3\sqrt{x} \Rightarrow$ inverse $x = 3\sqrt{y}$ i.e. $x^2 = 9y$

$$y = \frac{x^2}{9}$$

$f^{-1}(3) = \dfrac{3^2}{9} = 1$ [2 marks]

(ii) $y = \dfrac{3}{\sqrt{x}} \Rightarrow$ inverse $x = \dfrac{3}{\sqrt{y}}$ i.e. $x^2 = \dfrac{9}{y} \Rightarrow y = \dfrac{9}{x^2}$

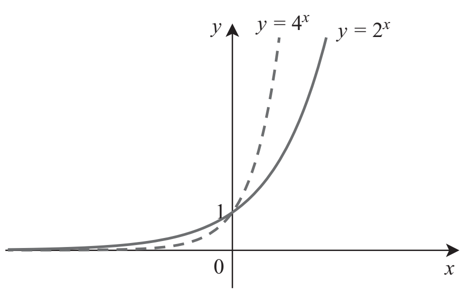
$f^{-1}(3) = \dfrac{9}{3^2} = 1$ [2 marks]

[Total: 4 marks]

11 $2x + 3 = \dfrac{x-3}{2} \Rightarrow 4x + 6 = x - 3 \Rightarrow 3x = -9 \Rightarrow x = -3$ [3 marks]

Exercise 3.8 Graphs of exponential functions

1 [2 marks]



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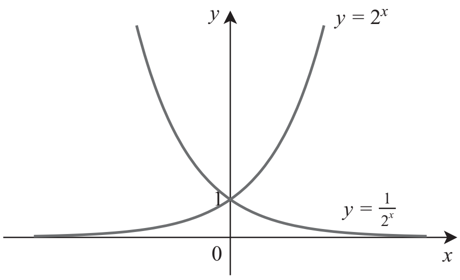
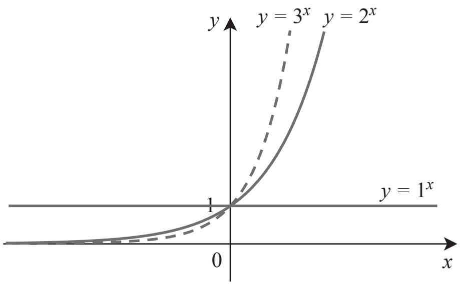


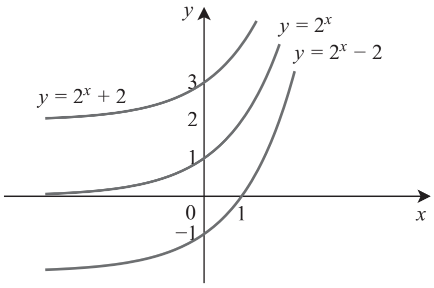
[2 marks]

3



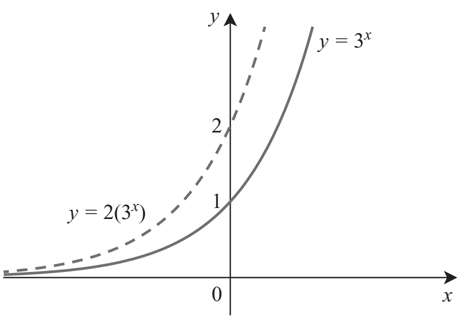
[3 marks]

4



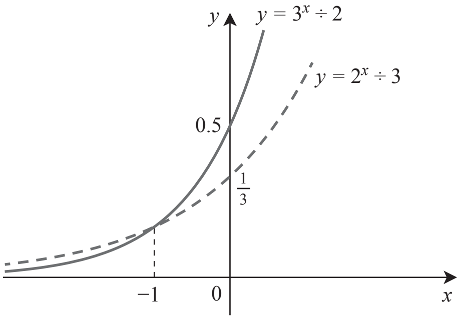
[3 marks]

5



[2 marks]

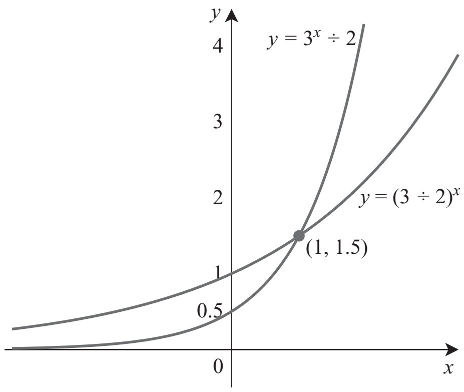
6 $y = 3^x \div 2$ and $y = 2^x \div 3$



[2 marks]

7 $y = 3^x \div 2$ and $y = (3 \div 2)^x$

[3 marks]



8 **(i)** $v = 5 + 20 \times 3^{-0.1t}$

Parachute opens when $t = 0 \Rightarrow v = 25\,\text{m}\,\text{s}^{-1}$ [1 mark]

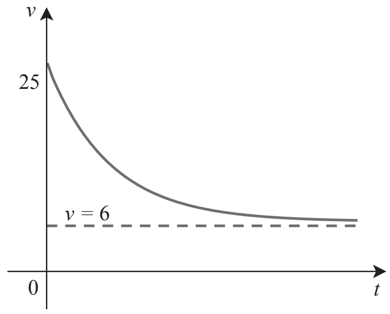
(ii) $t = 10 \Rightarrow v = 5 + 20 \times 3^{-1}$

$= 11\frac{2}{3}\,\text{m}\,\text{s}^{-1}$ [1 mark]

(iii) 1min = 60sec $t = 60 \Rightarrow v = 5 + 20 \times 3^{-6}$

$= 5.03\,\text{m}\,\text{s}^{-1}$ (2 d.p.) [1 mark]

(iv) [1 mark]



[Total: 4 marks]

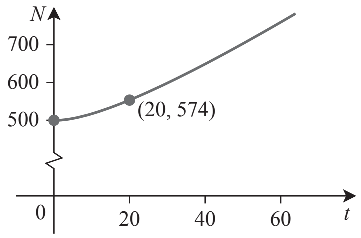
9 $N = 500 \times 2^{0.01t}$

(i) $t = 0 \Rightarrow N = 500$ [1 mark]

(ii) $t = 20 \Rightarrow N = 500 \times 2^{0.01 \times 20}$

$= 574$ [1 mark]

(iii) [2 marks]

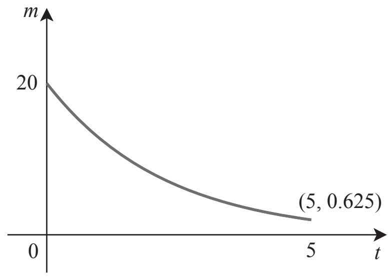


[Total: 4 marks]

10 $m = 20 \times (0.5)^t$

(i) $t = 0 \Rightarrow \text{mass} = 20\,\text{g}$ [1 mark]

(ii) $t = 5 \Rightarrow m = 20 \times (0.5)^5 = 0.625$ [2 marks]



(iii) $t = 7 \Rightarrow m = 0.15625$

$t = 8 \Rightarrow m = 0.0781 \Rightarrow$ Less than 0.1 g after 8 days [2 marks]

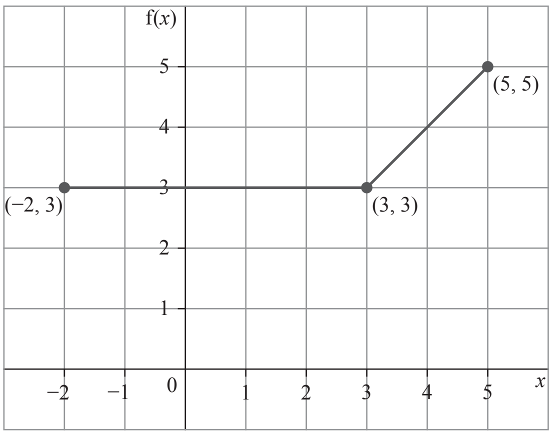
[Total: 5 marks]

11 $3 = ab^0 \quad \Rightarrow \quad a = 3$

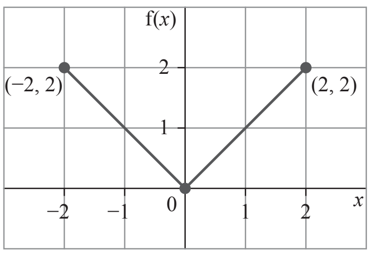
$48 = 3b^4 \quad \Rightarrow \quad b = \pm 2 \quad$ but $b > 0 \quad$ so $b = 2$ [4 marks]

Exercise 3.9 Graphs of functions with up to three parts to their domains

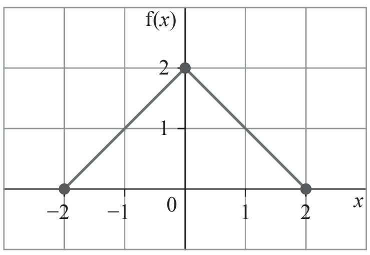
1 [2 marks]

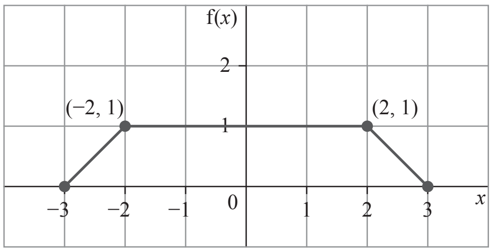


2 [2 marks]

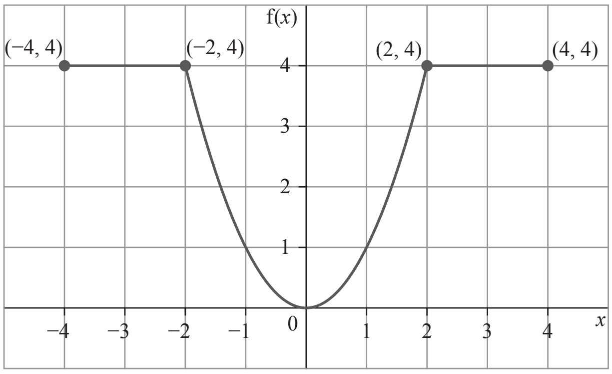


3 [2 marks]

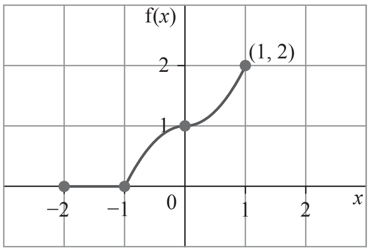


4 

[3 marks]

5 

[5 marks]

6 

[5 marks]

7 **(i)** $f(x) = x + 1 \qquad -1 \leqslant x \leqslant 0$

$= 1 - x \qquad 0 \leqslant x \leqslant 1$

$= 2x - 2 \qquad 1 \leqslant x \leqslant 3$ [3 marks]

(ii) Range of $f(x)$ is $0 \leqslant f(x) \leqslant 4$ [1 mark]

(iii) $f(x) = 2 \Rightarrow x = 2$ [1 mark]

[Total: 5 marks]

8 **(i)** $f(x) = x + 2 \qquad -2 \leqslant x \leqslant 0$

$= 2 \qquad 1 \leqslant x \leqslant 4$

$= 10 - 2x \qquad 4 \leqslant x \leqslant 5$ [3 marks]

(ii) Range $0 \leqslant f(x) \leqslant 2$ [2 marks]

(iii) Area $= \frac{1}{2}(4+7)\times 2$ (Area of trapezium)

$= 11$ units^2 [1 mark]

[Total: 6 marks]

9 **(i)** $f(x) = 0 \qquad -4 \leqslant x \leqslant -2$

$= 4 - x^2 \qquad -2 \leqslant x \leqslant 2$

$= x - 2 \qquad 2 \leqslant x \leqslant 4$ [3 marks]

(ii) Range $0 \leqslant f(x) \leqslant 4$ [1 mark]

[1 mark]

(iii) $f(1) = 4 - 1^2 = 3$ [1 mark]

(iv) $f(x) = 2 : 4 - x^2 = 2 \Rightarrow x^2 = 2$

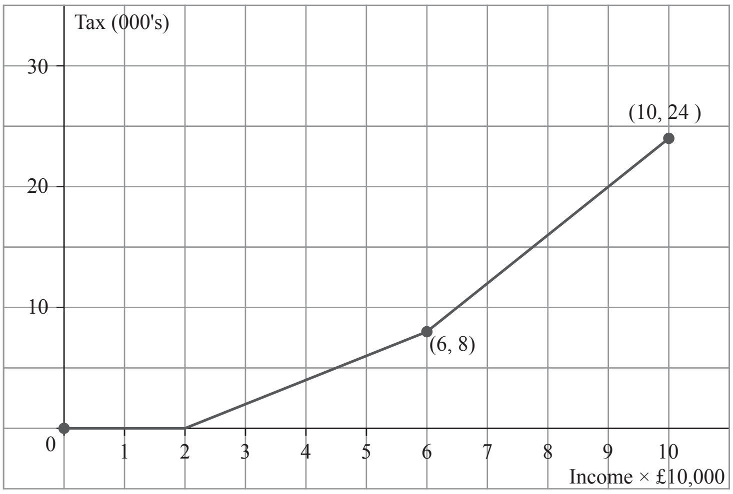
$$\Rightarrow x = \pm\sqrt{2}$$

$$x - 2 = 2 \Rightarrow x = 4$$

Solution $x = -\sqrt{2}, \sqrt{2}, 4$ [3 marks]

[Total: 9 marks]

10 (i) [3 marks]



(ii) £36 000 salary
£20 000 tax free
Tax = 20% × £16 000 = £3200 [1 mark]

(iii) £75 000 salary
£20 000 tax free
£40 000 × 20% = £8000
rem £15 000 × 40% = £6000
Total = £14 000 [2 marks]

(iv) £16 000 tax means £8000 paid at 40% level

$\Rightarrow$ Income in this band = £$\dfrac{8000}{0.4}$

= £20 000

Gross income = £20 000 + £40 000 + £20 000
tax free at 20% at 40%

$\Rightarrow$ Income = £80 000 pa [3 marks]

[Total: 9 marks]

11 When $x = 0$, $9 = a - 0^2 \Rightarrow a = 9$

$c = 9$ and $m = \dfrac{0-9}{3-0} = -3$

When $-3 \leqslant x \leqslant 0$ $\quad 9 - x^2 = 5 \Rightarrow x^2 = 4 \Rightarrow x = -2$

When $0 < x \leqslant 3$ $\quad -3x + 9 = 5 \Rightarrow 3x = 4 \Rightarrow x = \dfrac{4}{3}$ [5 marks]

3

Algebra III

4 Algebra IV

Exercise 4.1 Solving quadratic equations by factorising, completing the square or using the quadratic formula

1 (i) $(x-2)(x-4)=0 \Rightarrow x-2=0$ or $x-4=0 \Rightarrow x=2$ or $x=4$ [2 marks]

(ii) $(y-10)(y+1)=0 \Rightarrow y-10=0$ or $y+1=0 \Rightarrow y=10$ or $y=-1$ [2 marks]

(iii) $w^2-5w-6=0 \Rightarrow (w-6)(w+1)=0 \Rightarrow w=6$ or $w=-1$ [2 marks]

(iv) $(2p-3)(p+5)=0 \Rightarrow p=\frac{3}{2}$ or $p=-5$ [2 marks]

(v) $3n^2-20n-7=0 \Rightarrow (3n+1)(n-7)=0 \Rightarrow n=-\frac{1}{3}$ or $n=7$ [2 marks]

(vi) $2m^2-5m-3=0 \Rightarrow (2m+1)(m-3)=0 \Rightarrow m=-\frac{1}{2}$ or $m=3$ [2 marks]

[Total: 12 marks]

2 (i) $(p-4)^2-4^2+15=0 \Rightarrow (p-4)^2=1 \Rightarrow p-4=\pm 1$

$\Rightarrow p-4=1$ or $p-4=-1 \Rightarrow p=5$ or $p=3$ [3 marks]

(ii) $(x-3)^2-3^2-16=0 \Rightarrow (x-3)^2=25 \Rightarrow x-3=\pm 5$

$\Rightarrow x-3=5$ or $x-3=-5 \Rightarrow x=8$ or $x=-2$ [3 marks]

(iii) $(m+1)^2-1^2-3=0 \Rightarrow (m+1)^2=4 \Rightarrow m+1=\pm 2$

$\Rightarrow m+1=2$ or $m+1=-2 \Rightarrow m=1$ or $m=-3$ [3 marks]

(iv) $2\left[t^2-3t\right]-7=0 \Rightarrow 2\left[\left(t-\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]=7 \Rightarrow 2\left(t-\frac{3}{2}\right)^2-\frac{9}{2}=7$

$\Rightarrow 2\left(t-\frac{3}{2}\right)^2=\frac{23}{2} \Rightarrow \left(t-\frac{3}{2}\right)^2=\frac{23}{4} \Rightarrow t-\frac{3}{2}=\pm\sqrt{\frac{23}{4}}$

$\Rightarrow t=\frac{3}{2}\pm\frac{\sqrt{23}}{2} \Rightarrow t=\frac{3\pm\sqrt{23}}{2}$ [3 marks]

(v) $2\left[y^2-\frac{11}{2}y\right]+5=0 \Rightarrow 2\left[\left(y-\frac{11}{4}\right)^2-\left(\frac{11}{4}\right)^2\right]=-5$

$\Rightarrow 2\left(y-\frac{11}{4}\right)^2-\frac{121}{8}=-5 \Rightarrow 2\left(y-\frac{11}{4}\right)^2=\frac{81}{8} \Rightarrow \left(y-\frac{11}{4}\right)^2=\frac{81}{16}$

$\Rightarrow y-\frac{11}{4}=\pm\frac{9}{4} \Rightarrow y=5$ or $y=\frac{1}{2}$ [3 marks]

(vi) $3\left[n^2-\frac{4}{3}n\right]=2 \Rightarrow \left(n-\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2=\frac{2}{3} \Rightarrow \left(n-\frac{2}{3}\right)^2=\frac{10}{9}$

$\Rightarrow n=\frac{2}{3}\pm\sqrt{\frac{10}{9}} \Rightarrow n=\frac{2\pm\sqrt{10}}{3}$ [3 marks]

[Total: 18 marks]

3 (i) $y = \dfrac{-5 \pm \sqrt{5^2 - 4 \times 1 \times -9}}{2 \times 1} = \dfrac{-5 \pm \sqrt{61}}{2}$ [2 marks]

(ii) $p = \dfrac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times -5}}{2 \times 1} = \dfrac{3 \pm \sqrt{29}}{2}$ [2 marks]

(iii) $m = \dfrac{-8 \pm \sqrt{8^2 - 4 \times 1 \times 2}}{2 \times 1} = \dfrac{-8 \pm \sqrt{56}}{2} = -4 \pm \sqrt{14}$ [2 marks]

(iv) $r = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 2 \times -9}}{2 \times 2} = \dfrac{-6 \pm \sqrt{108}}{4} = \dfrac{-3 \pm 3\sqrt{3}}{2}$ [2 marks]

(v) $5n^2 - 2n - 7 = 0$

$\Rightarrow n = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \times 5 \times -7}}{2 \times 5} = \dfrac{2 \pm \sqrt{144}}{10} = \dfrac{2 \pm 12}{10}$

$\Rightarrow n = 1.4$ or -1 [2 marks]

(vi) $3w^2 + 8w - 2 = 0$

$\Rightarrow w = \dfrac{-8 \pm \sqrt{8^2 - 4 \times 3 \times -2}}{2 \times 3} = \dfrac{-8 \pm \sqrt{88}}{6} = \dfrac{-4 \pm \sqrt{22}}{3}$ [2 marks]

[Total: 12 marks]

4 (i) $\dfrac{12x}{x} + \dfrac{6x(x-2)}{3} = \dfrac{42x}{6}$

$\Rightarrow 12 + 2x(x-2) = 7x$

$\Rightarrow 2x^2 - 11x + 12 = 0$

$\Rightarrow (2x-3)(x-4) = 0$

$\Rightarrow x = \dfrac{3}{2}$ or $x = 4$ [4 marks]

(ii) $\dfrac{3m(m-2)}{3} + \dfrac{36(m-2)}{m-2} = 15(m-2)$

$\Rightarrow m^2 - 2m + 36 = 15m - 30$

$\Rightarrow m^2 - 17m + 66 = 0$

$\Rightarrow (m-11)(m-6) = 0$

$\Rightarrow m = 11$ or $m = 6$ [4 marks]

(iii) $\dfrac{4a(a+1)}{4} - \dfrac{24(a+1)}{a+1} = 12(a+1)$

$\Rightarrow a^2 + a - 24 = 12a + 12$

$\Rightarrow a^2 - 11a - 36 = 0$

$\Rightarrow a = \dfrac{11 \pm \sqrt{(-11)^2 - 4 \times 1 \times -36}}{2 \times 1} = \dfrac{11 \pm \sqrt{265}}{2}$ [4 marks]

(iv) $\dfrac{2(y+3)(2y-5)}{2} + \dfrac{8(2y-5)}{2y-5} = 14(2y-5)$

$\Rightarrow 2y^2 + y - 15 + 8 = 28y - 70$

$\Rightarrow 2y^2 - 27y + 63 = 0$

$\Rightarrow (2y-21)(y-3) = 0$

$\Rightarrow y = \dfrac{21}{2}$ or $y = 3$ [4 marks]

(v) $\dfrac{6n(n-1)}{n} + \dfrac{4n(n-1)}{n-1} = 7n(n-1)$

$\Rightarrow 6n - 6 + 4n = 7n^2 - 7n$

$\Rightarrow 0 = 7n^2 - 17n + 6$

$\Rightarrow (n-2)(7n-3) = 0$

$\Rightarrow n = 2 \quad \text{or} \quad n = \dfrac{3}{7}$ [4 marks]

(vi) $\dfrac{5(p+2)(2p+1)}{p+2} + \dfrac{1(p+2)(2p+1)}{2p+1} = 6(p+2)(2p+1)$

$\Rightarrow 10p + 5 + p + 2 = 12p^2 + 30p + 12$

$\Rightarrow 0 = 12p^2 + 19p + 5$

$\Rightarrow (3p+1)(4p+5) = 0$

$\Rightarrow p = -\dfrac{1}{3} \quad \text{or} \quad p = -\dfrac{5}{4}$ [4 marks]

[Total: 24 marks]

5 **(i)** $(x+3)(x+4) = (2x-1)(13-x)$

$\Rightarrow x^2 + 7x + 12 = -2x^2 + 27x - 13$

$\Rightarrow 3x^2 - 20x + 25 = 0$

$\Rightarrow (3x-5)(x-5) = 0$

$\Rightarrow x = \dfrac{5}{3} \quad \text{or} \quad x = 5$ [4 marks]

(ii) $(w+2)(2w+3) = (2w-1)(w-1)$

$\Rightarrow 2w^2 + 7w + 6 = 2w^2 - 3w + 1$

$\Rightarrow 10w + 5 = 0$

$\Rightarrow w = -0.5$ [4 marks]

(iii) $(p-3)(5-p) = (1-2p)(p+2)$

$\Rightarrow -p^2 + 8p - 15 = -2p^2 - 3p + 2$

$\Rightarrow p^2 + 11p - 17 = 0$

$\Rightarrow p = \dfrac{-11 \pm \sqrt{11^2 - 4 \times 1 \times -17}}{2 \times 1} = \dfrac{-11 \pm \sqrt{189}}{2} = \dfrac{-11 \pm 3\sqrt{21}}{2}$ [4 marks]

(iv) $(2y-5)(y-3) = (3y+7)(y+1)$

$\Rightarrow 2y^2 - 11y + 15 = 3y^2 + 10y + 7$

$\Rightarrow y^2 + 21y - 8 = 0$

$\Rightarrow y = \dfrac{-21 \pm \sqrt{21^2 - 4 \times 1 \times -8}}{2 \times 1} = \dfrac{-21 \pm \sqrt{473}}{2}$ [4 marks]

(v) $(5t-1)(t-2) = (2t+3)(t+1)$

$\Rightarrow 5t^2 - 11t + 2 = 2t^2 + 5t + 3$

$\Rightarrow 3t^2 - 16t - 1 = 0$

$\Rightarrow t = \dfrac{16 \pm \sqrt{16^2 - 4 \times 3 \times -1}}{2 \times 3} = \dfrac{16 \pm \sqrt{268}}{6} = \dfrac{8 \pm \sqrt{67}}{3}$ [4 marks]

(vi) $(3a+1)(4-2a)=(6-a)(a-3)$

$\Rightarrow -6a^2+10a+4=-a^2+9a-18$

$\Rightarrow 5a^2-a-22=0$

$\Rightarrow (5a-11)(a+2)=0$

$\Rightarrow a=\frac{11}{5}$ or $a=-2$ [4 marks]

[Total: 24 marks]

6 **(i)** $2x$ is a length $\therefore$ $2x>0 \Rightarrow x>0$ $\therefore$ $2x+1>x+1$

Also $2x+1>2x$ $\therefore$ $2x+1$ is the longest side $\therefore$ $2x+1$ is the hypotenuse [1 mark]

(ii) $(x+1)^2+(2x)^2=(2x+1)^2 \Rightarrow x^2+2x+1+4x^2=4x^2+4x+1$

$\Rightarrow x^2-2x=0 \Rightarrow x(x-2)=0 \Rightarrow x=0$ or $x=2$

but $x>0$ $\therefore$ $x=2$

$\therefore$ area $=\frac{1}{2}(x+1)\times 2x=\frac{1}{2}\times 3\times 4=6\,\text{cm}^2$ [4 marks]

[Total: 5 marks]

7 $4k^2+2vk-v^2=0 \Rightarrow k=\frac{-2v\pm\sqrt{4v^2+16v^2}}{8}=\frac{-2v\pm\sqrt{20v^2}}{8}=\frac{-1\pm\sqrt{5}}{4}v$ [2 marks]

8 $x=\frac{-5\pm\sqrt{25-12p}}{6}$ $\therefore$ $\frac{-5+\sqrt{25-12p}}{6}=\frac{-5-\sqrt{25-12p}}{6}$

$\Rightarrow \sqrt{25-12p}=-\sqrt{25-12p}$

$\Rightarrow 2\sqrt{25-12p}=0 \Rightarrow 25-12p=0 \Rightarrow p=\frac{25}{12}$ [3 marks]

9 Sum $=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$ [3 marks]

Exercise 4.2 Solving simultaneous equations

1 **(i)** adding $\Rightarrow 7x=14 \Rightarrow x=2 \therefore 6+y=8 \Rightarrow y=2$ [3 marks]

(ii) subtracting $\Rightarrow 3q=-3 \Rightarrow q=-1 \therefore 3p-8=7 \Rightarrow p=5$ [3 marks]

(iii) $20m+4n=56$ and $3m-4n=-10$ adding $\Rightarrow 23m=46 \Rightarrow m=2$
$\therefore 10+n=14 \Rightarrow n=4$ [4 marks]

(iv) $2a-12b=22$ and $2a+3b=7$ subtracting $\Rightarrow 15b=-15 \Rightarrow b=-1$
$\therefore 2a-3=7 \Rightarrow a=5$ [4 marks]

(v) $6r+9s=-3$ and $6r-10s=54$ subtracting $\Rightarrow 19s=-57 \Rightarrow s=-3$
$2r-9=-1 \Rightarrow r=4$ [4 marks]

(vi) $4c-3d=11$ and $5c-7d=13$ $\therefore 20c-15d=55$ and $20c-28d=52$

subtracting $\Rightarrow 13d=3 \Rightarrow d=\frac{3}{13} \therefore 4c-\frac{9}{13}=11 \Rightarrow c=\frac{38}{13}$ [4 marks]

[Total: 22 marks]

2 **(i)** $2x-3=x+1 \Rightarrow x=4 \therefore y=4+1=5$ [3 marks]

(ii) $7x-4=3x+4 \Rightarrow 4x=8 \Rightarrow x=2 \therefore y=3\times 2+4=10$ [3 marks]

(iii) $2x+3(5x+1)=20 \Rightarrow 17x=17 \Rightarrow x=1 \therefore y=5\times 1+1=6$ [3 marks]

(iv) $4x+3(2x-13)=11 \Rightarrow 10x=50 \Rightarrow x=5 \therefore y=2\times 5-13=-3$ [3 marks]

(v) $y=7-2x \therefore 3x+2(7-2x)=15 \Rightarrow x=-1 \therefore y=7-2\times -1=9$ [4 marks]

(vi) $x=-9-7y \therefore 3(-9-7y)-4y=23 \Rightarrow -27-25y=23 \Rightarrow y=-2$

$x=-9-7\times -2=5$ [4 marks]

[Total: 20 marks]

3 **(i)** $x^2-x-5=x+3 \Rightarrow x^2-2x-8=0 \Rightarrow (x-4)(x+2)=0$

$\Rightarrow x=4$ or $x=-2$

when $x=4, y=4+3=7$ and when $x=-2,\ y=-2+3=1$ [4 marks]

(ii) $x^2-3x-7=x-2 \Rightarrow x^2-4x-5=0 \Rightarrow (x-5)(x+1)=0$

$\Rightarrow x=5$ or $x=-1$

when $x=5,\ y=5-2=3$ and when $x=-1,\ y=-1-2=-3$ [4 marks]

(iii) $x^2+3x-3=2x-1 \Rightarrow x^2+x-2=0 \Rightarrow (x-1)(x+2)=0$

$\Rightarrow x=1$ or $x=-2$

when $x=1,\ y=2\times 1-1=1$ and when $x=-2, y=2\times -2-1=-5$ [4 marks]

(iv) $x^2-(x+1)^2=7 \Rightarrow -2x-1=7 \Rightarrow x=-4 \therefore y=-4+1=-3$ [4 marks]

(v) $y=3-2x \therefore x^2+(3-2x)^2=5 \Rightarrow x^2+9-12x+4x^2=5$

$\Rightarrow 5x^2-12x+4=0 \Rightarrow (5x-2)(x-2)=0 \Rightarrow x=\frac{2}{5}$ or $x=2$

when $x=\frac{2}{5},\ y=3-2\times\frac{2}{5}=\frac{11}{5}$ and when $x=2,\ y=3-2\times 2=-1$ [5 marks]

(vi) $x=-4-3y \therefore 2(-4-3y)^2-y^2=41 \Rightarrow 17y^2+48y-9=0$

$\Rightarrow (17y-3)(y+3)=0 \Rightarrow y=\frac{3}{17}$ or $y=-3$

when $y=\frac{3}{17}$, $x=-4-3\times\frac{3}{17}=-\frac{77}{17}$ and when $y=-3,\ x=-4-3\times -3=5$ [5 marks]

[Total: 26 marks]

4 $9x^2+4(x+4)^2=180 \Rightarrow 9x^2+4\left(x^2+8x+16\right)=180$

$\Rightarrow 13x^2+32x-116=0 \Rightarrow (13x+58)(x-2)=0 \Rightarrow x=-\frac{58}{13}$ or $x=2$

when $x=-\frac{58}{13},\ y=-\frac{58}{13}+4=-\frac{6}{13}$ and when $x=2,\ y=2+4=6$

Intersection points are $\left(-\frac{58}{13}\ ,\ -\frac{6}{13}\right)$ and $(2,\ 6)$ [5 marks]

5 Let $l=$ length and $w=$ width $\therefore lw=9$ and $l^2+w^2=30$

$\therefore \left(\frac{9}{w}\right)^2+w^2=30 \Rightarrow 81+w^4=30w^2 \Rightarrow w^4-30w^2+81=0$

$\Rightarrow \left(w^2-3\right)\left(w^2-27\right)=0 \Rightarrow w=\sqrt{3}$ or $w=\sqrt{27}=3\sqrt{3}$ as $w>0$

when $w=\sqrt{3}, l=\frac{9}{\sqrt{3}}=3\sqrt{3}$ and when $l=3\sqrt{3},\ w=\frac{9}{3\sqrt{3}}=\sqrt{3}$

$\therefore$ perimeter $=2\left(\sqrt{3}+3\sqrt{3}\right)=8\sqrt{3}=\sqrt{8^2\times 3}=\sqrt{192}$ cm [5 marks]

6 $(x-y)^2 = x^2 - 2xy + y^2 = x^2 + 2xy + y^2 - 4xy = (x+y)^2 - 4xy = 6^2 - 4\times 7 = 8$

$\therefore\ x - y = \sqrt{8} = 2\sqrt{2}$ [4 marks]

7 $4+\sqrt{34}$ and $4-\sqrt{34}$ are the roots of $\left(x-\left(4+\sqrt{34}\right)\right)\left(x-\left(4-\sqrt{34}\right)\right)=0$

$\Rightarrow\ x^2 - x\left(4+\sqrt{34}+4-\sqrt{34}\right)+\left(16+4\sqrt{34}-4\sqrt{34}-34\right)=0$

$\Rightarrow\ x^2 - 8x - 18 = 0 \therefore a = -8$ and $b = -18$ [5 marks]

8 $x = \dfrac{8\ \pm\ \sqrt{(-8)^2 - 4\times 1\times a}}{2\times 1} = \dfrac{8\ \pm\ \sqrt{64-4a}}{2} = 4\pm\sqrt{16-a}$

$\therefore p - q = 4+\sqrt{16-a}-\left(4-\sqrt{16-a}\right) = 2\sqrt{16-a} = 14 \Rightarrow \sqrt{16-a} = 7$

$\Rightarrow 16 - a = 49\ \Rightarrow\ a = -33$ [4 marks]

9 $2 = ab^{-1}$ and $16 = ab^2$ so $\dfrac{ab^2}{ab^{-1}} = \dfrac{16}{2}\quad\Rightarrow\quad b^3 = 8\quad\Rightarrow\quad b = 2$

So $2 = a\times 2^{-1}\quad\Rightarrow\quad a = 4$ [4 marks]

Exercise 4.3 Factor theorem

1 **(i)** $1^3 - 4\times 1 + 3 = 1 - 4 + 3 = 0\ \therefore\ (x-1)$ is a factor [2 marks]

(ii) $(-2)^4 - 16 = 16 - 16 = 0 \therefore\ (x+2)$ is a factor [2 marks]

(iii) $3^3 - 3 - 24 = 27 - 3 - 24 = 0\ \therefore\ (x-3)$ is a factor [2 marks]

(iv) $(-4)^3 + 2\times(-4)^2 + 32 = -64 + 32 + 32 = 0\ \therefore\ (x+4)$ is a factor [2 marks]

[Total: 8 marks]

2 **(i)** $2\times\left(\dfrac{1}{2}\right)^3 + 3\times\left(\dfrac{1}{2}\right)^2 - 1 = \dfrac{1}{4} + \dfrac{3}{4} - 1 = 0\ \therefore\ (2x-1)$ is a factor [2 marks]

(ii) $2\times\left(-\dfrac{3}{2}\right)^3 - 5\times\left(-\dfrac{3}{2}\right)^2 - 4\times -\dfrac{3}{2} + 12 = -\dfrac{27}{4} - \dfrac{45}{4} + 6 + 12 = 0$

$\therefore (2x+3)$ is a factor [2 marks]

(iii) $9\times\left(-\dfrac{1}{3}\right)^4 - \left(-\dfrac{1}{3}\right)^2 + 6\times -\dfrac{1}{3} + 2 = \dfrac{1}{9} - \dfrac{1}{9} - 2 + 2 = 0 \therefore \left(3x+1\right)$ is a factor [2 marks]

(iv) $8\times\left(-\dfrac{1}{2}\right)^4 + 2\times\left(-\dfrac{1}{2}\right)^2 - 1 = \dfrac{1}{2} + \dfrac{1}{2} - 1 = 0\ \therefore\ \left(2x+1\right)$ is a factor [2 marks]

[Total: 8 marks]

3 **(i)** $1^3 - 3\times 1 + 2 = 1 - 3 + 2 = 0 \therefore\ (x-1)$ is a factor [2 marks]

(ii) $x^3 - 3x + 2 = (x-1)\left(x^2 + x - 2\right) = (x-1)(x+2)(x-1)$ [3 marks]

[Total: 5 marks]

4 **(i)** $2^3 - 4\times 2^2 + 2 + 6 = 8 - 16 + 2 + 6 = 0$ [1 mark]

(ii) $x^3 - 4x^2 + x + 6 = (x-2)\left(x^2 - 2x - 3\right) = (x-2)(x-3)(x+1)$ [3 marks]

[Total: 4 marks]

5 **(i)** $3^3 - 5 \times 3^2 + 5 \times 3 + 3 = 27 - 45 + 15 + 3 = 0 \therefore (x-3)$ is a factor [2 marks]

(ii) $g(x) = (x-3)(x^2 - 2x - 1) = 0 \Rightarrow x = 3$ or $x = \dfrac{2 \pm \sqrt{2^2 - 4 \times 1 \times -1}}{2 \times 1} = 1 \pm \sqrt{2}$ [3 marks]

[Total: 5 marks]

6 $(-2)^5 - a \times (-2)^2 + 8 = 0 \Rightarrow -32 - 4a + 8 = 0 \Rightarrow -24 = 4a \Rightarrow a = -6$ [2 marks]

7 $(-2)^3 + 3 \times (-2)^2 + p \times -2 + q = 0 \Rightarrow -8 + 12 - 2p + q = 0 \Rightarrow 2p - q = 4$

$4^3 + 3 \times 4^2 + p \times 4 + q = 0 \Rightarrow 64 + 48 + 4p + q = 0 \Rightarrow 4p + q = -112$

adding: $2p + 4p = 4 + -112 \Rightarrow 6p = -108 \Rightarrow p = -18 \therefore q = -40$

If $(x+m)$ is the third factor then $2 \times -4 \times m = -40 \Rightarrow m = 5$

$\therefore$ the third factor is $(x+5)$ [5 marks]

8 $(-1)^4 + 3 \times (-1)^3 + p \times -1 + q = 0 \Rightarrow q = p + 2$

$2^3 - p \times 2^2 + q \times 2 + 9 = 0 \Rightarrow 2q - 4p = -17 \therefore 2(p+2) - 4p = -17$

$\Rightarrow -2p = -21 \Rightarrow p = 10.5 \therefore q = 10.5 + 2 = 12.5$ [4 marks]

9 **(i)** $3\left(\frac{1}{3}\right)^3 - 13\left(\frac{1}{3}\right)^2 + 10\left(\frac{1}{3}\right) + a = 0 \Rightarrow \frac{1}{9} - \frac{13}{9} + \frac{10}{3} + a = 0 \Rightarrow a = -2$ [2 marks]

(ii) $3x^3 - 13x^2 + 10x - 2 = 0 \Rightarrow (3x-1)(x^2 - 4x + 2) = 0$

$\Rightarrow x = \frac{1}{3}, \quad x = \dfrac{4 \pm \sqrt{16 - 4 \times 1 \times 2}}{2} = 2 \pm \sqrt{2}$ [4 marks]

[Total: 6 marks]

Exercise 4.4 Linear inequalities

1 **(i)** $3x > 5 - 7 \Rightarrow 3x > -2 \Rightarrow x > -\frac{2}{3}$ [1 mark]

(ii) $-2x < 9 - 5 \Rightarrow -2x < 4 \Rightarrow x > -2$ [1 mark]

(iii) $4x - 7x \leqslant -2 - 3 \Rightarrow -3x \leqslant -5 \Rightarrow x \geqslant \frac{5}{3}$ [1 mark]

(iv) $6x - 8 \geqslant x + 4 \Rightarrow 5x \geqslant 12 \Rightarrow x \geqslant 2.4$ [1 mark]

(v) $7x - 14 > 3x + 15 \Rightarrow 4x > 29 \Rightarrow x > 7\frac{1}{4}$ [2 marks]

(vi) $4x + 12 \leqslant 6x - 14 + 6 \Rightarrow 12 + 14 - 6 \leqslant 6x - 4x \Rightarrow 20 \leqslant 2x$

$\Rightarrow x \geqslant 10$ [2 marks]

[Total: 8 marks]

2 $x > -\frac{1}{2}$ and $x \leqslant \frac{5}{3} \therefore -\frac{1}{2} < x \leqslant \frac{5}{3}$ [2 marks]

3 **(i)** $2x + 3 < 35 \Rightarrow 2x < 32 \Rightarrow x < 16$ [2 marks]

(ii) $24 \leqslant 3 - x \Rightarrow x \leqslant -21$ [2 marks]

(iii) $x + 1 \geqslant 3x - 6 \Rightarrow -2x \geqslant -7 \Rightarrow x \leqslant 3.5$ [2 marks]

(iv) $4x + 2 > x - 4 \Rightarrow 3x > -6 \Rightarrow x > -2$ [2 marks]

(v) $9 - 3x < 10 - 4x \Rightarrow x < 1$ [2 marks]

(vi) $10 - 2x \geqslant 6x + 6 \Rightarrow -8x \geqslant -4 \Rightarrow x \leqslant 0.5$ [2 marks]

[Total: 12 marks]

4 **(i)** $2-3 \leqslant x \leqslant 5-3 \;\Rightarrow\; -1 \leqslant x \leqslant 2$ [1 mark]

(ii) $3 < 3x \leqslant 9 \;\Rightarrow\; 1 < x \leqslant 3$ [1 mark]

(iii) $-3 < -x < 6 \;\Rightarrow\; 3 > x > -6 \;\Rightarrow\; -6 < x < 3$ [2 marks]

(iv) $-4 \leqslant -2x < 6 \;\Rightarrow\; 2 \geqslant x > -3 \;\Rightarrow\; -3 < x \leqslant 2$ [2 marks]

(v) $15 < x-2 \leqslant 20 \;\Rightarrow\; 17 < x \leqslant 22$ [2 marks]

(vi) $-4 \leqslant x-4 < 36 \;\Rightarrow\; 0 \leqslant x < 40$ [2 marks]

[Total: 10 marks]

5 **(i)** $1+2 \leqslant p+q \leqslant 3+7 \;\Rightarrow\; 3 \leqslant p+q \leqslant 10$ [2 marks]

(ii) $1-7 \leqslant p-q \leqslant 3-2 \;\Rightarrow\; -6 \leqslant p-q \leqslant 1$ [2 marks]

(iii) $2\times1+2 \leqslant 2p+q \leqslant 2\times3+7 \;\Rightarrow\; 4 \leqslant 2p+q \leqslant 13$ [2 marks]

(iv) $1-3\times7 \leqslant p-3q \leqslant 3-3\times2 \;\Rightarrow\; -20 \leqslant p-3q \leqslant -3$ [2 marks]

[Total: 8 marks]

6 **(i)** $0^2 \leqslant m^2 \leqslant (-4)^2 \;\Rightarrow\; 0 \leqslant m^2 \leqslant 16$ [2 marks]

(ii) $-4+(-2) \leqslant m+n < 3+5 \;\Rightarrow\; -6 \leqslant m+n < 8$ [2 marks]

(iii) $0^2-5^2 < m^2-n^2 \leqslant (-4)^2-0^2 \;\Rightarrow\; -25 < m^2-n^2 \leqslant 16$ [2 marks]

(iv) $(-2)^3 \leqslant n^3 < 5^3 \;\Rightarrow\; -8 \leqslant n^3 < 125$ [2 marks]

[Total: 8 marks]

7 **(i)** $x < 0 \;\Rightarrow\; 2x < 0 \;\therefore\; 2x > 1$ is NEVER TRUE [1 mark]

(ii) $-3+2 < 0$ but $-3+4 > 0 \;\therefore\; x+y < 0$ is SOMETIMES TRUE [1 mark]

(iii) negative × positive = negative $\therefore xy < 0$ is ALWAYS TRUE [1 mark]

(iv) positive − negative = positive + positive = positive $\therefore y-x > 0$ is ALWAYS TRUE [1 mark]

(v) $x \neq 0 \;\therefore\; x^2 > 0$ is ALWAYS TRUE [1 mark]

(vi) $3 < 5$ but $7 > 5 \;\therefore\; y < 5$ is SOMETIMES TRUE [1 mark]

[Total: 6 marks]

8 **(i)** $0 < w < 1 \;\Rightarrow\; 0 < 3w < 3 \;\therefore\; 3w < 1$ is SOMETIMES TRUE [1 mark]

(ii) wx = positive × negative = negative, but y could be positive or negative
$\therefore wxy < 0$ is SOMETIMES TRUE [1 mark]

(iii) negative ÷ positive = negative $\therefore\; \dfrac{wx}{y} < 0 \;\therefore\; \dfrac{wx}{y} \leqslant -6$ is SOMETIMES TRUE [1 mark]

(iv) $wx < 0 \;\therefore\; 0 < wx < 1$ is NEVER TRUE [1 mark]

(v) positive ÷ negative = negative $\therefore\; \dfrac{w}{x} < 0$ is ALWAYS TRUE [1 mark]

(vi) $0.5+(-1)+0.5=0$ but $0.5+(-1)+(-3)=-3.5 \;\therefore\; -3 < w+x+y < 3$ is SOMETIMES TRUE [1 mark]

[Total: 6 marks]

9 $$\left.\begin{array}{l} -1 < 2x-1 < 5 \;\Rightarrow\; 0 < 2x < 6 \;\Rightarrow\; 0 < x < 3 \\ 4 \leqslant 3x+1 \leqslant 16 \;\Rightarrow\; 3 \leqslant 3x \leqslant 15 \;\Rightarrow\; 1 \leqslant x \leqslant 5 \end{array}\right\} \Rightarrow\; 1 \leqslant x < 3$$ [4 marks]

Exercise 4.5 Quadratic inequalities

1 **(i)** $(x-1)(x-3)>0 \Rightarrow x<1$ or $x>3$ [4 marks]

(ii) $(x+6)(x+1)\leqslant 0 \Rightarrow -6\leqslant x\leqslant -1$ [4 marks]

(iii) $(x-4)(x+1)\geqslant 0 \Rightarrow x\leqslant -1$ or $x\geqslant 4$ [4 marks]

(iv) $x^2-5x-14\leqslant 0 \Rightarrow (x-7)(x+2)\leqslant 0 \Rightarrow -2\leqslant x\leqslant 7$ [4 marks]

(v) $x^2-8x+7<0 \Rightarrow (x-1)(x-7)<0 \Rightarrow 1<x<7$ [4 marks]

(vi) $x^2+2x-80<0 \Rightarrow (x+10)(x-8)<0 \Rightarrow -10<x<8$ [4 marks]

[Total: 24 marks]

2 **(i)** $(4x+1)(x-1)\geqslant 0 \Rightarrow x\leqslant -\frac{1}{4}$ or $x\geqslant 1$ [4 marks]

(ii) $(2x+1)(x-1)<0 \Rightarrow -\frac{1}{2}<x<1$ [4 marks]

(iii) $2x^2-x-3<0 \Rightarrow (2x-3)(x+1)<0 \Rightarrow -1<x<\frac{3}{2}$ [4 marks]

(iv) $3x^2-7x+2\geqslant 0 \Rightarrow (3x-1)(x-2)\geqslant 0 \Rightarrow x\leqslant \frac{1}{3}$ or $x\geqslant 2$ [4 marks]

(v) $x^2+3x-7>0 \Rightarrow$ critical values are $\frac{-3\pm\sqrt{37}}{2} \Rightarrow x<\frac{-3-\sqrt{37}}{2}$ or $x>\frac{-3+\sqrt{37}}{2}$ [4 marks]

(vi) $2x^2-3x-9\leqslant 0 \Rightarrow (2x+3)(x-3)\leqslant 0 \Rightarrow -\frac{3}{2}\leqslant x\leqslant 3$ [4 marks]

[Total: 24 marks]

3 **(i)** $x^2-3x+2<0 \Rightarrow (x-1)(x-2)<0 \Rightarrow 1<x<2$ [4 marks]

(ii) $x^2-1<0 \Rightarrow (x-1)(x+1)<0 \Rightarrow -1<x<1$ [4 marks]

(iii) $2x^2-3x+1<0 \Rightarrow (2x-1)(x-1)<0 \Rightarrow \frac{1}{2}<x<1$ [4 marks]

(iv) $9-x^2<0 \Rightarrow x^2-9>0 \Rightarrow (x-3)(x+3)>0 \Rightarrow x<-3$ or $x>3$ [4 marks]

(v) $3+2x-5x^2<0 \Rightarrow 5x^2-2x-3>0 \Rightarrow (5x+3)(x-1)>0$

$\Rightarrow x<-\frac{3}{5}$ or $x>1$ [4 marks]

(vi) $2+x-4x^2<0 \Rightarrow 4x^2-x-2>0 \Rightarrow$ critical values are $\frac{1\pm\sqrt{33}}{8}$

$\Rightarrow x<\frac{1-\sqrt{33}}{8}$ or $x>\frac{1+\sqrt{33}}{8}$ [4 marks]

[Total: 24 marks]

4 $\frac{1}{2}(2x-1)(x+3)>(2x+1)(x-2) \Rightarrow 2x^2+5x-3>2\left(2x^2-3x-2\right)$

$\Rightarrow 0>2x^2-11x-1 \Rightarrow$ critical values are $\frac{11\pm\sqrt{129}}{4}$ $\therefore$ $\frac{11-\sqrt{129}}{4}<x<\frac{11+\sqrt{129}}{4}$

but $x-2>0 \Rightarrow x>2 \therefore 2<x<\frac{11+\sqrt{129}}{4}$ [5 marks]

5 **(i)** $3y+6>y-12 \Rightarrow 3y-y>-12-6 \Rightarrow 2y>-18 \Rightarrow y>-9$ [1 mark]

(ii) $y\leqslant -2$ or $y\geqslant 2$ [1 mark]

(iii) $-9<y\leqslant -2$ or $y\geqslant 2$ [2 marks]

[Total: 4 marks]

6 **(i)** $-4 < x < 2$ [3 marks]

(ii) $-1 \leqslant x \leqslant 5$ [3 marks]

(iii) $-1 \leqslant x < 2$ [2 marks]

[Total: 8 marks]

7 **(i)** $(x-1)(x-3) \leqslant 0 \quad \Rightarrow \quad 1 \leqslant x \leqslant 3$ [4 marks]

(ii) $(y-2)(y-4) < 0 \quad \Rightarrow \quad 2 < y < 4$ [4 marks]

(iii) $1+2 < x+y < 3+4 \quad \Rightarrow \quad 3 < w < 7$ [2 marks]

[Total: 10 marks]

8 $p^2 < 9 \quad \Rightarrow \quad -3 < p < 3$

$q^2 - 3q - 10 \leqslant 0 \quad \Rightarrow \quad (q+2)(q-5) \leqslant 0 \quad \Rightarrow \quad -2 \leqslant q \leqslant 5$

$2 \times -3 - 5 < 2p - q < 2 \times 3 - (-2) \quad \Rightarrow \quad -11 < m < 8$ [3 marks]

9 $2x^2 - 5 < x^2 - x - 3 \quad \Rightarrow \quad x^2 + x - 2 < 0 \quad \Rightarrow \quad (x-1)(x+2) < 0 \quad \Rightarrow \quad -2 < x < 1$ [5 marks]

10 $2x^2 + 2 < 5x \quad \Rightarrow \quad 2x^2 - 5x + 2 < 0 \quad \Rightarrow \quad (2x-1)(x-2) < 0 \quad \Rightarrow \quad \frac{1}{2} < x < 2$ [4 marks]

Exercise 4.6 Indices

1 **(i)** $x^{2+7-5} = x^4$ [1 mark]

(ii) $x^{3-7-4} = x^{-8}$ [1 mark]

(iii) $x^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = x^1$ [1 mark]

(iv) $\sqrt{x^7} = x^{\frac{7}{2}}$ [1 mark]

(v) $\sqrt[3]{x^{15}} = x^5$ [1 mark]

(vi) $\sqrt{x^{18}} = x^9$ [1 mark]

(vii) $\left(x^{\frac{5}{2}}\right)^4 = x^{10}$ [1 mark]

(viii) $\sqrt[4]{x^{-3}} = x^{-\frac{3}{4}}$ [1 mark]

[Total: 8 marks]

2 **(i)** $x^2 \times x^3 + x^2 \times x^{-4} = x^5 + x^{-2}$ [2 marks]

(ii) $2x^{\frac{1}{2}+\frac{1}{2}} + x^{\frac{1}{2}-\frac{1}{2}} = 2x + 1$ [2 marks]

(iii) $x^{-1} \times x^1 - x^{-1} \times 3x^{-2} = 1 - 3x^{-3}$ [2 marks]

(iv) $x^{\frac{1}{3}+\frac{2}{3}} - 4x^{\frac{1}{3}+\frac{5}{3}} = x - 4x^2$ [2 marks]

(v) $3x^{\frac{1}{4}+\frac{7}{4}} + x^{\frac{1}{4}-\frac{1}{4}} = 3x^2 + 1$ [2 marks]

[Total: 10 marks]

3 **(i)** $\frac{1}{x}=7 \Rightarrow x=\frac{1}{7}$ [1 mark]

(ii) $\sqrt{x}=9 \Rightarrow x=9^2=81$ [1 mark]

(iii) $\frac{1}{x^2}=\frac{1}{4} \Rightarrow x^2=4 \Rightarrow x=\pm 2$ [2 marks]

(iv) $x^{\frac{1}{3}}=-\frac{3}{2} \Rightarrow x=\left(-\frac{3}{2}\right)^3=-\frac{27}{8}$ [2 marks]

(v) $\frac{1}{x^3}=8 \Rightarrow x^3=\frac{1}{8} \Rightarrow x=\frac{1}{2}$ [2 marks]

(vi) $x^{\frac{1}{3}}=\pm 4 \Rightarrow x=\pm 64$ [2 marks]

(vii) $x^{\frac{1}{2}}=\frac{1}{4} \Rightarrow x=\frac{1}{16}$ [2 marks]

(viii) $x^{\frac{3}{2}}=\frac{1}{27} \Rightarrow x^{\frac{1}{2}}=\frac{1}{3} \Rightarrow x=\frac{1}{9}$ [2 marks]

[Total: 14 marks]

4 **(i)** $3^y=1 \Rightarrow y=0$ [1 mark]

(ii) $3^x=81$ or $3^{x+2}=1 \Rightarrow x=4$ or $3^{x+2}=3^0 \Rightarrow x+2=0 \Rightarrow x=-2$ [3 marks]

[Total: 4 marks]

5 **(i)** Let $y=2^x \therefore 4^x=\left(2^2\right)^x=2^{2x}=\left(2^x\right)^2=y^2$ and $2^{x+1}=2^1\times 2^x=2y$

$\therefore y^2=2y+8 \Rightarrow y^2-2y-8=0$ [3 marks]

(ii) $(y-4)(y+2)=0 \Rightarrow y=4$ or $y=-2 \therefore 2^x=4$ or $2^x=-2$

but $2^x>0 \therefore 2^x=2^2 \Rightarrow x=2$ [3 marks]

[Total: 6 marks]

6 $x^2+3x+2=0 \Rightarrow (x+1)(x+2)=0 \Rightarrow x=-1$ or $x=-2$ [4 marks]

7 Let $y=\sqrt{x} \therefore y^2+6=5y \Rightarrow y^2-5y+6=0 \Rightarrow (y-2)(y-3)=0$

$\Rightarrow y=2$ or $y=3 \therefore \sqrt{x}=2$ or $\sqrt{x}=3 \therefore x=4$ or $x=9$ [5 marks]

8 $x-3=0$ or $x^2-3=1 \therefore x=3$ or $x=\pm 2$

Note: $x^2-3=-1$ is not a solution as the index would need to be an even integer. [3 marks]

9 $1\left(x^{\frac{3}{2}}\right)^4\left(-x^{-\frac{1}{2}}\right)^0+4\left(x^{\frac{3}{2}}\right)^3\left(-x^{-\frac{1}{2}}\right)^1+6\left(x^{\frac{3}{2}}\right)^2\left(-x^{-\frac{1}{2}}\right)^2+4\left(x^{\frac{3}{2}}\right)^1\left(-x^{-\frac{1}{2}}\right)^3+1\left(x^{\frac{3}{2}}\right)^0\left(-x^{-\frac{1}{2}}\right)^4$

$=x^6-4x^4+6x^2-4+x^{-2}$ [3 marks]

Exercise 4.7 Algebraic proof

1 **(i)** All odd numbers can be written in the form $2m-1$, where m is a positive integer.
Product of two odd numbers $=(2m-1)(2n-1)$, where m and n are positive integers.

$=4mn-2m-2n+1=2(2mn-m-n)+1=2\times\text{integer}+1=\text{odd number}$ [3 marks]

(ii) All even numbers can be written in the form $2m$, where m is a positive integer.
Product of two even numbers $=2m\times 2n$, where m and n are positive integers.

$=4mn=2\times 2mn=2\times\text{integer}=\text{even number}$ [3 marks]

[Total: 6 marks]

2 **(i)** $x^2-2x+3=(x-1)^2-1^2+3=(x-1)^2+2$ [2 marks]

(ii) $(x-1)^2 \geqslant 0$ for all values of x

$\therefore\ (x-1)^2+2 \geqslant 2 \quad \Rightarrow \quad x^2-2x+3>0$ for all values of x [2 marks]

[Total: 4 marks]

3 $f(y+4)-f(y) = (y+4)^2-y^2 = y^2+8y+16-y^2 = 8y+16 = 8(y+2) = 8\times$ integer
$=$ a multiple of 8 [4 marks]

4 Any product of two consecutive integers $= n(n+1)$, where n is an integer

required sum $= n(n+1)+(n+1) = n^2+n+n+1 = n^2+2n+1 = (n+1)^2 = (\text{integer})^2$
$=$ a square number [4 marks]

5 **(i)** Even square number $=$ even$^2 = (2m)^2$ where m is an integer
$= 4m^2 = 4\times$ integer $=$ a multiple of 4 [3 marks]

(ii) Odd square number $=$ odd$^2 = (2m+1)^2$, where m is an integer

$= 4m^2+4m+1 = 4(m^2+m)+1 = 4\times$ integer $+1$ $=$ one more than a multiple of 4 [3 marks]

[Total: 6 marks]

6 All integers can be written as $3n$ or $3n+1$ or $3n+2$, where n is an integer
$\therefore$ all square numbers take one of these forms: $(3n)^2$ or $(3n+1)^2$ or $(3n+2)^2$

$(3n)^2 = 9n^2 = 3\times 3n^2 = 3\times$ integer $=$ a multiple of 3

$(3n+1)^2 = 9n^2+6n+1 = 3(3n^2+2n)+1 = 3\times$ integer $+1$ $=$ one more than a multiple of 3

$(3n+2)^2 = 9n^2+12n+4 = 3(3n^2+4n+1)+1 = 3\times$ integer $+1$ $=$ one more than a multiple of 3

$\therefore$ a square number can never be one less than a multiple of 3 [5 marks]

7 **(i)** $x^2-2xy+y^2 = (x-y)^2 \geqslant 0$ for all values of x and y [2 marks]

(ii) $(x-y)^2 = x^2-2xy+y^2 \ \Rightarrow\ x^2+2xy+y^2-4xy = (x+y)^2-4xy$
$\therefore\ (x+y)^2-4xy \geqslant 0 \quad \Rightarrow \quad (x+y)^2 \geqslant 4xy$ for all values of x and y [4 marks]

(iii) $\dfrac{(x+y)^2}{4} \geqslant xy \quad \Rightarrow \quad \dfrac{x+y}{2} \geqslant \sqrt{xy}$ for all positive values of x and y [2 marks]

[Total: 6 marks]

8 **(i)** $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{x^2+y^2}{xy}$ [1 mark]

(ii) $x^2-2xy+y^2 = (x-y)^2 \geqslant 0$ [1 mark]

(iii) $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{x^2+y^2}{xy}=\dfrac{(x-y)^2+2xy}{xy}=\dfrac{(x-y)^2}{xy}+2 \geqslant 0+2$ if x and y are positive

$\therefore\ \dfrac{x}{y}+\dfrac{y}{x} \geqslant 2$ for all positive x and y [3 marks]

[Total: 5 marks]

9 Product $=\left(\dfrac{-b+\sqrt{b^2-4c}}{2}\right)\left(\dfrac{-b-\sqrt{b^2-4c}}{2}\right)=\dfrac{b^2-b\sqrt{b^2-4c}+b\sqrt{b^2-4c}-(b^2-4c)}{4}=\dfrac{4c}{4}=c$ [3 marks]

Exercise 4.8 Linear sequences

1 **(i)** $3n + 2$ [1 mark]

(ii) $4n + 2$ [1 mark]

(iii) $10n$ [1 mark]

(iv) $-5n + 12$ [1 mark]

(v) $-3n + 3$ [1 mark]

(vi) $-4n - 1$ [1 mark]

[Total: 6 marks]

2 **(i)** nth term $= 2n + 2 \therefore$ 100th term is $2 \times 100 + 2 = 202$ [2 marks]

(ii) nth term $= 5n + 1 \therefore$ 100th term is $5 \times 100 + 1 = 501$ [2 marks]

(iii) nth term $= 9n + 1 \therefore$ 100th term is $9 \times 100 + 1 = 901$ [2 marks]

(iv) nth term $= -7n + 13 \therefore$ 100th term is $-7 \times 100 + 13 = -687$ [2 marks]

(v) nth term $= -8n + 8 \therefore$ 100th term is $-8 \times 100 + 8 = -792$ [2 marks]

(vi) nth term $= -2n - 5 \therefore$ 100th term is $-2 \times 100 - 5 = -205$ [2 marks]

[Total: 12 marks]

3 **(i)** nth term $= 6n - 4 \;\therefore\; 6n - 4 > 250 \;\Rightarrow\; n > 42\frac{1}{3}$

43rd term $= 6 \times 43 - 4 = 254$ [3 marks]

(ii) nth term $= 3n + 4 \;\therefore\; 3n + 4 > 250 \;\Rightarrow\; n > 82$

83rd term $= 3 \times 83 + 4 = 253$ [3 marks]

(iii) nth term $= 9n + 2 \;\therefore\; 9n + 2 > 250 \;\Rightarrow\; n > 27\frac{5}{9}$

28th term $= 9 \times 28 + 2 = 254$ [3 marks]

(iv) nth term $= 5n - 12 \;\therefore\; 5n - 12 > 250 \;\Rightarrow\; n > 52.4$

53rd term $= 5 \times 53 - 12 = 253$ [3 marks]

(v) nth term $= 3n - 3 \;\therefore\; 3n - 3 > 250 \;\Rightarrow\; n > 84\frac{1}{3}$

85th term $= 3 \times 85 - 3 = 252$ [3 marks]

(vi) nth term $= 4n - 9 \;\therefore\; 4n - 9 > 250 \;\Rightarrow\; n > 64\frac{3}{4}$

65th term $= 4 \times 65 - 9 = 251$ [3 marks]

[Total: 18 marks]

4 **(i)** nth term $= 82 - 8n \;\therefore\; 82 - 8n < 0 \;\Rightarrow\; n > 10\frac{1}{4}$

11th term $= 82 - 8 \times 11 = -6$ [3 marks]

(ii) nth term $= 103 - 3n \;\therefore\; 103 - 3n < 0 \;\Rightarrow\; n > 34\frac{1}{3}$

35th term $= 103 - 3 \times 35 = -2$ [3 marks]

(iii) nth term $= 4006 - 6n \ \therefore \ 4006 - 6n < 0 \quad \Rightarrow \quad n > 667\frac{2}{3}$

668th term = 4006 − 6 × 668 = −2 [3 marks]

(iv) nth term $= 2009 - 9n \ \therefore \ 2009 - 9n < 0 \quad \Rightarrow \quad n > 223\frac{2}{9}$

224th term = 2009 − 9 × 224 = −7 [3 marks]

(v) nth term $= 512 - 12n \ \therefore \ 512 - 12n < 0 \quad \Rightarrow \quad n > 42\frac{2}{3}$

43rd term = 512 − 12 × 43 = −4 [3 marks]

(vi) nth term $= 994 - 7n \ \therefore \ 994 - 7n < 0 \quad \Rightarrow \quad n > 142$

143rd term = 994 − 7 × 143 = − 7 [3 marks]

[Total: 18 marks]

5 **(i)** difference between terms is $q - p \ \therefore$ 3rd term is $q + (q - p) = 2q - p$ [2 marks]

(ii) nth term is $p - (q - p) + n(q - p) = (2 - n)p + (n - 1)q$ [3 marks]

(iii) $(2 - 10)p + (10 - 1)q = 74 \quad \Rightarrow \quad - 8p + 9q = 74$ [2 marks]

(iv) $2(q - p) = 88 - 74 = 14 \quad \Rightarrow \quad q - p = 7 \quad \Rightarrow \quad q = p + 7$

$\therefore \ -8p + 9\,(p + 7) = 74 \quad \Rightarrow \quad p + 63 = 74 \quad \Rightarrow \quad p = 11 \ \therefore \ q = 11 + 7 = 18$ [5 marks]

[Total: 12 marks]

6 If 654 is in the sequence, then $7n + 2 = 654 \quad \Rightarrow \quad n = 93\frac{1}{7}$ but n is an integer, so 654 cannot be in the sequence. [2 marks]

7 Difference between consecutive terms is $(n + 1)^2 - n^2$, where n is an integer
$= n^2 + 2n + 1 - n^2 = 2n + 1 =$ an odd number [4 marks]

8 $b - a = c - b \quad \Rightarrow \quad 2b = a + c \quad \Rightarrow \quad b = \dfrac{a + c}{2}$ [2 marks]

9 The sequences are 2, 5, 8, 11, … and 7, 11, … so the first common term is 11

The LCM of 3 and 4 is 12. So the nth term is $12n - 1$ [3 marks]

Exercise 4.9 Quadratic sequences and the limiting value of a sequence

1 **(i)** 11 + 5 = 16 [1 mark]

(ii) 32 + 12 = 44 [1 mark]

(iii) 27 + 11 = 38 [1 mark]

(iv) 1 − 6 = −5 [1 mark]

[Total: 4 marks]

2 **(i)** Differences = 3, 5, 7, 9, … $\therefore$ difference of differences = 2 $\therefore$ n^2 coefficient is 2 ÷ 2 = 1
1, 4, 9, 16, 25 = n^2

1, 1, 1, 1, 1 = 1

2, 5, 10, 17, 26 = $n^2 + 1$ [4 marks]

(ii) Differences = 4, 6, 8, 10, … $\therefore$ difference of differences = 2 $\therefore$ n^2 coefficient is 2 ÷ 2 = 1
1, 4, 9, 16, 25 = n^2

4, 5, 6, 7, 8 = $n + 3$

5, 9, 15, 23, 33 = $n^2 + n + 3$ [4 marks]

(iii) Differences = 2, 4, 6, 8, … $\therefore$ difference of differences = 2 $\therefore$ n^2 coefficient is 2 ÷ 2 = 1

$1, 4, 9, 16, 25 = n^2$

$0, -1, -2, -3, -4 = -1n + 1$

$1, 3, 7, 13, 21 = n^2 - n + 1$ [4 marks]

(iv) Differences = 2, 4, 6, 8, … $\therefore$ difference of differences = 2 $\therefore$ n^2 coefficient is 2 ÷ 2 = 1

$1, 4, 9, 16, 25 = n^2$

$2, 1, 0, -1, -2 = -n + 3$

$3, 5, 9, 15, 23 = n^2 - n + 3$ [4 marks]

(v) Differences = 4, 8, 12, 16, … $\therefore$ difference of differences = 4 $\therefore$ n^2 coefficient is 4 ÷ 2 = 2

$2, 8, 18, 32, 50 = 2n^2$

$3, 1, -1, -3, -5 = -2n + 5$

$5, 9, 17, 29, 45 = 2n^2 - 2n + 5$ [4 marks]

(vi) Differences = −1, −7, −13, −19, … $\therefore$ difference of differences = −6

$\therefore$ n^2 coefficient is −6 ÷ 2 = −3

$-3, -12, -27, -48, -75 = -3n^2$

$33, 41, 49, 57, 65 = 8n + 25$

$30, 29, 22, 9, -10 = -3n^2 + 8n + 25$ [4 marks]

[Total: 24 marks]

3 **(i)** Differences = $3,\ p - 8,\ 26 - p,\ 15$

$\therefore$ difference of differences $= p - 8 - 3,\quad 26 - p - (p - 8),\quad 15 - (26 - p)$

$= p - 11,\quad 34 - 2p,\quad p - 11$

$\therefore\ p - 11 = 34 - 2p \quad \Rightarrow\quad 3p = 45 \quad \Rightarrow \quad p = 15$ [3 marks]

(ii) n^2 coefficient is 4 ÷ 2 = 2

$2, 8, 18, 32, 50 = 2n^2$

$3, 0, -3, -6, -9 = -3n + 6$

$5, 8, 15, 26, 41 = 2n^2 - 3n + 6$ [4 marks]

[Total: 7 marks]

4 **(i)** 1st term $= \dfrac{3 - 2 \times 1}{3 \times 1 + 4} = \dfrac{1}{7}$

2nd term $= \dfrac{3 - 2 \times 2}{3 \times 2 + 4} = -\dfrac{1}{10}$

3rd term $= \dfrac{3 - 2 \times 3}{3 \times 3 + 4} = -\dfrac{3}{13}$ [1 mark]

(ii) $-\dfrac{2}{3}$ [1 mark]

[Total: 2 marks]

5 **(i)** $\dfrac{a}{2} = -3 \quad \Rightarrow \quad a = -6$ [1 mark]

(ii) 1st term $= \dfrac{-6 \times 1 + 5}{2 \times 1 - 1} = \dfrac{-1}{1} = -1$ [1 mark]

[Total: 2 marks]

6 n^2 coefficient is $6 \div 2 = 3$
$3, 12, 27 = 3n^2$
$0, -1, -2 = -n + 1$
$3, 11, 25 = 3n^2 - n + 1$ [4 marks]

7 $\dfrac{4n^2 - 5n + 6}{7 + 4n - 11n^2} = \dfrac{4 - \frac{5}{n} + \frac{6}{n^2}}{\frac{7}{n^2} + \frac{4}{n} - 11} \to \dfrac{4 - 0 + 0}{0 + 0 - 11} = -\dfrac{4}{11}$ as $n \to \infty$ [2 marks]

8 1st differences are $p-2$, $12-p$, $q-12$, $30-q$
2nd differences are $14-2p$, $p+q-24$, $42-2q$
$\therefore\ 14-2p = p+q-24$ and $14-2p = 42-2q$
$\Rightarrow\ 38 = 3p+q$ and $q = p+14$
$\therefore\ 38 = 3p+p+14 \Rightarrow 24 = 4p \Rightarrow p = 6 \therefore q = 6+14 = 20$ [4 marks]

9 **(i)** Let successive differences be: 7, $7 + d$, $7 + 2d$
So $x - 3 = 7 + d$ and $35 - x = 7 + 2d$
$d = x - 10$ so $35 - x = 7 + 2(x - 10) \Rightarrow 48 = 3x \Rightarrow x = 16$ [2 marks]

(ii) $d = 16 - 10 = 6$
Let the *n*th term be $an^2 + bn + c$
So $a = \frac{6}{2} = 3$ and $c =$ 0th term $= -4 - 1 = -5$
1st term $= a + b + c$ so $-4 = 3 + b + (-5) \Rightarrow b = -2$
So the *n*th term $= 3n^2 - 2n - 5$ [3 marks]
[Total: 5 marks]

Exercise 4.10 Simultaneous equations in three unknowns

1 subtracting equations 1 and 2 $\quad 2x - x = 23 - 12 \Rightarrow x = 11$
subtracting equations 1 and 3 $\quad 2y - y = 25 - 12 \Rightarrow y = 13$
substituting into equation 1 $\quad 11 + 13 + z = 12 \Rightarrow z = -12$ [3 marks]

2 adding equations 1 and 2 $\quad 4x + 3y = 2$
2 × 4 − 3 $\quad 4x - 2x + 8y + 3y = 24 - 4 \Rightarrow 2x + 11y = 20$
$3y - 22y = 2 - 40 \Rightarrow -19y = -38 \Rightarrow y = 2$
$\therefore 2x + 22 = 20 \Rightarrow x = -1$
$\therefore 3 \times -1 + 2 - z = -4 \Rightarrow -3 + 2 - z = -4 \Rightarrow z = 3$ [4 marks]

3 **(i)** $1^3 + p \times 1^2 + q \times 1 + r = 0 \Rightarrow p + q + r = -1$ 1
$(-1)^3 + p \times (-1)^2 + q \times -1 + r = 0 \Rightarrow p - q + r = 1$ 2
2 − 1 $\quad -q - q = 1 - -1 \Rightarrow -2q = 2 \Rightarrow q = -1$ [2 marks]

(ii) Substitute $q = -1$ into equation 1 $\therefore\ p - 1 + r = -1 \Rightarrow p = -r$
Substitute $p = -r$ into $p = r + 6$ gives $r + 6 = -r \Rightarrow r = -3$
$\therefore p = -3 + 6 = 3$ [3 marks]

[Total: 5 marks]

4 $2x+5y-z=19$ ①
$3x-2y+4z=-11$ ② ① × 4 + ② $11x+18y=65$ ④
$x+3y-2z=14$ ③ ① × 2 − ③ $3x+7y=24$ ⑤
④ × 3 − ⑤ ×11 $54y-77y=195-264 \Rightarrow -23y=-69 \Rightarrow y=3$
Substitute $y=3$ into ⑤ $3x+7\times3=24 \Rightarrow 3x=3 \Rightarrow x=1$
Substitute $x=1$ and $y=3$ into equation ① $2+15-z=19 \Rightarrow z=-2$ [4 marks]

5 Substitute $z=5-3y$ into $x=3-4z$ $\therefore$ $x=3-4(5-3y)=12y-17$
But $y=x-5$ $\therefore$ $x=12(x-5)-17 \Rightarrow x=12x-77 \Rightarrow 77=11x \Rightarrow x=7$
$\therefore$ $y=7-5=2$ and $z=5-3\times2=-1$ [3 marks]

6 limiting value $=\frac{a}{2}=4 \Rightarrow a=8$
1st term $=\frac{a+1}{2+b}=-3$ $\therefore$ $8+1=-3(2+b) \Rightarrow -3=2+b \Rightarrow b=-5$
$\therefore$ 3rd term is $\frac{8\times3+1}{2\times3-5}=25$ [4 marks]

7 Equation ② ÷ 2 $2x+y-3z=2.5$, which contradicts equation ①
$\therefore$ there are no solutions common to all three equations [2 marks]

8 **(i)** $a\times2^2+b\times2+c=-4 \Rightarrow 4a+2b+c=-4$ ①
$a\times3^2+b\times3+c=-4 \Rightarrow 9a+3b+c=-4$ ②
$a\times5^2+b\times5+c=2 \Rightarrow 25a+5b+c=2$ ③ [1 mark]

(ii) ② − ① $5a+b=0$ ④
③ − ① $21a+3b=6 \Rightarrow 7a+b=2$ ⑤
⑤ − ④ $2a=2 \Rightarrow a=1$ $\therefore$ $7+b=2 \Rightarrow b=-5$
Substitute into equation ① $4-10+c=-4 \Rightarrow c=2$
nth term $=n^2-5n+2$ [4 marks]

[Total: 5 marks]

9 $2(3x-7)+5(11-4x)=13 \Rightarrow 14x=28 \Rightarrow x=2$

$y=3\times2-7=-1$ and $z=11-4\times2=3$ [5 marks]

5 Coordinate geometry

Exercise 5.1 Parallel and perpendicular lines: distances, midpoints and gradients

1 **(i)** **(a)** Gradient AB $= \dfrac{7-1}{4-2} = 3$ [1 mark]

(b) Gradient of perpendicular $= -\dfrac{1}{3}$ [1 mark]

(c) Length AB $= \sqrt{(4-2)^2 + (7-1)^2} = 2\sqrt{10}$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{2+4}{2}, \dfrac{1+7}{2}\right) = (3,\ 4)$ [1 mark]

(ii) **(a)** Gradient AB $= \dfrac{9-3}{5-2} = 2$ [1 mark]

(b) Gradient of perpendicular $= -\dfrac{1}{2}$ [1 mark]

(c) Length AB $= \sqrt{(5-2)^2 + (9-3)^2} = 3\sqrt{5}$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{2+5}{2}, \dfrac{3+9}{2}\right) = (3.5,\ 6)$ [1 mark]

(iii) **(a)** Gradient AB $= \dfrac{2-2}{-3-5} = 0$ [1 mark]

(b) Gradient of perpendicular is undefined as you can't divide by 0 [1 mark]

(c) Length AB $= \sqrt{(2-2)^2 + (-3-5)^2} = 8$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{5-3}{2}, \dfrac{2+2}{2}\right) = (1,\ 2)$ [1 mark]

(iv) **(a)** Gradient AB $= \dfrac{8-7}{5-0} = 0.2$ [1 mark]

(b) Gradient of perpendicular $= -\dfrac{1}{0.2} = -5$ [1 mark]

(c) Length AB $= \sqrt{(5-0)^2 + (8-7)^2} = \sqrt{26}$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{0+5}{2}, \dfrac{7+8}{2}\right) = (2.5,\ 7.5)$ [1 mark]

(v) **(a)** Gradient AB $= \dfrac{3-6}{4-(-2)} = -\dfrac{1}{2}$ [1 mark]

(b) Gradient of perpendicular $= -\dfrac{1}{-0.5} = 2$ [1 mark]

(c) Length AB $= \sqrt{(4-(-2))^2 + (3-6)^2} = 3\sqrt{5}$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{-2+4}{2}, \dfrac{6+3}{2}\right) = (1,\ 4.5)$ [1 mark]

(vi) **(a)** Gradient AB $= \dfrac{4-(-3)}{2-(-5)} = 1$ [1 mark]

(b) Gradient of perpendicular $= -\dfrac{1}{1} = -1$ [1 mark]

(c) Length AB $= \sqrt{(2-(-5))^2 + (4-(-3))^2} = 7\sqrt{2}$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{-5+2}{2}, \dfrac{-3+4}{2}\right) = (-1.5,\ 0.5)$ [1 mark]

(vii) **(a)** Gradient AB $= \dfrac{-2-2}{0-(-9)} = -\dfrac{4}{9}$ [1 mark]

(b) Gradient of perpendicular $= -\dfrac{1}{-\frac{4}{9}} = \dfrac{9}{4}$ [1 mark]

(c) Length AB $= \sqrt{(0-(-9))^2 + ((-2)-2)^2} = \sqrt{97}$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{-9+0}{2}, \dfrac{2+(-2)}{2}\right) = (-4.5, 0)$ [1 mark]

(viii) **(a)** Gradient AB $= \dfrac{(-4)-(-6)}{6-4} = 1$ [1 mark]

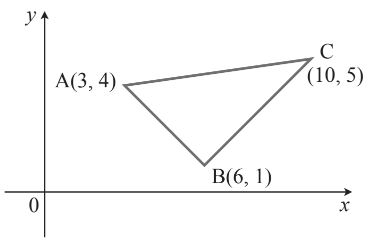
(b) Gradient of perpendicular $= -\dfrac{1}{1} = -1$ [1 mark]

(c) Length AB $= \sqrt{(6-4)^2 + ((-4)-(-6))^2} = 2\sqrt{2}$ [1 mark]

(d) Midpoint AB $= \left(\dfrac{4+6}{2}, \dfrac{(-6)+(-4)}{2}\right) = (5, -5)$ [1 mark]

[Total: 32 marks]

2 **(i)**



[2 marks]

(ii) AB $= \sqrt{(6-3)^2 + (4-1)^2} = 3\sqrt{2}$

BC $= \sqrt{(10-6)^2 + (5-1)^2} = 4\sqrt{2}$

AC $= \sqrt{(10-3)^2 + (5-4)^2} = 5\sqrt{2}$

$(3\sqrt{2})^2 + (4\sqrt{2})^2 = 50 = (5\sqrt{2})^2$

$\Rightarrow \angle ABC = 90°$ (converse Pythagoras theorem) [4 marks]

(iii) $\left.\begin{aligned} \text{Gradient AB} &= \dfrac{1-4}{6-3} = -1 \\ \text{Gradient BC} &= \dfrac{5-1}{10-6} = 1 \end{aligned}\right\} (-1) \times 1 = -1$

$\Rightarrow$ AB and BC perpendicular

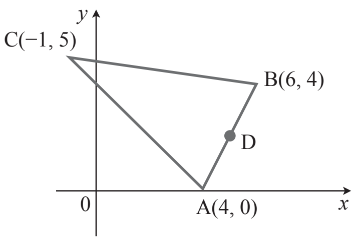
$\Rightarrow \angle ABC = 90°$ [2 marks]

(iv) Area $= \dfrac{1}{2} \times AB \times BC$

$= \dfrac{1}{2} \times 3\sqrt{2} \times 4\sqrt{2}$

$= 12$ units^2 [2 marks]

[Total: 10 marks]

3 **(i)** 

[2 marks]

(ii) $AB^2 = (6 - 4)^2 + (4 - 0)^2 = 20$

$BC^2 = (6 - (-1))^2 + (4 - 5)^2 = 50$

$AC^2 = (4 - (-1))^2 + (0 - 5)^2 = 50$

$BC = AC = \sqrt{50} \Rightarrow$ triangle is isosceles [3 marks]

(iii) Let D be the midpoint of AB

D has coordinates $\left(\frac{4+6}{2}, \frac{0+4}{2}\right) = (5, 2)$

CD is a line of symmetry and $CD = \sqrt{(5-(-1))^2 + (2-5)^2}$

$= 3\sqrt{5}$

$AB = \sqrt{20} = 2\sqrt{5}$

$\Rightarrow \text{Area} = \frac{1}{2} AB \times CD$

$= \frac{1}{2} \times 2\sqrt{5} \times 3\sqrt{5}$

$= 15 \text{ units}^2$

[3 marks]

[Total: 8 marks]

4 **(i)** A(−3, 4), B(7, 4), C(6, 1)

$$\left.\begin{aligned} AB^2 &= (7-(-3))^2 + (4-4)^2 = 100 \\ BC^2 &= (7-6)^2 + (4-1)^2 = 10 \\ AC^2 &= (6-(-3))^2 + (1-4)^2 = 90 \end{aligned}\right\} \Rightarrow \begin{aligned} AB &= 10 \\ BC &= \sqrt{10} \\ AC &= 3\sqrt{10} \end{aligned} \Rightarrow BC^2 + AC^2 = AB^2$$

$\Rightarrow$ right angle at C [3 marks]

(ii) $\text{Area} = \frac{1}{2} \times BC \times AC$

$= \frac{1}{2} \times \sqrt{10} \times \sqrt{90}$

$= 15 \text{ units}^2$

[2 marks]

[Total: 5 marks]

5 **(i)** A(1, 4), B(x, 1), C(9, 0)

Collinear $\Rightarrow$ gradient AB = gradient AC

$$\Rightarrow \frac{1-4}{x-1} = \frac{0-4}{9-1}$$

$$\Rightarrow (-3)(8) = -4(x-1)$$

$$\Rightarrow -24 = -4x + 4$$

$$\Rightarrow -28 = -4x$$

$$\Rightarrow x = 7$$

[3 marks]

(ii) $\text{AB} = \sqrt{(7-1)^2 + (1-4)^2} = 3\sqrt{5}$

$\text{BC} = \sqrt{(9-7)^2 + (0-1)^2} = \sqrt{5}$

AB : BC = 3 : 1

[2 marks]

[Total: 5 marks]

6 A(0, 0), B(4, 4), C(6, 2), D(2, −2)

$$\left.\begin{aligned} \text{AB} &= \sqrt{4^2 + 4^2} = 4\sqrt{2} \\ \text{BC} &= \sqrt{(4-6)^2 + (4-2)^2} = 2\sqrt{2} \\ \text{CD} &= \sqrt{(6-2)^2 + (2-(-2))^2} = 4\sqrt{2} \\ \text{AD} &= \sqrt{2^2 + (-2)^2} = 2\sqrt{2} \end{aligned}\right\} \Rightarrow \text{opposite sides equal in length}$$

$$\left.\begin{aligned} \text{Gradient AB} &= \frac{4}{4} = 1 \\ \text{Gradient BC} &= \frac{2-4}{6-4} = -1 \\ \text{Gradient CD} &= \frac{-2-2}{2-6} = 1 \\ \text{Gradient AD} &= \frac{-2}{2} = -1 \end{aligned}\right\} \Rightarrow \text{opposite sides parallel}$$

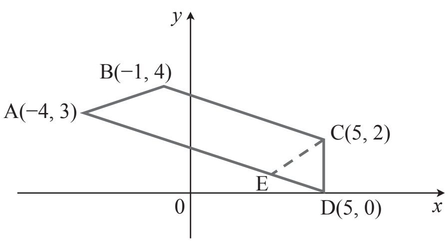
Adjacent sides have product of gradients = −1

$\Rightarrow$ adjacent sides perpendicular

$\Rightarrow$ it is a rectangle

[5 marks]

7 **(i)**



[2 marks]

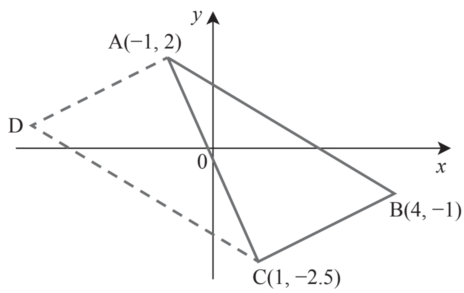
$$\left.\begin{aligned}\text{Gradient BC} &= \frac{2-4}{5-(-1)} = -\frac{1}{3}\\ \text{Gradient AD} &= \frac{0-3}{5-(-4)} = -\frac{1}{3}\end{aligned}\right\} \Rightarrow \text{opposite sides parallel; clearly not equal}$$

(ii) $\Rightarrow$ it is a trapezium [2 marks]

(iii) B to C is 6 across and 2 down
$\Rightarrow$ A to E is 6 across and 2 down $\Rightarrow$ E is $(-4 + 6, 3 - 2)$
E is (2, 1) [2 marks]

[Total: 6 marks]

8 (i)

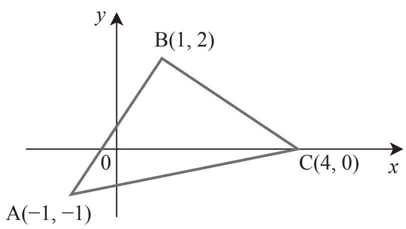


[2 marks]

(ii) B $\rightarrow$ A is 5 left, 3 up
So, C $\rightarrow$ D is 5 left, 3 up.
D is $(1 - 5, -2.5 + 3) = (-4, 0.5)$ [1 mark]

[Total: 3 marks]

9 (i)



[2 marks]

(ii) $AB = \sqrt{(2-(-1)^2 + (1-(-1))^2} = \sqrt{13}$

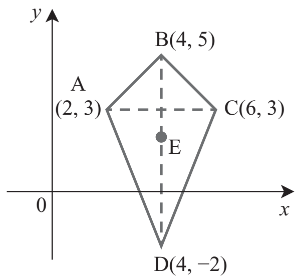
$BC = \sqrt{(0-2)^2 + (4-1)^2} = \sqrt{13}$

$$\left.\begin{aligned}\text{Gradient AB} &= \frac{2-(-1)}{1-(-1)} = \frac{3}{2}\\ \text{Gradient BC} &= \frac{0-2}{4-1} = \frac{-2}{3}\end{aligned}\right\} \frac{3}{2} \times \frac{-2}{3} = -1$$

$\Rightarrow$ triangle has a right angle at B and is isosceles with AB = BC [4 marks]

[Total: 6 marks]

10 (i)



[2 marks]

(ii) $\left.\begin{array}{l} AB = \sqrt{(5-3)^2 + (4-2)^2} = 2\sqrt{2} \\ BC = \sqrt{(3-5)^2 + (6-4)^2} = 2\sqrt{2} \end{array}\right\} AB = BC$

$\left.\begin{array}{l} AD = \sqrt{((-2)-3)^2 + (4-2)^2} = \sqrt{29} \\ CD = \sqrt{(3-(-2))^2 + (6-4)^2} = \sqrt{29} \end{array}\right\} AD = CD$

$\Rightarrow$ ABCD is a kite [4 marks]

(iii) Area ABC $= \dfrac{1}{2} \times 4 \times 2 = 4$ units^2

Area ACD $= \dfrac{1}{2} \times 4 \times 5 = 10$ units^2

$\Rightarrow$ Area of kite $= 14$ units^2 [2 marks]

(iv) Area ACE $= \dfrac{1}{2} \times 4 \times 1 = 2$ units^2

$\Rightarrow$ Area ABCE $= 4 + 2 = 6$ units^2

Area AECD $= 14 - 6 = 8$ units^2 [2 marks]

[Total: 10 marks]

11 (i) A(7, 3) B(–4, 1) C(–3, –2)

$AB = \sqrt{(7-(-4))^2 + (3-1)^2} = 5\sqrt{5}$

$BC = \sqrt{(-4-(-3))^2 + (1-(-2))^2} = \sqrt{10}$

$AC = \sqrt{(7-(-3))^2 + (3-(-2))^2} = 5\sqrt{5}$

AB = AC $\Rightarrow$ triangle isosceles [2 marks]

(ii) Midpoint BC $= \left(\dfrac{-4+(-3)}{2}, \dfrac{1+(-2)}{2}\right) = (-3.5, -0.5)$ [2 marks]

(iii) Perpendicular height = distance from (7, 3) to (–3.5, –0.5)

$$= \sqrt{\left(7-(-3.5)\right)^2 + \left(3-(-0.5)\right)^2}$$

$$= \frac{7\sqrt{10}}{2}$$

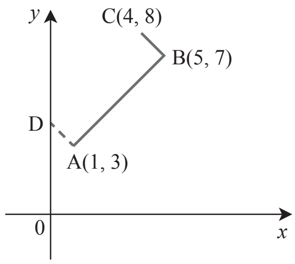
Base BC = $\sqrt{10}$

Area = $\frac{1}{2} \times \frac{7\sqrt{10}}{2} \times \sqrt{10}$

= 17.5 units^2 [3 marks]

[Total: 7 marks]

12 (i) A(1, 3) B(5, 7) C(4, 8) D(x, y)



[2 marks]

(ii) From the diagram B → C is 1 unit left, 1 up

A → D is 1 unit left, 1 up

D is (0, 4)

BC = $\sqrt{(5-4)^2 + (7-8)^2} = \sqrt{2}$

AB = $\sqrt{(5-1)^2 + (7-3)^2} = 4\sqrt{2}$

→ Area of rectangle = $\sqrt{2} \times 4\sqrt{2}$

= 8 units^2 [3 marks]

[Total: 5 marks]

13 (i) A(p, q), B(q, p), midpoint $\left(\frac{p+q}{2}, \frac{p+q}{2}\right)$ [1 mark]

(ii) Gradient of AB is $\frac{p-q}{q-p} = -1$

⇒ Gradient of perpendicular bisector is +1

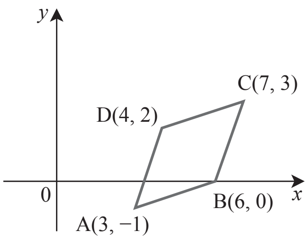
Eq^n is $y - \left(\frac{p+q}{2}\right) = 1\left(x - \frac{p+q}{2}\right)$

⇒ $y = x$

∴ origin is on the perpendicular bisector [4 marks]

[Total: 5 marks]

14 (i)



AB gradient $= \dfrac{0-(-1)}{6-3} = \dfrac{1}{3}$ BC gradient $= \dfrac{3-0}{7-6} = 3$

DC gradient $= \dfrac{3-2}{7-4} = \dfrac{1}{3}$ AD gradient $= \dfrac{2-(-1)}{4-3} = 3$

Opposite sides parallel

AB $= \sqrt{1^2+3^2} = \sqrt{10}$ BC $= \sqrt{1^2+3^2} = \sqrt{10}$

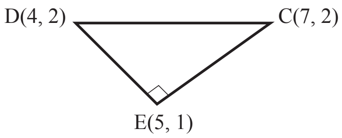
Adjacent sides equal $\Rightarrow$ it is a rhombus [5 marks]

(ii) $\left.\begin{array}{l}\text{Midpoint BD} = \left(\dfrac{4+6}{2}, \dfrac{2+0}{2}\right) = (5,1) \\ \text{Midpoint AC} = \left(\dfrac{3+7}{2}, \dfrac{-1+3}{2}\right) = (5,1)\end{array}\right\}$ Diagonals bisect each other

Gradient AC $= \dfrac{3-(-1)}{7-3} = 1$ Gradient BD $= \dfrac{0-2}{6-4} = -1$

$\Rightarrow$ AC $\perp$ BD so diagonals bisect at right angles [4 marks]

(iii) Let E be (5, 1) (where diagonals bisect each other)



DE $= \sqrt{(5-4)^2+(1-2)^2} = \sqrt{2}$

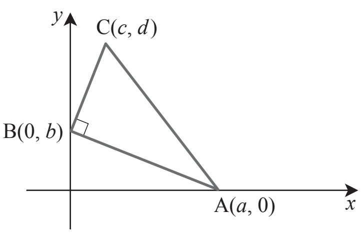
EC $= \sqrt{(7-5)^2+(2-1)^2} = \sqrt{5}$

Area ΔCED $= \dfrac{1}{2}\sqrt{2}\times\sqrt{5} = \dfrac{1}{2}\sqrt{10}$

Area of rhombus $= 4 \times \dfrac{1}{2}\sqrt{10} = 2\sqrt{10}$ units^2 [4 marks]

[Total: 13 marks]

15



Gradient BC $= \dfrac{d-b}{c-0}$ Gradient BA $= \dfrac{0-b}{a-0}$

Lines perpendicular $\dfrac{d-b}{c} \times \dfrac{-b}{a} = -1$

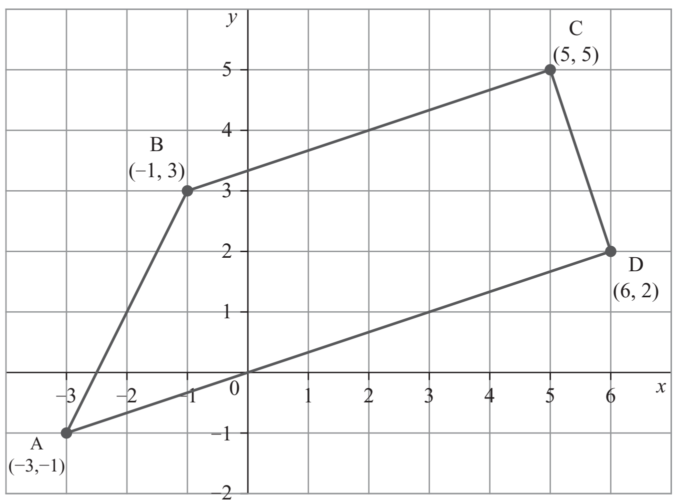
$$\Rightarrow \frac{d-b}{c} = \frac{a}{b}$$

$$\Rightarrow bd - b^2 = ac$$

$$b^2 + ac - bd = 0$$

[4 marks]

16



From the graph: Gradient BC $= \dfrac{5-3}{5-(-1)} = \dfrac{2}{6} = \dfrac{1}{3}$

Gradient AD $= \dfrac{2-(-1)}{6-(-1)} = \dfrac{3}{9} = \dfrac{1}{3}$

BC // AD

Gradient CD $= \dfrac{2-5}{6-5} = -3$

Gradient AD × Gradient CD $= \dfrac{1}{3} \times -3$

$= -1 \Rightarrow$ AD ⊥ CD

Figure is a trapezium.

Length BC $= \sqrt{(5-(-1))^2 + (5-3)^2} = 2\sqrt{10}$

Length AD $= \sqrt{(6-(-3))^2 + (2-(-1))^2} = 3\sqrt{10}$

Length CD $= \sqrt{(6-5)^2 + (2-5)^2} = \sqrt{10}$

Area $= \dfrac{1}{2} \times$ sum of parallel sides × distance between

$= \dfrac{1}{2} \times \left(2\sqrt{10} + 3\sqrt{10}\right) \times \sqrt{10}$

$= 25$ units^2

[8 marks]

17 (i) AB $= \sqrt{(1-8)^2 + (4-7)^2} = \sqrt{58}$, BC $= \sqrt{(8--2)^2 + (7-11)^2} = \sqrt{116}$

AC $= \sqrt{(1--2)^2 + (4-11)^2} = \sqrt{58}$ so AB = AC so the triangle is isosceles

AB^2 + AC^2 $= 58 + 58 = 116 =$ BC^2 so the triangle is right-angled at A

[5 marks]

(ii) Area $= \dfrac{1}{2} \times \sqrt{58} \times \sqrt{58} = 29$

[1 mark]

[Total: 6 marks]

Exercise 5.2 Equations of straight lines

1 **(i)**

$$\left.\begin{array}{ll} x = 3 & \text{(parallel to } y\text{-axis)} \\ y = 3 & \text{(parallel to } x\text{-axis)} \end{array}\right\} \Rightarrow \text{perpendicular}$$

[2 marks]

(ii)

$$\left.\begin{array}{ll} \text{Gradient } y = 3x + 2 & \text{is } 3 \\ \text{Gradient } y = 3x - 2 & \text{is } 3 \end{array}\right\} \Rightarrow \text{parallel}$$

[2 marks]

(iii)

$$\left.\begin{array}{ll} \text{Gradient } y = 4 - 2x & \text{is } -2 \\ \text{Gradient } y = 4 + 2x & \text{is } +2 \end{array}\right\} \Rightarrow \text{neither}$$

[2 marks]

(iv)

$$\left.\begin{array}{ll} \text{Gradient } x + 2y = 4 & \text{is } -\dfrac{1}{2} \\ \text{Gradient } x - 2y = 4 & \text{is } +\dfrac{1}{2} \end{array}\right\} \Rightarrow \text{neither}$$

[2 marks]

(v)

$$\left.\begin{array}{ll} \text{Gradient } x - 2y = 3 & \text{is } +\dfrac{1}{2} \\ \text{Gradient } y - 2x = 3 & \text{is } +2 \end{array}\right\} \Rightarrow \text{neither}$$

[2 marks]

(vi)

$$\left.\begin{array}{ll} \text{Gradient } 3x - 2y + 5 = 0, \text{ i.e. } y = \dfrac{3x}{2} + \dfrac{5}{2} & \text{is } +\dfrac{3}{2} \\ \text{Gradient } 3x - 2y - 7 = 0, \text{ i.e. } y = \dfrac{3x}{2} - \dfrac{7}{2} & \text{is } +\dfrac{3}{2} \end{array}\right\} \Rightarrow \text{parallel}$$

[2 marks]

(vii)

$$\left.\begin{array}{ll} \text{Gradient } x = 2y & \text{is } +\dfrac{1}{2} \\ \text{Gradient } 2x + y = 0 & \text{is } -2 \end{array}\right\} \Rightarrow \text{perpendicular}$$

[2 marks]

(viii)

$$\left.\begin{array}{ll} \text{Gradient } 2x + 5y = 7, \text{ i.e. } y = -\dfrac{2x}{5} + \dfrac{7}{5} & \text{is } -\dfrac{2}{5} \\ \text{Gradient } 5x + 2y = 7, \text{ i.e. } y = -\dfrac{5x}{2} + \dfrac{7}{2} & \text{is } -\dfrac{5}{2} \end{array}\right\} \Rightarrow \text{neither}$$

[2 marks]

[Total: 16 marks]

2 **(i)** Parallel to $y = 4x \Rightarrow$ gradient $= 4$

Through $(3, -1)$

$y - (-1) = -4(x - 3)$

$\Rightarrow y + 1 = -4x + 12$

$\Rightarrow 4x + y - 11 = 0$ [2 marks]

(ii) Parallel to $y = 5x + 3 \Rightarrow$ gradient $= 5$

Through $(1, -2)$

$y - (-2) = 5(x - 1)$

$\Rightarrow y + 2 = 5x - 5$

$\Rightarrow y = 5x - 7$ [2 marks]

(iii) Parallel to $y = 2 - 4x \Rightarrow$ gradient $= -4$

Through $(5, -3)$

$y - (-3) = -4(x - 5)$

$\Rightarrow y + 3 = -4x + 20$

$\Rightarrow 4x + y - 17 = 0$ [2 marks]

(iv) Parallel to $2x - 3y = 5$, i.e. $y = \frac{2x}{3} - \frac{5}{3} \Rightarrow$ gradient $= \frac{2}{3}$

Through (2, 6)

$\Rightarrow y - 6 = \frac{2}{3}(x - 2)$

$\Rightarrow 3y - 18 = 2x - 4$

$\Rightarrow 2x - 3y + 14 = 0 \quad \left(\text{or } y = \frac{2}{3}x + 4\frac{2}{3}\right)$ [2 marks]

(v) Parallel to $x - 2y + 4 = 0$, i.e. $y = \frac{1}{2}x + 2 \Rightarrow$ gradient $= \frac{1}{2}$

Through $(-3, -2)$

$\Rightarrow y - (-2) = \frac{1}{2}(x - (-3))$

$\Rightarrow 2y + 4 = x + 3$

$\Rightarrow x - 2y - 1 = 0 \quad \left(\text{or } y = \frac{1}{2}x - \frac{1}{2}\right)$ [2 marks]

(vi) Parallel to $2x + 3y - 6$, i.e. $y = -\frac{2}{3}x + 2 \Rightarrow$ gradient $= -\frac{2}{3}$

Through $(5, -1)$

$\Rightarrow y + 1 = -\frac{2}{3}(x - 5)$

$\Rightarrow 3y + 3 = -2x + 10$

$\Rightarrow 2x + 3y - 7 = 0 \quad \left(\text{or } y = -\frac{2}{3}x + 2\frac{1}{3}\right)$ [2 marks]

[Total: 12 marks]

3 (i) Perpendicular to $y = 3x \Rightarrow$ gradient $= -\frac{1}{3}$

Through (0, 2)

$\Rightarrow y - 2 = -\frac{1}{3}(x - 0)$

$\Rightarrow 3y - 6 = -x$

$\Rightarrow x + 3y - 6 = 0 \quad \left(\text{or } y = -\frac{1}{3}x + 2\right)$ [2 marks]

(ii) Perpendicular to $y = 2x - 3 \Rightarrow$ gradient $= -\frac{1}{2}$

Through $(3, -1)$

$\Rightarrow y - (-1) = -\frac{1}{2}(x - 3)$

$\Rightarrow y + 1 = -\frac{x}{2} + 1\frac{1}{2}$

$\Rightarrow 2y + x - 1 = 0 \quad \left(\text{or } y = -\frac{1}{2}x + \frac{1}{2}\right)$ [2 marks]

(iii) Perpendicular to $y + x = 2$ (i.e. $y = -x + 2$) $\Rightarrow$ gradient $= 1$

Through $(-2, 4)$

$\Rightarrow y - 4 = 1(x - (-2))$

$\Rightarrow y = x + 6$ [2 marks]

(iv) Perpendicular to $2x + 3y = 5$ $\left(\text{i.e. } y = -\frac{2}{3}x + \frac{5}{3}\right) \Rightarrow$ gradient $= +\frac{3}{2}$

Through (0, 0)

$\Rightarrow y = \frac{3x}{2}$ [2 marks]

(v) Perpendicular to $3x - 2y = 1$ $\left(\text{i.e. } y = \frac{3}{2}x - \frac{1}{2}\right) \Rightarrow$ gradient $= -\frac{2}{3}$
Through $(-3, 2)$

$\Rightarrow y - 2 = -\frac{2}{3}\,(x - (-3))$

$\Rightarrow 3y - 6 = -2x - 6$

$\Rightarrow 2x + 3y = 0 \quad \left(\text{or } y = -\frac{2}{3}x\right)$ [2 marks]

(vi) Perpendicular to $4x - 3y = 0$ $\left(\text{i.e. } y = \frac{4}{3}x\right) \Rightarrow$ gradient $= -\frac{3}{4}$
Through $(4, -1)$

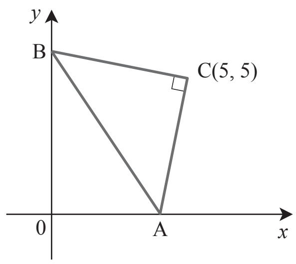
$\Rightarrow y - (-1) = -\frac{3}{4}\,(x - 4)$

$\Rightarrow 4y + 4 = -3x + 12$

$\Rightarrow 3x + 4y - 8 = 0 \quad \left(\text{or } y = -\frac{3}{4}x + 2\right)$ [2 marks]

[Total: 12 marks]

4 **(i)**



[2 marks]

(ii) AB: $3x + 2y - 12 = 0$

$x = 0 \Rightarrow y = 6 \Rightarrow$ B is $(0, 6)$

$y = 0 \Rightarrow x = 4 \Rightarrow$ A is $(4, 0)$ [2 marks]

(iii) Length AC $= \sqrt{(5-4)^2 + (5-0)^2} = \sqrt{26}$

Length BC $= \sqrt{(5-6)^2 + (5-0)^2} = \sqrt{26}$

Area ABC $= \frac{1}{2}\sqrt{26}\sqrt{26}$

$= 13$ units^2 [2 marks]

[Total: 6 marks]

5 **(i)** At A, $y = 0 \Rightarrow 3x + 12 = 0$

$\Rightarrow x = -4$

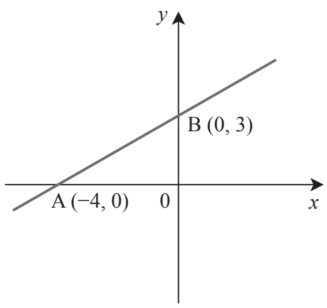
$\Rightarrow$ A is the point $(-4, 0)$

At B, $x = 0 \Rightarrow -4y + 12 = 0$

$\Rightarrow y = 3$

$\Rightarrow$ B is the point $(0, 3)$ [1 mark]

(ii)



[2 marks]

(iii) Area of triangle AOB = $\frac{1}{2} \times 3 \times 4 = 6$ units^2 [1 mark]

(iv) Gradient of AB = $\frac{3}{4} \Rightarrow$ gradient of the line perpendicular to AB = $-\frac{4}{3}$

Equation of the line through O with gradient $-\frac{4}{3}$ is $y = -\frac{4}{3}x$

$\Rightarrow 4x + 3y = 0$ [2 marks]

(v) Length of AB = $\sqrt{4^2 + 3^2} = 5$

Let OD be the shortest (i.e. perpendicular) distance from O to AB

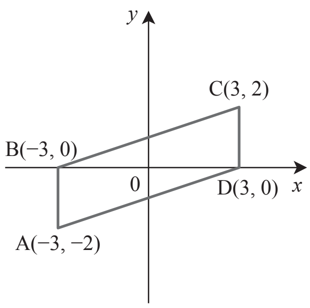
Area of triangle AOB = 6 (from (iii)) and also = $\frac{1}{2} \times 5 \times \text{OD}$

$\Rightarrow 6 = 2.5 \times \text{OD}$

$\Rightarrow \text{OD} = 2.4$ units [2 marks]

[Total: 8 marks]

6 **(i)**



[1 mark]

(ii) AB and DC are vertical so they are parallel and their gradient is undefined.

Gradient AD = $\frac{0-(-2)}{3-(-3)} = \frac{1}{3}$

Gradient BC = $\frac{2-0}{3-(-3)} = \frac{1}{3}$ so AD and BC are parallel.

Opposite sides are parallel so ABCD is a parallelogram. [2 marks]

(iii) Area of the quadrilateral = 2 × Area BDC

$= 2 \times \frac{1}{2} \times 6 \times 2$

$= 12$ units^2 [1 mark]

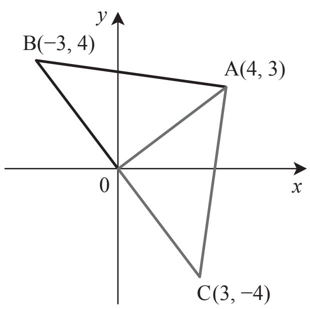
(iv) Gradient of AC = $\frac{2-(-2)}{3-(-3)} = \frac{4}{6} = \frac{2}{3}$

Diagonal BD is horizontal ⇒ angle between the diagonals = $\tan^{-1}\frac{2}{3}$

= 33.7° (1 d.p.) [2 marks]

[Total: 6 marks]

7 (i)



[1 mark]

(ii) Length of OA = $\sqrt{4^2+3^2} = 5$

OA = OB ⇒ length of OB = 5

Right angle at O ⇒ AB = $\sqrt{5^2+5^2} = 5\sqrt{2}$ [2 marks]

(iii) Gradient of OA = $\frac{3}{4}$ ⇒ equation of OA is $y = \frac{3}{4}x$

OB is perpendicular to OA ⇒ gradient of OB = $-\frac{4}{3}$

Equation of OB is $y = -\frac{4}{3}x$

OA = OB so coordinates (x, y) of B satisfy $x^2 + y^2 = 5^2$

$\Rightarrow x^2 + \left(-\frac{4}{3}x\right)^2 = 25$

$\Rightarrow \frac{25}{9}x^2 = 25$

$x = \pm 3$

B has a negative x co-ordinate so B is the point when $x = -3$

$y = -\frac{4}{3}x = +4$ so B is (−3, 4)

AB is the line through A(4, 3) and B(−3, 4)

Gradient = $\frac{4-3}{(-3)-4} = -\frac{1}{7}$

Through A so equation is $y - 3 = -\frac{1}{7}(x-4)$

$\Rightarrow 7y - 21 = -x + 4$

$\Rightarrow x + 7y = 25$ [7 marks]

(iv) B is 3 units to the left of O and 4 units above O

⇒ C is 3 units to the right of O and 4 units below O

⇒ C is the point (3, −4) [2 marks]

(v) AO is the line of symmetry for the triangle, so triangle ABC is isosceles.

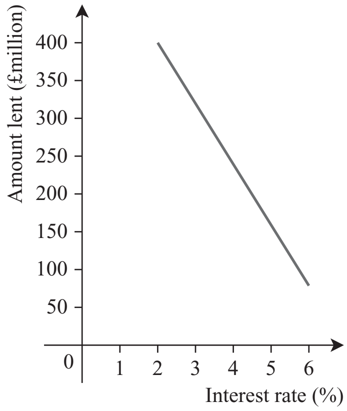
Length of OB $= \sqrt{4^2 + 3^2} = 5$

Length of BC $= 2 \times$ OB $= 10$

Area of triangle ABC $= \frac{1}{2} \times 10 \times 5 = 25$ units^2 [2 marks]

[Total: 14 marks]

8 (i)



[1 mark]

(ii) The line passes through the points (4, 240) and (2.4, 400)

Gradient $= \dfrac{400-240}{2.4-4} = -100$

Using the point (4, 240) the equation of the line is $y - 240 = -100(x - 4)$

$\Rightarrow y - 240 = -100x + 400$

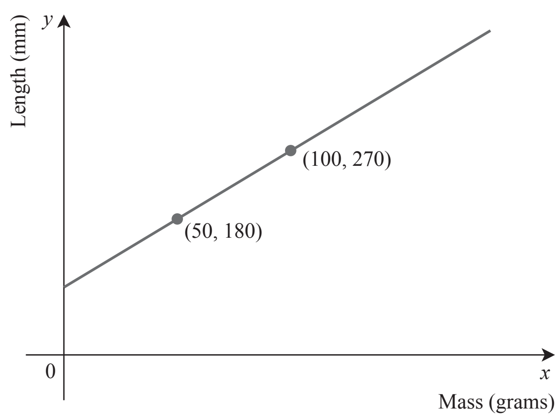
$\Rightarrow y = 640 - 100x$ [3 marks]

(iii) (a) When $x = 2$, $y = 440$ so the amount lent is £440 million.

(b) When $x = 5$, $y = 140$ so the amount lent is £140 million. [2 marks]

[Total: 6 marks]

9 (i)



[1 mark]

(ii) The line passes through the points (50, 180) and (100, 270) so the gradient is $\dfrac{270-180}{100-50} = 1.8$

Using the point (50, 180) the equation is $y - 180 = 1.8(x - 50)$

$\Rightarrow y - 180 = 1.8x - 90$

$\Rightarrow y = 1.8x + 90$ [2 marks]

(iii) The unstretched length is when $x = 0$

$\Rightarrow$ unstretched length is 90 mm. [1 mark]

(iv) When the extension is 150 mm, the stretched length is 150 + 90 = 240 mm

$y = 240 \Rightarrow 240 = 1.8x + 90$

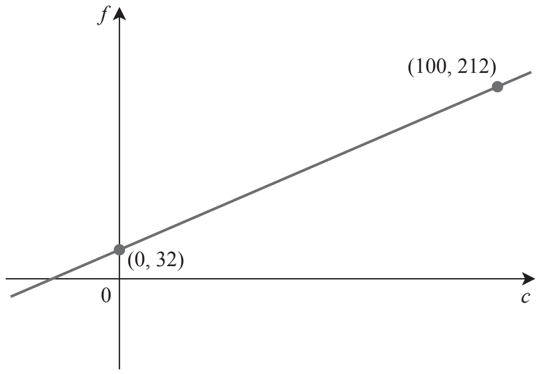
$\Rightarrow 1.8x = 150$

$\Rightarrow x = 83\frac{1}{3}$ so the load is $83\frac{1}{3}$ g [2 marks]

(v) Adding a load of 1 kg is likely to cause the elastic band to snap. [1 mark]

[Total: 7 marks]

10 (i) First draw the straight line joining (0, 32) to (100, 212)



[1 mark]

(ii) The gradient of the line is $\frac{212-32}{100-0} = 1.8$

Using the point (0, 32) the equation is $f - 32 = 1.8(c - 0)$

$\Rightarrow f = 1.8c + 32$ [2 marks]

(iii) Let t be the temperature that has the same numerical value in both scales.

This gives $t = 1.8t + 32$

$\Rightarrow -0.8t = -32$

$\Rightarrow t = \frac{32}{-0.8} = -40$

i.e. –40°C = –40°F [2 marks]

[Total: 5 marks]

11 (i) $2(1-3y)+y=3 \quad\Rightarrow\quad -1=5y \quad\Rightarrow\quad y=-\frac{1}{5} \quad \text{so} \quad 2x-\frac{1}{5}=3 \quad\Rightarrow\quad x=\frac{8}{5}$

$2(4-3y)+y=7 \quad\Rightarrow\quad 1=5y \quad\Rightarrow\quad y=\frac{1}{5} \quad \text{so} \quad 2x+\frac{1}{5}=7 \quad\Rightarrow\quad x=\frac{17}{5}$

Gradient is $\frac{0.2--0.2}{3.4-1.6}=\frac{2}{9}$ so an equation is $y-\frac{1}{5}=\frac{2}{9}\left(x-\frac{17}{5}\right) \quad\Rightarrow\quad 45y-9=10x-34$

$\Rightarrow \quad 2x-9y=5$

$2(1-3y)+y=7 \quad\Rightarrow\quad -5=5y \quad\Rightarrow\quad y=-1 \quad \text{so} \quad 2x-1=7 \quad\Rightarrow\quad x=4$

$2(4-3y)+y=3 \quad\Rightarrow\quad 5=5y \quad\Rightarrow\quad y=1 \quad \text{so} \quad 2x+1=3 \quad\Rightarrow\quad x=1$

Gradient is $\frac{1--1}{1-4}=-\frac{2}{3}$ so an equation is $y-1=-\frac{2}{3}(x-1) \quad\Rightarrow\quad 3y-3=-2x+2$

$\Rightarrow \quad 2x+3y=5$ [7 marks]

(ii) The gradient product is $\frac{2}{9}\times-\frac{2}{3}=-\frac{-4}{27}\neq -1$ so the diagonals are not perpendicular

So the shape is not a rhombus. [1 mark]

[Total: 8 marks]

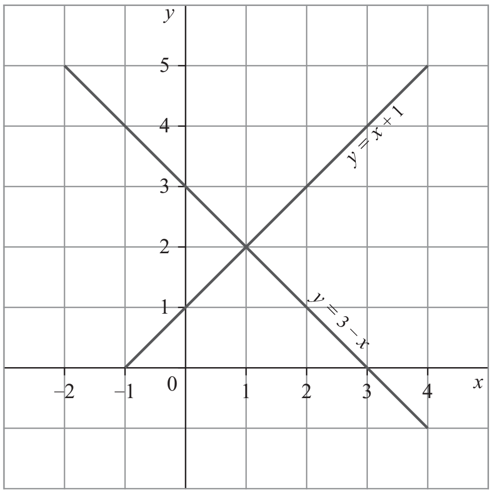
Exercise 5.3 The intersection of two lines

1 (i)

x	–1	0	1	2	3	4
$y = x + 1$	0	1	2	3	4	5
$y = 3 - x$	4	3	2	1	0	–1

$x + y = 3$

$\Rightarrow y = 3 - x$



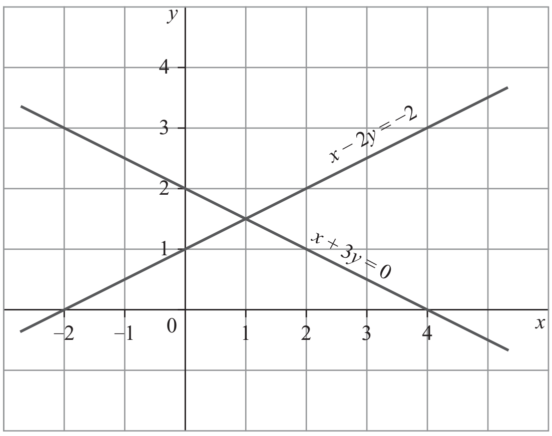
sol^{n} $x = 1, y = 2$ [4 marks]

(ii)

x	–2	–1	0	1	2	3	4
$y = 2 - \frac{x}{2}$	3		2		1		0
$y = \frac{x}{2} + 1$	0		1		2		3

$x + 2y = 4 \Rightarrow y = 2 - \frac{x}{2}$

$x - 2y = -2 \Rightarrow y = \frac{x}{2} + 1$



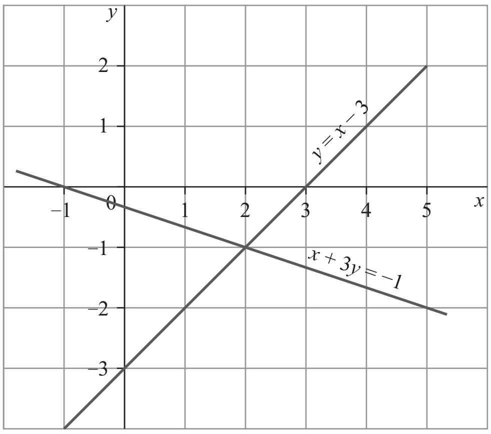
sol^{n} $x = 1, y = 1.5$ [4 marks]

[Total: 8 marks]

2 (i) $x + 3y = -1$

$\Rightarrow y = -\frac{1}{3}x - \frac{1}{3}$

x	–1	0	1	2	3	4	5
$y = x - 3$	–4	–3	–2	–1	0	1	2
$y = -\frac{1}{3}x - \frac{1}{3}$	0			–1			–2

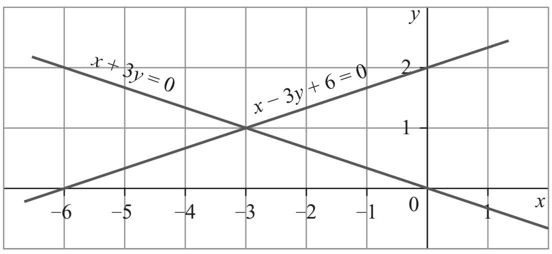


sol^{n} $x = 2, y = -1$ [4 marks]

(ii) $x + 3y = 0 \Rightarrow y = -\frac{x}{3}$

$x - 3y + 6 = 0 \Rightarrow y = \frac{x}{3} + 2$

x	–6	–5	–4	–3	–2	–1	0	1
$y = -\frac{x}{3}$	2			1			0	
$y = \frac{x}{3} + 2$	0			1			2	

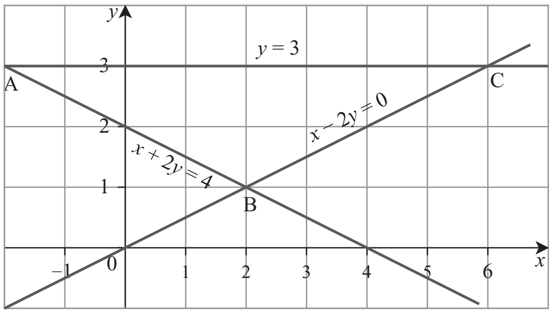


sol^{n} $x = -3, y = 1$ [4 marks]

[Total: 8 marks]

3 (i)

x	–2	–1	0	1	2	3	4	5	6
$x - 2y = 0 \Rightarrow y = \frac{x}{2}$	–1		0		1		2		3
$x + 2y = 4 \Rightarrow y = -\frac{x}{2}$	3		2		1		0		–1



[5 marks]

(ii) $\left.\begin{aligned} y &= 3 \\ x + 2y &= 4 \end{aligned}\right\} \Rightarrow (-2, 3)$

$\left.\begin{aligned} y &= 3 \\ x - 2y &= 0 \end{aligned}\right\} \Rightarrow (6, 3)$

$\left.\begin{aligned} x + 2y &= 4 \\ x - 2y &= 0 \end{aligned}\right\} \Rightarrow (2, 1)$

Points of intersection (–2, 3) (2, 1) (6, 3) [3 marks]

(iii) Area = $\frac{1}{2}$ base × height

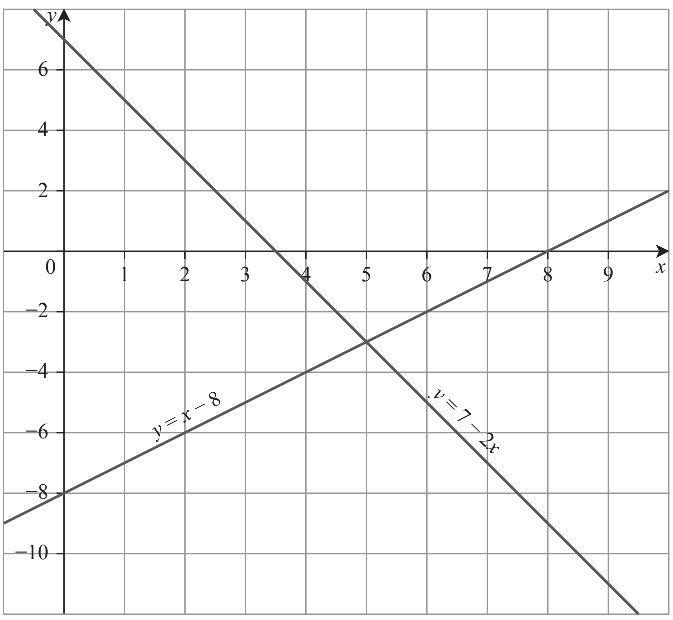
Take AC as base ⇒ Area = $\frac{1}{2}(8)(2) = 8$ units^2 [2 marks]

(iv) From the graph AB = BC ⇒ triangle, isosceles [1 mark]

[Total: 11 marks]

4 (i)

x	0	1	2	3	4	5	6	7	8	9
$y = x - 8$	–8			–5			–2			1
$y = 7 - 2x$	7			1			–5			–11



[1 mark]

(ii) Area between lines and x-axis is a triangle.

For $y = 7 - 2x$, when $y = 0$, $x = 3.5$

Base = 4.5 Height = 3 Area = $\frac{1}{2} \times 4.5 \times 3$

$= 6.75$ units^2 [2 marks]

(iii) Area between lines and y-axis is a triangle

Base = 15 Height = 5 Area = $\frac{1}{2} \times 15 \times 5$

$= 37.5$ units^2 [2 marks]

[Total: 5 marks]

5 **(i)** A(–1, 2), B(3, 4), C(4, 2)

$AB = \sqrt{(4-2)^2 + (3-(-1))^2} = 2\sqrt{5}$

$BC = \sqrt{(2-4)^2 + (4-3)^2} = \sqrt{5}$

$AC = \sqrt{(2-2)^2 + (4-(-1))^2} = 5$ [2 marks]

(ii) Gradient AB $= \frac{4-2}{3-(-1)} = 0.5$

Gradient BC $= \frac{2-4}{4-3} = -2$

Gradient AC $= \frac{2-2}{4-(-1)} = 0$ [2 marks]

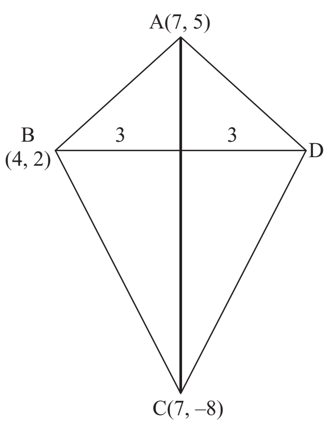
(iii) Gradient AB × Gradient BC = –1 ⇒ Triangle right angled [1 mark]

(iv) Right angle at B ⇒ Area = $\frac{1}{2}$AB × BC

$= \frac{1}{2}\left(2\sqrt{5}\right)\left(\sqrt{5}\right) = 5$ units^2 [1 mark]

[Total: 6 marks]

6 **(i)**



D is (10, 2) [2 marks]

(ii) Gradient AB $= \frac{5-2}{7-4} = 1$ Gradient AD $= \frac{5-2}{7-10} = -1$

AB and AD are perpendicular

(Since it is a kite, only one pair of sides can be perpendicular.) [2 marks]

(iii) Area ABD $= \frac{1}{2} \times 6 \times 3 = 9$

Area BCD $= \frac{1}{2} \times 6 \times 10 = 30$

Total area = 39 units^2

$= 39 \times 12^2\,\text{cm}^2 = 5616\,\text{cm}^2$ [2 marks]

[Total: 6 marks]

7 Let Amanda's wage be £x per day $\Rightarrow$ Belinda's wage is £$(x + 2)$ per day

$14x + 10(x + 2) = 224$

$\Rightarrow 24x + 20 = 224$

$\Rightarrow 24x = 204$

$\Rightarrow x = 8.5$

Amanda earns £8.50 per day and Belinda earns £10.50 per day. [3 marks]

8 $\left.\begin{array}{l}\text{Let } x = \text{no. of 10p coins} \\ \quad\;\; y = \text{no. of 20p coins}\end{array}\right\} \Rightarrow x + y = 48$ (number of coins) ①

Amount of money (pence): $10x + 20y = 780$

$\Rightarrow x + 2y = 78$ ②

② – ① $\Rightarrow y = 30$

Sub in ① $\Rightarrow x = 18$

i.e. 18 10p coins and 30 20p coins. [3 marks]

9 Let man be m years now and the grandson g years now.

$m + g = 80$ ①

10 years later $(m + 10) = 4(g + 10) \Rightarrow m = 4g + 30$ ②

Sub from ② into ① $(4g + 30) + g = 80$

$\Rightarrow g = 10$

$m + g = 80 \quad \Rightarrow m = 70$

Man is 70 years old and grandson is 10 years old. [4 marks]

10 (i) Let a pork chop cost £p and a lamb chop cost £l

$4p + 2l = 12$ ①

$6p + 3l = 18$ ②

From ① $2p + l = 6 \Rightarrow l = 6 - 2p$

Sub into ② $6p + 3(6 - 2p) = 18$

$\Rightarrow 0 = 0$

Individual prices cannot be found from this information, since there are not two 'different' equations – the second is 1.5 times the first. They represent lines that are parallel. [4 marks]

(ii) (a) $\left.\begin{array}{l} 2x + y + 3z = 19 \text{ ①} \\ x + y + 4z = 17 \text{ ②}\end{array}\right\}$ subtract $\Rightarrow x - z = 2$

① $x = z + 2$

① – 2 ② $\Rightarrow 2x + y + 3z - 2(x + y + 4z) = 19 - 34$

$\Rightarrow -y - 5z = -15$

$\Rightarrow y = 15 - 5z$ [3 marks]

(b) Sub into ② $(z + 2) + (15 - 5z) + 4z = 17$

$\Rightarrow 0 = 0$

Sub into ① $2(z + 2) + (15 - 5z) + 3z = 19$

$\Rightarrow 0 = 0$ [2 marks]

(c) Each equation will represent a plane and the two planes have no common point – they are parallel. [1 mark]

[Total: 10 marks]

11 (i) $3y + x = -5 \quad \Rightarrow \quad y = -\frac{1}{3}(x+5)$

So the gradients of the lines are $3,\ -\frac{1}{3},\ 1$ respectively [2 marks]

(ii) $3x + 5 = x + 6 \quad \Rightarrow \quad 2x = 1 \quad \Rightarrow \quad x = \frac{1}{2}$ so $y = \frac{1}{2} + 6 = \frac{13}{2}$

So the intersection of $y = 3x + 5$ and $y = x + 6$ is $\left(\frac{1}{2}, \frac{13}{2}\right)$

$3(x+6) + x = -5 \quad \Rightarrow \quad 4x = -23 \quad \Rightarrow \quad x = -\frac{23}{4}$ so $y = -\frac{23}{4} + 6 = \frac{1}{4}$

So the intersection of $3y + x = -5$ and $y = x + 6$ is $\left(-\frac{23}{4}, \frac{1}{4}\right)$

$3(3x + 5) + x = -5 \quad \Rightarrow \quad 10x = -20 \quad \Rightarrow \quad x = -2$ so $y = 3(-2) + 5 = -1$

So the intersection of $3y + x = -5$ and $y = 3x + 5$ is $(-2, -1)$ [4 marks]

(iii) $\sqrt{\left(\frac{1}{2} - -\frac{23}{4}\right)^2 + \left(\frac{13}{2} - \frac{1}{4}\right)^2} = \frac{25}{4}\sqrt{2}, \quad \sqrt{\left(\frac{1}{2} - -2\right)^2 + \left(\frac{13}{2} - -1\right)^2} = \frac{5}{2}\sqrt{10}$

$\sqrt{\left(-\frac{23}{4} - -2\right)^2 + \left(\frac{1}{4} - -1\right)^2} = \frac{5}{4}\sqrt{10}$ [2 marks]

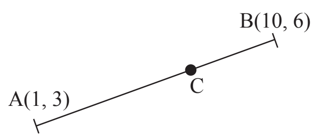
(iv) $y = 3x + 5$ and $3y + x = -5$ are perpendicular to each other

So area is $\frac{1}{2} \times \frac{5}{2}\sqrt{10} \times \frac{5}{4}\sqrt{10} = \frac{125}{8}$ [2 marks]

[Total: 10 marks]

Exercise 5.4 Dividing of a line in a given ratio

1 (i)

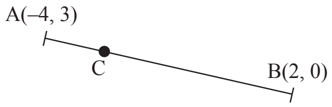


AC : CB = 2 : 1 (3 parts)

$\Rightarrow$ C is $(1 + \frac{2}{3}(10 - 1), 3 + \frac{2}{3}(6 - 3))$

$\Rightarrow$ C is (7, 5) [3 marks]

(ii)

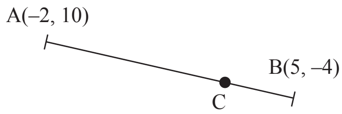


AC : CB = 1 : 2 (3 parts)

$\Rightarrow$ C is $\left(-4 + \frac{1}{3}(2-(-4)), 3 + \frac{1}{3}(0-3)\right)$

$\Rightarrow$ C is (–2, 2) [3 marks]

(iii)

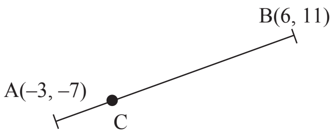


AC : CB = 5 : 2 (7 parts)

$\Rightarrow$ C is $\left(-2 + \frac{5}{7}(5-(-2)), 10 + \frac{5}{7}(-4-10)\right)$

$\Rightarrow$ C is (3, 0) [3 marks]

(iv)



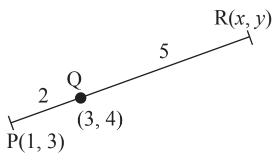
AC : CB = 2 : 7 (9 parts)

$\Rightarrow$ C is $\left(-3 + \frac{2}{9}(6-(-3)), -7 + \frac{2}{9}(11-(-7))\right)$

$\Rightarrow$ C is (–1, –3) [3 marks]

[Total: 12 marks]

2 **(i)**

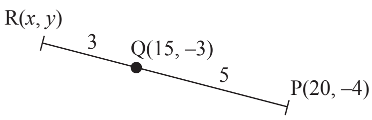


$x = \frac{(5+2)}{2}(3-1) + 1 = 8$

$y = \frac{(5+2)}{2}(4-3) + 3 = 6.5$

R is (8, 6.5) [3 marks]

(ii)

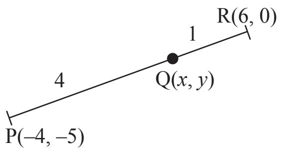


$x = -\frac{(5+3)}{5}(20-15) + 20 = 12$

$y = -\frac{(5+3)}{5}(-4-(-3)) + (-4) = -2.4$

R is (12, –2.4) [3 marks]

(iii)

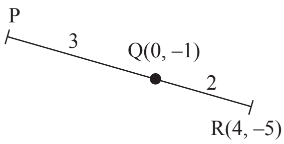


PQ : QR = 4 : 1 (5 parts)

Q is $\left(-4+\frac{4}{5}(6-(-4)), -5+\frac{4}{5}(0-(-5))\right)$

Q is (4, –1) [3 marks]

(iv)



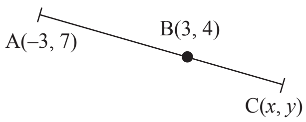
PQ : QR = 3 : 2 (5 parts)

P is $(0-\frac{3}{2}(4-0), -1-\frac{3}{2}(-5+1))$

P is (–6, 5) [3 marks]

[Total: 12 marks]

3 **(i)**



[1 mark]

(ii) 50% longer ⇒ AB = $1\frac{1}{2}$ × BC

⇒ AB : BC = 3 : 2 [2 marks]

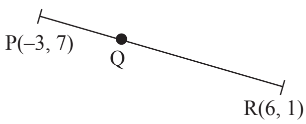
(iii) C is (x, y) where $x = \frac{(2+3)}{3}(3-(-3)) + (-3) = 7$

$y = \frac{(2+3)}{3}(4-7) + 7 = 2$

C is (7, 2) [2 marks]

[Total: 5 marks]

4 **(i)** PR = 3PQ ⇒ PQ : QR = 1 : 2



[2 marks]

(ii) Q is $\left(-3+\frac{1}{3}(6-(-3)), 7+\frac{1}{3}(1-7)\right)$

⇒ Q is (0, 5) [2 marks]

[Total: 4 marks]

5 **(i)** Circumference : diameter $= 2\pi r : 2r$
$= \pi : 1$ [1 mark]

(ii) Square of side x; perimeter : length of side $= 4x : x$
$= 4 : 1$ [1 mark]

(iii) Area of circle with $r = 5\,\text{cm} = \pi r^2 = 25\pi\,\text{cm}^2$
Area of circle with $r = 6\,\text{cm} = \pi r^2 = 36\pi\,\text{cm}^2$
Ratio $= 25 : 36$ [1 mark]

(iv) Area of square with side $5\,\text{cm} = 25\,\text{cm}^2$
$6\,\text{cm} = 36\,\text{cm}^2$
Ratio $= 25 : 36$ [1 mark]

[Total: 4 marks]

6 $22 + 26 + 27 = 75$

Anna receives $\frac{22}{75} \times £150\,000 = £44\,000$

Brian receives $\frac{26}{75} \times £150\,000 = £52\,000$

Charlotte receives $\frac{27}{75} \times £150\,000 = £54\,000$ [2 marks]

7 $4 + 6 + 8 = 18 \quad 360° \div 18 = 20°$
$\Rightarrow$ Age 4 receives $4 \times 20° = 80°$
Age 6 receives $6 \times 20° = 120°$
Age 8 receives $8 \times 20° = 160°$ [2 marks]

8 Materials : wages : admin costs $= 3 : 6 : 1$

(i) House selling for £250 000

$\Rightarrow$ Materials $= \frac{3}{10} \times £250\,000 = £75\,000$

Wages $= \frac{6}{10} \times £250\,000 = £150\,000$

Admin $= \frac{1}{10} \times £250\,000 = £25\,000$ [2 marks]

(ii) New materials cost $= \frac{105}{100} \times £75\,000 = £78\,750$

New wages cost $= \frac{104}{100} \times £150\,000 = £156\,000$

New admin cost $= \frac{106}{100} \times £25\,000 = £26\,500$

Total new cost $= £78\,750 + £156\,000 + 26\,500$
$= £261\,250$

Increase $= £11\,250 \Rightarrow$ % increase $= \frac{11\,250}{250\,000} \times 100\%$
$= 4.5\%$ [4 marks]

[Total: 6 marks]

9 $\dfrac{10-3}{p-3}=\dfrac{5}{2} \Rightarrow 14=5(p-3) \Rightarrow p=\dfrac{29}{5}=5.8$

$\dfrac{q-4}{7-4}=\dfrac{5}{2} \Rightarrow 2(q-4)=15 \Rightarrow q=\dfrac{23}{2}=11.5$ [4 marks]

Exercise 5.5 Equation of a circle

1 **(i)** Centre (0, 1), radius 3, equation $x^2+(y-1)^2=9$ [1 mark]

(ii) Centre (3, 0), radius 5, equation $(x-3)^2+y^2=25$ [1 mark]

(iii) Centre (–2, 5), radius 2, equation $(x+2)^2+(y-5)^2=4$ [1 mark]

(iv) Centre (4, –3), radius 3, equation $(x-4)^2+(y+3)^2=9$ [1 mark]

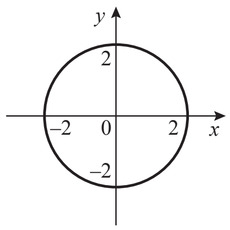
(v) Centre (–6, –2), radius 4, equation $(x+6)^2+(y+2)^2=16$ [1 mark]

[Total: 5 marks]

2 **(i)** **(a)** Centre (0, 0) [1 mark]

(b) Radius = 2 [1 mark]

(c)

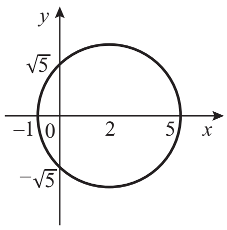


[1 mark]

(ii) **(a)** Centre (2, 0) [1 mark]

(b) Radius = 3 [1 mark]

(c)

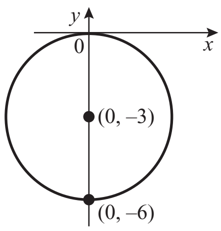


[1 mark]

(iii) **(a)** Centre (0, −3) [1 mark]

(b) Radius = 3 [1 mark]

(c)

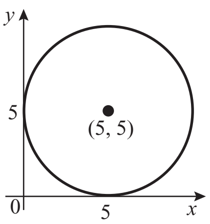


[1 mark]

(iv) (a) Centre (5, 5) [1 mark]

(b) Radius 5 [1 mark]

(c)

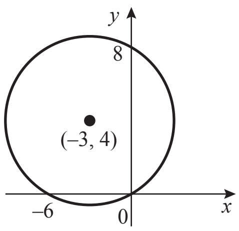


[1 mark]

(v) (a) Centre (−3, 4) [1 mark]

(b) Radius 5 [1 mark]

(c)



[1 mark]

[Total: 15 marks]

3 (i) $(x-2)^2 + (y-3)^2 = r^2$

Through (0, 3) $\Rightarrow (-2)^2 + 0^2 = r^2$

$\Rightarrow r = 2$

Eq^n $(x-2)^2 + (y-3)^2 = 4$ [2 marks]

(ii) $(x+1)^2 + (y-2)^2 = r^2$

Through (2, −1) $\Rightarrow 3^2 + (-3)^2 = r^2$

$\Rightarrow r = 3\sqrt{2}$

Eq^n $(x+1)^2 + (y-2)^2 = 18$ [2 marks]

(iii) $(x-1)^2 + (y+1)^2 = r^2$

Through (1, 2) $\Rightarrow 0^2 + 3^2 = r^2$

$\Rightarrow r = 3$

Eq^n $(x-1)^2 + (y+1)^2 = 9$ [2 marks]

[Total: 6 marks]

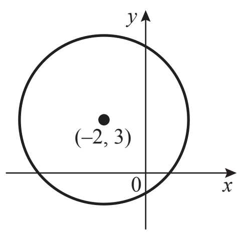
4 $x^2 + y^2 + 4x - 6y - 12 = 0$

$\Rightarrow x^2 + 4x + y^2 - 6y = 12$

$\Rightarrow (x+2)^2 + (y-3)^2 = 12 + 4 + 9$

$\Rightarrow (x+2)^2 + (y-3)^2 = 25$

Centre (−2, 3), radius 5

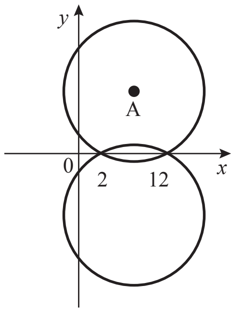


[4 marks]

5 $x^2 + y^2 + 4x - 6y + 19 = 0$
$\Rightarrow x^2 + 4x + y^2 - 6y = -19$
$\Rightarrow (x + 2)^2 + (y - 3)^2 = -19 + 4 + 9$
$\Rightarrow (x + 2)^2 + (y - 3)^2 = -6$
For a circle, RHS must be positive $\Rightarrow$ not a circle [4 marks]

6 A(−2, 4), B(6, 10)
A and B are ends of a diameter $\Rightarrow$ centre $= \left(\dfrac{-2+6}{2}, \dfrac{4+10}{2}\right)$
Centre = (2, 7)
Equation $(x - 2)^2 + (y - 7)^2 = r^2$
Passes through (−2, 4) $\Rightarrow (-4)^2 + (-3)^2 = r^2$
$\Rightarrow r = 5$
Equation is $(x - 2)^2 + (y - 7)^2 = 25$ [4 marks]

7 (i)



[1 mark]

(ii) The x coordinate of A is 7 (i.e. $\dfrac{2+12}{2}$)
The radius = 13, so $(7-2)^2 + y^2 = 13^2$
$\Rightarrow y^2 = 144$
$\Rightarrow y = \pm 12$
A is the point (7, 12) [2 marks]

(iii) The centre of the bottom circle is the point (7, −12) [1 mark]

(iv) The equation of the top circle is $(x-7)^2 + (y-12)^2 = 169$
The equation of the bottom circle is $(x-7)^2 + (y+12)^2 = 169$ [2 marks]

(v) The top circle intersects the y-axis when $x = 0 \Rightarrow 49 + (y-12)^2 = 169$
$\Rightarrow y - 12 = \pm\sqrt{120}$
$\Rightarrow y = 1.05$ and $y = 22.95$ (2 d.p.)
So points are (0, 1.05) and (0, 22.95)
By symmetry the bottom circle intersects the y-axis at (0, –1.05) and (0, –22.95) (2 d.p.) [2 marks]
[Total: 8 marks]

8 $x^2 + y^2 - 6x + 4y + 4 = 4$
$\Rightarrow x^2 - 6x + y^2 + 4y = 0$
$\Rightarrow (x - 3)^2 + (y + 2)^2 = 0 + 9 + 4$
$\Rightarrow (x - 3)^2 + (y + 2)^2 = 13$ i.e. a circle

Centre (3, −2), radius $\sqrt{13}$
Concentric circle with radius 5 is $(x - 3)^2 + (y + 2)^2 = 25$ [4 marks]

9 **(i)** Centre $(-1, 1)$ through $(2, 5)$

$\text{Eq}^n\ (x + 1)^2 + (y - 1)^2 = r^2$

Through $(2, 5) \Rightarrow 3^2 + 4^2 = r^2$

$\Rightarrow r = 5$

Circle $(x + 1)^2 + (y - 1)^2 = 25$ [3 marks]

(ii) Same centre and double radius $\Rightarrow r = 10$

$(x + 1)^2 + (y - 1)^2 = 100$ [1 mark]

[Total: 4 marks]

10 A(1, 8), B(3, 14), midpoint C is $\left(\dfrac{1 + 3}{2}, \dfrac{8 + 14}{2}\right) = (2, 11)$

$\text{Eq}^n\ (x - 2)^2 + (y - 11)^2 = r^2$

Through $(3, 14) \Rightarrow 1^2 + 3^2 = r^2$

$\Rightarrow r = \sqrt{10}$

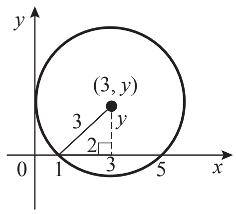
$\text{Eq}^n\ (x - 2)^2 + (y - 11)^2 = 10$ [3 marks]

11 Passes through $(1, 0)$ and $(5, 0)$

$\Rightarrow$ Centre on line $x = 3$

y-axis is a tangent $\Rightarrow$ radius $= 3$

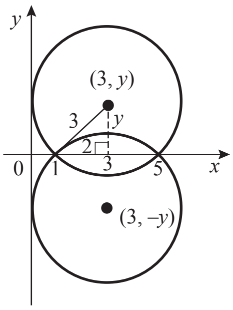
From diagram $2^2 + y^2 = 3^2$



$\Rightarrow y = \pm\sqrt{5}$

Circle $(x - 3)^2 + (y - \sqrt{5})^2 = 9$ (top)

Or $(x - 3)^2 + (y + \sqrt{5})^2 = 9$ (bottom) [6 marks]



12 $(7-2y-3)^2+(y-2)^2=5 \quad \Rightarrow \quad (4-2y)^2+(y-2)^2=5 \quad \Rightarrow \quad 16-16y+4y^2+y^2-4y+4=5$

$\Rightarrow \quad 5y^2-20y+15=0 \quad \Rightarrow \quad y^2-4y+3=0 \quad \Rightarrow \quad (y-1)(y-3)=0 \quad \Rightarrow \quad y=1,\ y=3$

When $y=1$, $x=7-2\times1=5$ and when $y=3$, $x=7-2\times3=1$

The intersection points are $(5, 1)$ and $(1, 3)$ [6 marks]

5

Coordinate geometry

Exercise 5.6 Circle geometry, including tangents and chords

1 AB diameter $\Rightarrow$ APB = 90°
Gradient AP × Gradient BP = −1

$\Rightarrow$ Gradient AP = $\dfrac{8-(-2)}{2-4} = -5$

$\Rightarrow$ Gradient BP = $+\dfrac{1}{5} = 0.2$ [3 marks]

2 A(6, 3), B(10, 1), centre C(11, 8)

(i) Midpoint AB = $\left(\dfrac{6+10}{2}, \dfrac{3+1}{2}\right) = (8, 2)$ [1 mark]

(ii) Distance from chord AB to centre = $\sqrt{(11-8)^2+(8-2)^2} = 3\sqrt{5}$ [1 mark]

(iii) Using A, $r = \sqrt{(11-6)^2+(8-3)^2} = 5\sqrt{2}$ [1 mark]

(iv) Eq^n $(x-11)^2 + (y-8)^2 = 50$ [1 mark]

[Total: 4 marks]

3 $(x-2)^2 + (y+3)^2 = 100$

(i) Radius = $\sqrt{100} = 10$; centre $(2, -3)$ [1 mark]

(ii) $(8-2)^2 + (y+3)^2 = 10^2$
$36 + y^2 + 6y + 9 = 100$
$\Rightarrow y^2 + 6y - 55 = 0$
$\Rightarrow (y+11)(y-5) = 0$
$\Rightarrow y_1 = 5$ and $y_2 = -11$ [3 marks]

(iii) ΔCPQ has height 6 and base $5 - (-11) = 16$

Area = $\dfrac{1}{2} \times 6 \times 16 = 48$ units^2 [1 mark]

[Total: 5 marks]

4 **(i)** Radius$^2 = (2-1)^2 + (-1-1)^2$
$= 5$

Radius = $\sqrt{5}$ [1 mark]

(ii) Radius ⊥ tangent

Gradient of radius = $\dfrac{1-(-1)}{1-2} = -2$

$\Rightarrow$ Gradient of tangent = $\dfrac{1}{2}$

Equation of tangent $(y-1) = \dfrac{1}{2}(x-1)$

$y - 1 = \dfrac{1}{2}x - \dfrac{1}{2}$

$x - 2y + 1 = 0$ [3 marks]

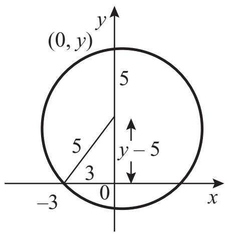
[Total: 4 marks]

5 (i) $3^2 + (y - 5)^2 = 5^2$

$\Rightarrow (y - 5)^2 = 16$

$\Rightarrow y - 5 = 4$ (it is positive)

$\Rightarrow y = 9$ [3 marks]



(ii) Centre of circle is at (0, 4)

Line joining (−3, 0) to (0, 4) has gradient $\frac{4}{3}$

$\Rightarrow$ tangent at (−3, 0) has gradient $-\frac{3}{4}$

Equation $(y - 0) = -\frac{3}{4}(x - (-3))$

$\Rightarrow 4y = -3x - 9$

$\Rightarrow 3x + 4y + 9 = 0$

Tangent at (0, 9) has equation $y = 9$

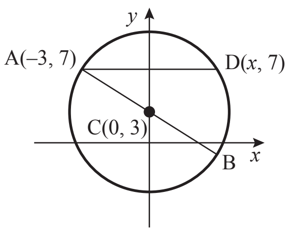
Tangents meet when $\left.\begin{array}{r} 3x + 4y + 9 = 0 \\ y = 9 \end{array}\right\} \begin{array}{l} \Rightarrow 3x + 45 = 0 \\ \Rightarrow x = -15 \end{array}$

Point is (−15, 9) [5 marks]

[Total: 8 marks]

6 (i) $AC^2 = (0 - (-3))^2 + (7 - 3)^2$

$\Rightarrow$ AC = 5, so radius is 5



[2 marks]

(ii) A $\rightarrow$ C is 3 right, 4 down

$\Rightarrow$ C $\rightarrow$ B is 3 right, 4 down

B is $(0 + 3, 3 - 4) = (3, -1)$

AD horizontal $\Rightarrow$ DB vertical (∠ subtended by diameter = 90°)

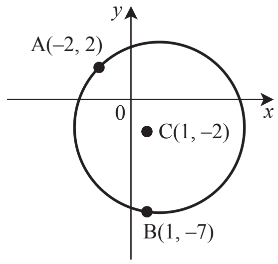
$\Rightarrow$ B and D have same x coordinate

$\Rightarrow$ D is (3, 7) [4 marks]

(iii) ABD is a right-angled triangle [1 mark]

[Total: 7 marks]

7 (i) $(x - 1)^2 + (y + 2)^2 = 25$ has centre $(1, -2)$, radius 5



[2 marks]

(ii) Gradient AC $= \dfrac{(-2)-2}{1-(-2)} = -\dfrac{4}{3}$

$\Rightarrow$ gradient of tangent $= \dfrac{3}{4}$

At A, equation of tangent $y - 2 = \dfrac{3}{4}(x + 2)$

$4y - 8 = 3x + 6$

$3x - 4y + 14 = 0$

At B, BC vertical $\Rightarrow$ tangent horizontal

$\Rightarrow$ tangent is $y = -7$ [4 marks]

(iii) Meet where $3x - 4(-7) + 14 = 0$

$\Rightarrow 3x = -42$

$\Rightarrow x = -14$, meet $(-14, -7)$ [2 marks]

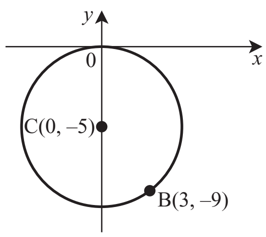
[Total: 8 marks]

8 (i) $x^2 + y^2 + 10y = 0$

$\Rightarrow x^2 + (y + 5)^2 - 5^2 = 0$

$\Rightarrow x^2 + (y + 5)^2 = 25$

Centre $(0, -5)$, radius 5



[2 marks]

(ii) Gradient BC $= \dfrac{-9-(-5)}{3-0} = -\dfrac{4}{3}$

$\Rightarrow$ gradient of tangent $= \dfrac{3}{4}$

Equation $(y - (-9)) = \dfrac{3}{4}(x - 3)$

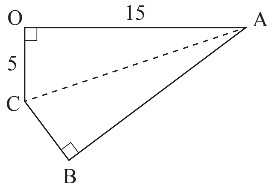
$4y + 36 = 3x - 9$

$3x - 4y - 45 = 0$

Intersects x-axis when $y = 0 \Rightarrow 3x - 45 = 0$

$\Rightarrow x = 15$

Point A is $(15, 0)$ [4 marks]

(iii) 

CBA = 90° (tangent ⊥ radius)

Area OABC = 2 × area ΔOAC

$= 2 \times \frac{1}{2} \times 15 \times 5$

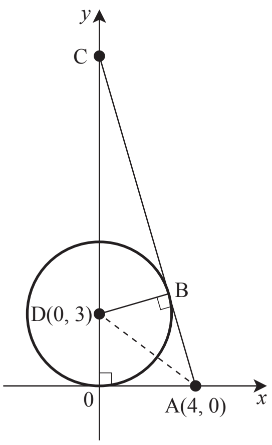
$= 75 \text{ units}^2$ [3 marks]

[Total: 9 marks]

9 Centre D(0, 3), radius = 3, A(4, 0)

(i) Circle is $(x - 0)^2 + (y - 3)^2 = 3^2$

$\Rightarrow x^2 + y^2 - 6y = 0$ [1 mark]

(ii) 

[2 marks]

(iii) Area OABD = 2 area OAD

$= 2 \times \frac{1}{2} \times 4 \times 3$

$= 12 \text{ units}^2$ [4 marks]

[Total: 7 marks]

10 (i) $(3-4)^2 + (-1+5)^2 = (-1)^2 + 4^2 = 1+16 = 17$ so P(3, −1) lies on the circle [2 marks]

(ii) $\frac{-5--1}{4-3} = -4$ [2 marks]

(iii) $y--1 = \frac{1}{4}(x-3) \quad \Rightarrow \quad 4y+4 = x-3 \quad \Rightarrow \quad x-4y = 7$ [3 marks]

[Total: 7 marks]

6 Geometry I

Exercise 6.1 Circle theorems

1 Reflex angle at C = $360° - 122° = 2x$
$\Rightarrow x = 119°$ [1 mark]

2 $\angle$ PSR = 40° ($\angle$ in alt segment)
$\Rightarrow \angle$ PRS = $\angle$ RPS = 70° (isosceles triangle)
$\Rightarrow \angle$ PRQ = 180° − 70° = 110° (adjacent angles)
$\Rightarrow \angle$ PQR = x = 30° ($\angle$ sum of triangle) [3 marks]

3 $\angle$ QPS = $\angle$ QSP = $\dfrac{180° - 54°}{2}$ = 63° (isosceles triangle)
$\angle$ PSR = 90° ($\angle$ in semicircle)
$\angle$ PRS = 54° (subtended by PS)
$\Rightarrow \angle$ SPR = 36° ($\angle$ sum of triangle PRS)
$\Rightarrow x = 63° - 36° = 27°$ [3 marks]

4 Join PR
$\angle$ PRS = 90° ($\angle$ in a semicircle)
$\Rightarrow \angle$ RPS = 20° ($\angle$ sum of triangle PRS)
$\Rightarrow \angle$ QPR = 40° − 20° = 20°
$\Rightarrow$ PR bisects $\angle$ QPS [3 marks]

5 $\angle$TPR = $\angle$TRP = $\dfrac{180° - 40°}{2}$ = 70° (isosceles triangle)
$\angle$TRP = $\angle$ PQR = 70° ($\angle$ in alternate segment)
$\angle$ PRQ = 70° (isosceles triangle)
$\angle$ QPR = 40° ($\angle$ sum of triangle PQR) [3 marks]

6 $\angle$ BRO = $\angle$ BPO = 90° (tangent perpendicular to radius)
$\angle$ ROP = 360° − 90° − 90° − 50° ($\angle$ sum of quadrilateral)
= 130°
$\angle$ RQP = 65° ($\angle$ at centre = 2 × $\angle$ at circumference) [3 marks]

7 $\angle$ PQR = 58° ($\angle$ in alternate segment)
y = 122° (adjacent angles)
$\angle$ CPQ = 58° (CP = CQ (radii))
$\Rightarrow \angle$ QPB = 32° ($\angle$ CPB = 90° (tangent perpendicular to chord))
$\Rightarrow x = 180° - 122° - 32°$
= 26° [4 marks]

8 **(i)** $\angle$ PQR = $180° - 7x$ (angles on a straight line) [1 mark]

(ii) $\angle$ RSQ = 180° − 105° = 75°
$\Rightarrow 75° = 180° - 7x$ (angles in alt. segment)
$\Rightarrow 7x = 105°$
$\Rightarrow x = 15°$ [2 marks]

[Total: 3 marks]

9 $\angle$ QRT = 26° (angle in alt. segment)
$\angle$ QTR = 90° (angle subtended by diameter)
$\Rightarrow \angle$TQR = 64° (angle sum of ΔQRT)
$\angle$TQP = 116° (adjacent angles)
$\Rightarrow x = 180° - 116° - 26°$
= 38° [4 marks]

10 (i) $\angle BDC_2 = 180° - x$ (angles on a line)
$\angle C_2BD = 180° - x$ (isosceles Δ)
$\Rightarrow y + 360° - 2x = 180°$ (angle sum of triangle)
$\Rightarrow y = 2x - 180°$ [2 marks]

(ii) Reflex $\angle AC_1B = 2x$ (angle at centre = 2 × angle at circumference)
$\Rightarrow$ Angle $AC_1B = 360° - 2x$
Angle $AC_1B + y = (360° - 2x) + (2x - 180°)$
$= 180°$
$\Rightarrow AC_1BC_2$ is cyclic quadrilateral [3 marks]

[Total: 5 marks]

11 $\angle ADC = 34°$ (alternate segment theorem)
$\angle CAD = 90°$ (angle in a semi-circle)
$\angle ACD = 180 - (34 + 90) = 56°$ (angle sum in a triangle)
$\angle ACB = 180 - 56 = 124°$ (angles on a straight line)
$\angle ABC = 180 - (34 + 124) = 22°$ (angle sum in a triangle) [5 marks]

Exercise 6.2 Geometric proof

1 (i) $\angle CAB + \angle CBA = 180° - y$ (angle sum of triangle is 180°)
$\angle CAB = \angle CBA = 90° - \dfrac{y}{2}$ (base angles of isosceles triangle ABC are equal)
$\angle CBD + \angle CDB = 180° - x$ (angle sum of triangle is 180°)
$\angle CBD = \angle CDB = 90° - \dfrac{x}{2}$ (base angles of isosceles triangle BCD are equal) [2 marks]

(ii) In ΔABD, $\angle ABD + \left(90° - \dfrac{y}{2}\right) + \left(90° - \dfrac{x}{2}\right) = 180°$ (angle sum of triangle is 180°)
$\Rightarrow \angle ABD = \dfrac{x + y}{2}$
$x + y = 180°$ (angles on a straight line add to 180°)
$\Rightarrow \angle ABD = 90°$ [3 marks]

[Total: 5 marks]

2 $\angle BAC = x$ (base angles of an isosceles triangle are equal) $\Rightarrow \angle ACB = 180° - 2x$
(angle sum of triangle is 180°)
$\angle BDC = y$ (base angles of an isosceles triangle are equal) $\Rightarrow \angle DCB = 180° - 2y$
(angle sum of triangle is 180°)
$\angle ACD = 360° - (180° - 2x) - (180° - 2y)$ (angle sum at a point is 360°)
$= 2x + 2y$
$= 2(x + y)$
$\Rightarrow$ Angle at centre is double the angle at the circumference [3 marks]

3 $\angle ACB = 2x$ ($\angle$ at centre = 2 × angle at circumference)
and
$\angle ACB = 2y$ ($\angle$ at centre = 2 × angle at circumference)
$\Rightarrow 2x = 2y$
$\Rightarrow x = y$ i.e. angles in the same segment are equal [3 marks]

4 ΔACB is isosceles, (AC = CB = radius)

$$\left.\begin{array}{l} AC = BC \\ AM = MB \\ CM = CM \end{array}\right\} \Delta ACM \text{ congruent to } \Delta BCM$$

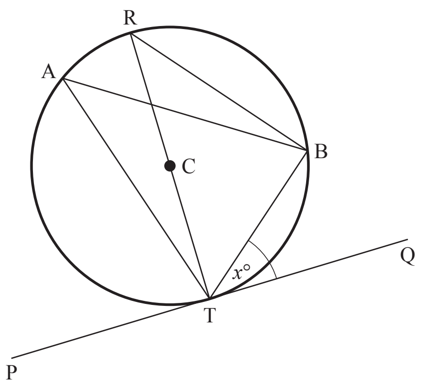
$\angle AMC = \angle BMC$ and $\angle AMC + \angle BMC = 180°$ (angle sum on a straight line is 180°)
$\Rightarrow x = 90°$ [3 marks]

5 $\angle$ BAD + $\angle$ ADC = 180° (1) (sum of angles between || lines)
$\angle$ ADC + $\angle$ DCB = 180° (2) (sum of angles between || lines)
(1) − (2) $\Rightarrow \angle$ BAD − $\angle$ DCB = 0°
$\Rightarrow \angle$ BAD = $\angle$ DCB
Similarly $\angle$ ADC = $\angle$ CBA
$\Rightarrow$ opposite angles of a parallelogram are equal [3 marks]

6 $\angle$ PCR = $2x$ (angle at centre = 2 × angle at circumference)
Reflex $\angle$ PCR = $2y$ (angle at centre = 2 × angle at circumference)
$\Rightarrow 360° = 2x + 2y$ (angle sum at a point is 360°)
$\Rightarrow x + y = 180°$ [3 marks]

7 <u>ΔABC</u> <u>ΔADC</u>
BC = CD (radii)
$\angle$ ABC = $\angle$ ADC = 90° (tangent perpendicular to radius)
AC = AC
$\Rightarrow$ ΔABC and ΔADC are congruent (2 sides and a 90° angle)
$\Rightarrow$ AB = AD [3 marks]

8



Let $\angle$ BTQ = $x°$
$\angle$ RTQ = 90° (tangent perpendicular to radius)
$\Rightarrow \angle$ RTB = 90° − $x°$
$\angle$ RBT = 90° ($\angle$ in semicircle)
$\Rightarrow \angle$ TRB = $x°$ ($\angle$ sum of ΔRBT)
$\angle$ TRB = $\angle$ TAB = $x°$ ($\angle$'s in same segment are equal)
$\Rightarrow \angle$ BTQ = $\angle$ BAT [3 marks]

9 Let angles EBC and ECB be θ (base angles of isosceles triangle BEC are equal)
$\angle ABC = \angle DCB = 180° - \theta$ (angle sum on a straight line is 180°)
$\angle ADC = \angle DAB = \theta$ (opposite angles of a cyclic quadrilateral add to 180°)
$AE = DE$ (base angles of isosceles triangle AED are equal) [3 marks]

Exercise 6.3 Trigonometry in two dimensions

1 **(i)** $\frac{x}{12} = \sin 30°$
$= \frac{1}{2} \Rightarrow x = 6.0$ cm [1 mark]

(ii) $\frac{8}{x} = \tan 40° \quad \Rightarrow \quad x = \frac{8}{\tan 40°} = 9.5$ cm (1 d.p.) [1 mark]

(iii) $\frac{5}{x} = \cos 50° \quad \Rightarrow \quad x = \frac{5}{\cos 50°} = 7.8$ cm (1 d.p.) [1 mark]

(iv) $\frac{9}{x} = \sin 62° \quad \Rightarrow \quad x = \frac{9}{\sin 62°} = 10.2$ cm (1 d.p.) [1 mark]

[Total: 4 marks]

2 **(i)** $\cos\theta = \dfrac{4.3}{7.2} \quad \Rightarrow \quad \theta = 53.3°$ (1 d.p.) [1 mark]

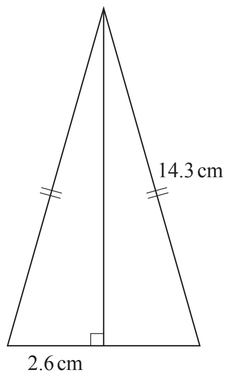
(ii) $\sin\theta = \dfrac{7.4}{9.2} \quad \Rightarrow \quad \theta = 53.5°$ (1 d.p.) [1 mark]

(iii) $\tan\theta = \dfrac{1.8}{2.7} \quad \Rightarrow \quad \theta = 33.7°$ (1 d.p.) [1 mark]

(iv) $\sin\theta = \dfrac{6.2}{8.1} \quad \Rightarrow \quad \theta = 49.9°$ (1 d.p.) [1 mark]

[Total: 4 marks]

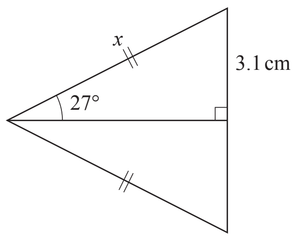
3 **(i)**



$5.2 \div 2 = 2.6$

$\cos\theta = \dfrac{2.6}{14.3} \Rightarrow \theta = 79.5°$ (1 d.p.) [2 marks]

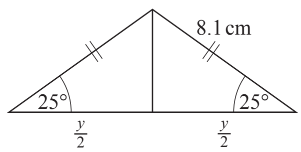
(ii)



$\Rightarrow \sin 27° = \dfrac{3.1}{x}$

$\Rightarrow x = \dfrac{3.1}{\sin 27°} = 6.8\,\text{cm}$ (1 d.p.) [2 marks]

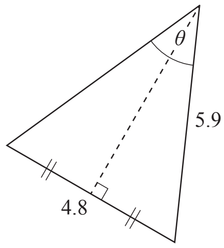
(iii)



$\dfrac{y/2}{8.1} = \cos 25° \Rightarrow \dfrac{y}{2} = 8.1\cos 25°$

$\Rightarrow y = 16.2 \cos 25° = 14.7\,\text{cm}$ (1 d.p.) [2 marks]

(iv)



$$\sin\frac{\theta}{2} = \frac{2.4}{5.9}$$
$$\Rightarrow \frac{\theta}{2} = 24.00°$$
$$\Rightarrow \theta = 48.0° \text{ (1 d.p.)}$$
[3 marks]

[Total: 9 marks]

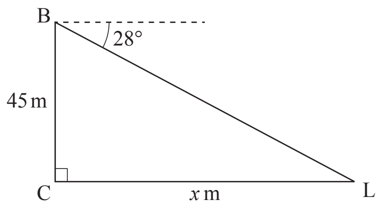
4 **(i)** $\cos \angle ADB = \dfrac{7.4}{12.3} \Rightarrow \angle ADB = 53.0°$ (1 d.p.) [1 mark]

(ii) $\tan \angle DCB = \dfrac{7.4}{8.1} \Rightarrow \angle DCB = 42.4°$ (1 d.p.) [1 mark]

(iii) $AB^2 + 7.4^2 = 12.3^2 \Rightarrow AB = 9.8\text{ cm}$ (1 d.p.)
$\Rightarrow AC = 9.8\text{ cm} + 8.1\text{ cm}$
$= 17.9\text{ cm}$ [2 marks]

[Total: 4 marks]

5 **(i)**



[2 marks]

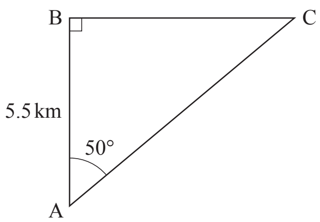
(ii) $\angle BLC = 28°$ (alternate angles)

$\tan 28° = \dfrac{45}{x} \Rightarrow x = \dfrac{45}{\tan 28°}$

$\Rightarrow x = 84.6\text{ m}$ [2 marks]

[Total: 4 marks]

6 **(i)**



[2 marks]

(ii) $\dfrac{5.5}{AC} = \cos 50° \Rightarrow AC = \dfrac{5.5}{\cos 50°}$ km
$= 8.55648 \ldots$ km
$= 8556\text{ m}$ (nearest metre) [1 mark]

(iii) Walks A → B → C

$\frac{BC}{5.5} = \tan 50° \Rightarrow BC = 5.5\tan 50°$ km

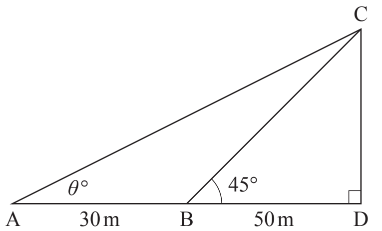
$= 6.55464 \ldots$ km

$= 6555$ m (nearest metre)

Travels (5500 + 6555) m = 12 055 m on foot [2 marks]

[Total: 5 marks]

7 **(i)**



[2 marks]

(ii) $\frac{CD}{50} = \tan 45° = 1 \Rightarrow CD = 50$ m [1 mark]

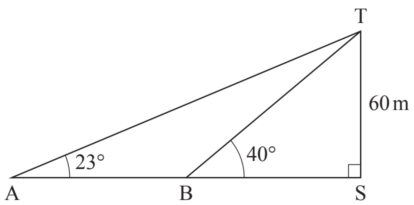
(iii) In triangle ACD $\tan\theta = \frac{CD}{80}$

$= \frac{50}{80}$

$\Rightarrow \theta = 32.0°$ [1 mark]

[Total: 4 marks]

8 **(i)**



[2 marks]

(ii) $\frac{60}{BS} = \tan 40° \Rightarrow BS = \frac{60}{\tan 40°}$

$= 71.505$

$= 71.51$ m [1 mark]

(iii) $\frac{60}{AS} = \tan 23° \Rightarrow AS = \frac{60}{\tan 23°}$

$= 141.351$

$= 141.35$ m

AB = 141.35 − 71.51 = 69.8 m (1 d.p.) [2 marks]

[Total: 5 marks]

9 **(i)** $\tan^{-1}\left(\frac{2}{5}\right) = 21.8°$ (1 d.p.) [2 marks]

(ii) $2 \times 21.8 = 43.6°$ (1 d.p.) [2 marks]

[Total: 4 marks]

Exercise 6.4 Angles of 45°, 30° and 60°

1 **(i)** $\sin 30° = \dfrac{x}{4\sqrt{3}-2} \Rightarrow x = \dfrac{1}{2}\left(4\sqrt{3}-2\right)$

$\Rightarrow x = 2\sqrt{3}-1$ [1 mark]

(ii) $\sin 60° = \dfrac{3\sqrt{3}}{x} \Rightarrow x = 3\sqrt{3} \div \dfrac{\sqrt{3}}{2}$

$= 3\sqrt{3} \times \dfrac{2}{\sqrt{3}}$

$\Rightarrow x = 6$ [1 mark]

(iii) $\cos 60° = \dfrac{x}{6\sqrt{2}} \Rightarrow x = 6\sqrt{2} \times \dfrac{1}{2}$

$\Rightarrow x = 3\sqrt{2}$ [1 mark]

(iv) $\sin 30° = \dfrac{5}{2\sqrt{3}} \div x \Rightarrow x = \dfrac{5}{2\sqrt{3}} \div \sin 30°$

$\Rightarrow x = \dfrac{5}{2\sqrt{3}} \div \dfrac{1}{2}$

$\Rightarrow x = \dfrac{5}{\sqrt{3}} = \dfrac{5\sqrt{3}}{3}$ [2 marks]

[Total: 5 marks]

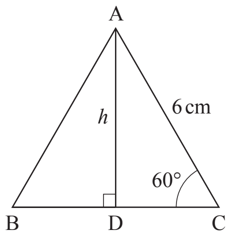
2 $\dfrac{\text{AB}}{2} = \tan 30° \Rightarrow \text{AB} = 2 \times \dfrac{1}{\sqrt{3}}$

$\Rightarrow \text{AB} = \dfrac{2\sqrt{3}}{3}$ cm

$\text{Area} = \dfrac{1}{2} \times 2 \times \dfrac{2\sqrt{3}}{3}$

$= \dfrac{2\sqrt{3}}{3}\ \text{cm}^2$ [3 marks]

3 AD is a line of symmetry so DC = 3 cm.



In ΔADC, $\sin 60° = \dfrac{h}{6}$

$\Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{h}{6}$

$\Rightarrow h = 3\sqrt{3}$ cm

$\text{Area ABC} = \dfrac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{1}{2} \times 6 \times 3\sqrt{3}$

$= 9\sqrt{3}\ \text{cm}^2$ [2 marks]

4 **(i)** $\dfrac{BD}{2} = \cos 45° \Rightarrow BD = 2 \times \dfrac{1}{\sqrt{2}}$

$\Rightarrow BD = \sqrt{2}$ cm [1 mark]

(ii) $\dfrac{DC}{BD} = \tan 60° \Rightarrow \dfrac{DC}{\sqrt{2}} = \sqrt{3}$

$\Rightarrow DC = \sqrt{6}$ cm [1 mark]

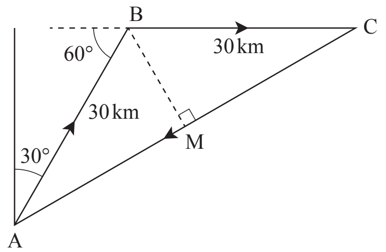
(iii) $AD = BD = \sqrt{2}\,\text{cm} \Rightarrow AC = \sqrt{2} + \sqrt{6}$ cm

$$\text{Area ABC} = \frac{1}{2} \times AC \times BD$$
$$= \frac{1}{2}\left(\sqrt{2} + \sqrt{6}\right)\sqrt{2}$$
$$= \left(1 + \sqrt{3}\right)\text{cm}^2$$

[2 marks]

[Total: 4 marks]

5 **(i)**



[2 marks]

(ii) ΔABC is isosceles

Adding M, the midpoint of AC to the sketch

$\angle ABC = 120° \Rightarrow \angle ABM = 60°$

$\dfrac{AM}{AB} = \sin \angle ABM \Rightarrow \dfrac{AM}{30} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow AM = 15\sqrt{3}$

$\Rightarrow AC = 30\sqrt{3}$ km [1 mark]

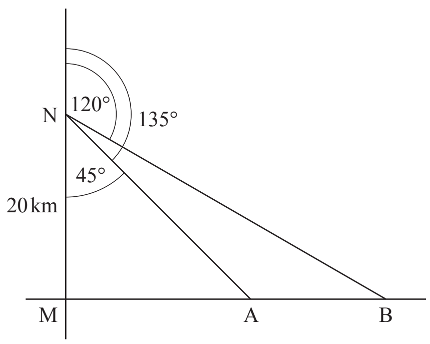
(iii) Total distance $= \left(60 + 30\sqrt{3}\right)$ km

$\Rightarrow$ total time $= \dfrac{60 + 30\sqrt{3}}{30}$ h

$= (2 + \sqrt{3})$h or 3h 44 min [2 marks]

[Total: 5 marks]

6 **(i)**



[2 marks]

(ii) MA = 20 km [1 mark]

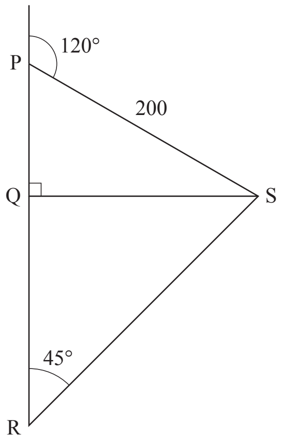
(iii) $\angle MNB = 60°$

In ΔMBN, $\dfrac{MB}{20} = \tan 60° = \sqrt{3} \Rightarrow MB = 20\sqrt{3}$ km

Distance apart $= AB = \left(20\sqrt{3} - 20\right)$ km [2 marks]

[Total: 5 marks]

7 (i)



[2 marks]

(ii) $\angle QPS = 60°$

$\sin 60° = \dfrac{QS}{200} \Rightarrow QS = 200 \sin 60°$

$= 200 \times \dfrac{\sqrt{3}}{2}$

$= 100\sqrt{3}$ m

$QR = QS = 100\sqrt{3} \Rightarrow SR^2 = \left(100\sqrt{3}\right)^2 + \left(100\sqrt{3}\right)^2$

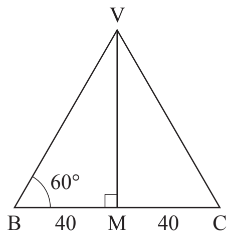
$\Rightarrow RS^2 = 30\,000 + 30\,000$

$\Rightarrow RS = \sqrt{60\,000} = 100\sqrt{6}$ m

[3 marks]

[Total: 5 marks]

8 (i)



$\dfrac{VM}{40} = \tan 60° = \sqrt{3} \Rightarrow VM = 40\sqrt{3}$

Area VBC $= \dfrac{1}{2} \times 80 \times 40\sqrt{3}$

$= 1600\sqrt{3}$ cm^2

[2 marks]

(ii) Total surface area = 4 sides + base

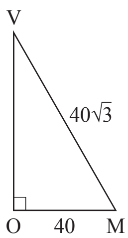
$= 4 \times 1600\sqrt{3} + 80^2$

$= 6400\sqrt{3} + 6400$

$= 6400\left(\sqrt{3} + 1\right)$ cm^2

[1 mark]

(iii)



$OM = \dfrac{1}{2}$ width of base = 40 cm

Height VO: $VO^2 = \left(40\sqrt{3}\right)^2 - 40^2$

$= 1600 \times 3 - 1600$

$= 1600 \times 2$

$\Rightarrow VO = 40\sqrt{2}$ cm

[2 marks]

[Total: 5 marks]

9 $2 \times \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 60^\circ$ [4 marks]

Exercise 6.5 Trigonometric functions for angles of any size, and trigonometric graphs

1 **(i)** $\sin\theta = 0.5 \Rightarrow \theta = 30^\circ$ (principal value)
or $\theta = 180^\circ - 30^\circ = 150^\circ$ [1 mark]

(ii) $\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ$ (principal value)
or $360^\circ - 30^\circ = 330^\circ$ [1 mark]

(iii) $\tan\theta = 2 \Rightarrow \theta = 63.4^\circ$ (principal value)
or $180^\circ + 63.4^\circ = 243.4^\circ$ [1 mark]

(iv) $\cos\theta = -0.5 \Rightarrow \theta = 120^\circ$ (principal value)
or $360^\circ - 120^\circ = 240^\circ$ [1 mark]

(v) $\tan\theta = -\sqrt{2} \Rightarrow \theta = -54.7^\circ$ (not in range)
or $-54.7^\circ + 180^\circ = 125.3^\circ$
or $125.3^\circ + 180^\circ = 305.3^\circ$ [1 mark]

(vi) $\sin\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = -60^\circ$ (not in range)
or $180^\circ + 60^\circ = 240^\circ$
or $360^\circ - 60^\circ = 300^\circ$ [1 mark]

[Total: 6 marks]

2 **(i)** $5\sin\theta = 2 \Rightarrow \sin\theta = 0.4$
$\Rightarrow \theta = 23.6^\circ$ (principal value)
or $180^\circ - 23.6^\circ = 156.4^\circ$ [2 marks]

(ii) $9\cos\theta = 4 \Rightarrow \cos\theta = \frac{4}{9}$
$\Rightarrow \theta = 63.6^\circ$ (principal value)
or -63.6° [2 marks]

(iii) $2\tan\theta = 7 \Rightarrow \tan\theta = 3.5$
$\Rightarrow \theta = 74.1^\circ$ (principal value)
or $74.1^\circ - 180^\circ = -105.9^\circ$ [2 marks]

(iv) $4\cos\theta + 3 = 0 \Rightarrow \cos\theta = -\frac{3}{4}$
$\Rightarrow \theta = 138.6^\circ$ (principal value)
or -138.6° [2 marks]

(v) $\tan\theta + 7 = 4 \Rightarrow \tan\theta = -3$
$\Rightarrow \theta = -71.6^\circ$ (principal value)
or $-71.6^\circ + 180^\circ = 108.4^\circ$ [2 marks]

(vi) $5\sin\theta + 2 = 0 \Rightarrow \sin\theta = -\frac{2}{5}$
$\Rightarrow \theta = -23.6^\circ$
or $-180^\circ + 23.6^\circ = -156.4^\circ$ [2 marks]

[Total: 12 marks]

3 **(i)** $\sin^2\theta = 0.6 \Rightarrow \sin\theta = \pm\sqrt{0.6}$
positive sign $\Rightarrow \theta = 50.8°$ or $180° - 50.8° = 129.2°$
negative sign $\Rightarrow \theta = 180° + 50.8° = 230.8°$ or $360° - 50.8° = 309.2°$ [2 marks]

(ii) $\cos^2\theta = 0.4 \Rightarrow \cos\theta = \pm\sqrt{0.4}$
$\cos\theta = \sqrt{0.4} \Rightarrow \theta = 50.8°$ or $360° - 50.8° = 309.2°$
$\cos\theta = -\sqrt{0.4} \Rightarrow \theta = 129.2°$ or $360° - 129.2° = 230.8°$ [2 marks]

(iii) $\tan^2\theta = 10 \Rightarrow \tan\theta = \pm\sqrt{10}$
$\tan\theta = \sqrt{10} \Rightarrow \theta = 72.5°$ or $180° + 72.5° = 252.5°$
$\tan\theta = -\sqrt{10} \Rightarrow \theta = -72.5°$ or $-72.5° + 180° = 107.5°$ or $-72.5° + 360° = 287.5°$ [2 marks]

[Total: 6 marks]

4 **(i)** $3x^2 - x - 2 = (3x + 2)(x - 1)$ [1 mark]

(ii) $(3x + 2)(x - 1) = 0 \Rightarrow x = 1$ or $x = -\frac{2}{3}$ [1 mark]

(iii) **(a)** Replace x by $\cos\theta$
$\cos\theta = 1 \Rightarrow \theta = -360°$ or $0°$ or $360°$
$\cos\theta = -\frac{2}{3} \Rightarrow \theta = 131.8°$ or $360° - 131.8° = 228.2°$
or $-131.8°$ or $\theta = -228.2°$ [2 marks]

(b) Replace x by $\sin\theta$
$\sin\theta = 1 \Rightarrow \theta = 90°$ or $\theta = -270°$
$\sin\theta = -\frac{2}{3} \Rightarrow \theta = -41.8°$ or $-180° + 41.8° = -138.2°$
or $\theta = 180° + 41.8° = 221.8°$
or $\theta = 360° - 41.8° = 318.2°$ [2 marks]

(c) Replace x by $\tan\theta$
$\tan\theta = 1 \Rightarrow \theta = 45°$ or $225°$ or $-135°$ or $-315°$
$\tan\theta = -\frac{2}{3} \Rightarrow \theta = -33.7°$ or $-213.7°$
or $180° - 33.7° = 146.3°$
or $360° - 33.7° = 326.3°$ [2 marks]

[Total: 8 marks]

5 **(i)** $2\sin^2\theta + \sin\theta = 1 \Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$
$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$
$\Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = -1$
$\sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$ or $180° - 30° = 150°$
$\sin\theta = -1 \Rightarrow \theta = -90°$ [3 marks]

(ii) $2\tan^2\theta - 3\tan\theta - 5 = 0 \Rightarrow (2\tan\theta - 5)(\tan\theta + 1) = 0$
$\Rightarrow \tan\theta = 2.5$ or $\tan\theta = -1$
$\tan\theta = 2.5 \Rightarrow \theta = 68.2°$ or $68.2° - 180° = -111.8°$
$\tan\theta = -1 \Rightarrow \theta = -45°$ or $-45° + 180° = 135°$ [3 marks]

(iii) $8\cos^2\theta - 2\cos\theta - 3 = 0 \Rightarrow (2\cos\theta + 1)(4\cos\theta - 3) = 0$

$$\Rightarrow \cos\theta = -\frac{1}{2} \text{ or } \cos\theta = \frac{3}{4}$$

$$\cos\theta = -\frac{1}{2} \Rightarrow \theta = 120° \text{ or } -120°$$

$$\cos\theta = \frac{3}{4} \Rightarrow \theta = 41.4° \text{ or } -41.4°$$

[3 marks]

(iv) $4 - 9\sin^2\theta = 0 \Rightarrow (2 - 3\sin\theta)(2 + 3\sin\theta) = 0$

$$\Rightarrow \sin\theta = \frac{2}{3} \text{ or } \sin\theta = -\frac{2}{3}$$

$$\sin\theta = \frac{2}{3} \Rightarrow \theta = 41.8° \text{ or } 180° - 41.8° = 138.2°$$

$$\sin\theta = -\frac{2}{3} \Rightarrow \theta = -41.8° \text{ or } -138.2°$$

[3 marks]

[Total: 12 marks]

6 $(2\sin\theta + 1)(\sin\theta - 2)(\sin\theta - 1) = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}$ or $\sin\theta = 2$ (not possible) or $\sin\theta = 1$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta = -30°$ (not in range)

or $180° + 30° = 210°$ or $360° - 30° = 330°$

$\sin\theta = 1 \Rightarrow \theta = 90°$ [3 marks]

7 **(i)** $\cos\theta = 1 \Rightarrow \theta = -360°$ or $0°$ or $360°$ [1 mark]

(ii) $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$ or $360° - 45°$ or $-45°$ or $-360° + 45°$

$= -315°, -45°, 45°, 315°$ [1 mark]

(iii) $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$ or $360° - 60°$ or $-60°$ or $-360° + 60°$

$= -300°, -60°, 60°, 300°$ [2 marks]

(iv) $\sin\theta = 1 \Rightarrow \theta = -270°$ or $90°$ [1 mark]

(v) $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$ or $180° - 45°$ or $-180° - 45°$ or $-360° + 45°$

$= -315°, -225°, 45°, 135°$ [2 marks]

(vi) $\sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$ or $180° - 30°$ or $-180° - 30°$ or $-360° + 30°$

$= -330°, -210°, 30°, 150°$ [2 marks]

(vii) $\tan\theta = 1 \Rightarrow \theta = 45°$ or $180° + 45°$ or $-180° + 45°$ or $-360° + 45°$

$= -315°, -135°, 45°, 225°$ [2 marks]

(viii) $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$ or $180° + 30°$ or $-180° + 30°$ or $-360° + 30°$

$= -330°, -150°, 30°, 210°$ [2 marks]

(ix) $\tan\theta = \sqrt{3} \Rightarrow \theta = 60°$ or $180° + 60°$ or $-180° + 60°$ or $-360° + 60°$

$= -300°, -120°, 60°, 240°$ [2 marks]

[Total: 15 marks]

8 $\sin x = 0 \quad \Rightarrow \quad x = 0^\circ, 180^\circ, 360^\circ$

$\tan x = -1 \quad \Rightarrow \quad x = 135^\circ, 315^\circ$

$\cos x = \frac{1}{2} \quad \Rightarrow \quad x = 60^\circ, 300^\circ$ [5 marks]

9 (i) $f(x) = 4x^3 + 4x^2 - x - 1$

$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{8}\right) + 4\left(\frac{1}{4}\right) - \frac{1}{2} - 1 = 0$ [1 mark]

(ii) $f\left(\frac{1}{2}\right) = 0 \Rightarrow \left(x - \frac{1}{2}\right)$ is a factor

i.e. $(2x - 1)$ is a factor

$4x^3 + 4x^2 - x - 1 = (2x - 1)(2x^2 + 3x + 1)$

$= (2x - 1)(2x + 1)(x + 1)$

i.e. solving $(2\cos\theta - 1)(2\cos\theta + 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ \text{ or } -60^\circ$

$\cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^\circ \text{ or } -120^\circ$

$\cos\theta = -1 \Rightarrow \theta = 180^\circ \text{ or } -180^\circ$

$\text{Sol}^n\ \theta = -180^\circ, -120^\circ, -60^\circ, 60^\circ, 120^\circ, 180^\circ$ [5 marks]

[Total: 6 marks]

Exercise 6.6 Trigonometric identities

1 (i) (a) $2\cos^2\theta + 3\sin^2\theta = 3$

$\Rightarrow 2\cos^2\theta + 3(1 - \cos^2\theta) = 3$

$\Rightarrow -\cos^2\theta = 0$ [2 marks]

(b) $\Rightarrow \cos\theta = 0 \quad \theta = 90^\circ$ [1 mark]

(ii) (a) $3\cos^2\theta + 2\sin^2\theta = 3$

$\Rightarrow 3\cos^2\theta + 2(1 - \cos^2\theta) = 3$

$\Rightarrow \cos^2\theta = 1$ [2 marks]

(b) $\Rightarrow \cos\theta = 1 \text{ or } \cos\theta = -1 \quad \theta = 0^\circ \text{ or } 180^\circ$ [1 mark]

(iii) (a) $\sin^2\theta - \sin\theta = \cos^2\theta$

$\Rightarrow \sin^2\theta - \sin\theta = 1 - \sin^2\theta$

$\Rightarrow 2\sin^2\theta - \sin\theta - 1 = 0$ [2 marks]

(b) $\Rightarrow (2\sin\theta + 1)(\sin\theta - 1) = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}$ no sol^n in range

or $\sin\theta = 1 \quad \theta = 90^\circ$ [2 marks]

(iv) (a) $\cos^2\theta - \cos\theta = \sin^2\theta$

$\Rightarrow \cos^2\theta - \cos\theta = 1 - \cos^2\theta$

$\Rightarrow 2\cos^2\theta - \cos\theta - 1 = 0$ [2 marks]

(b) $\Rightarrow (2\cos\theta + 1)(\cos\theta - 1) = 0$

$\Rightarrow \cos\theta = -\frac{1}{2} \quad \theta = 120^\circ$

or $\cos\theta = 1 \quad \theta = 0^\circ$ [2 marks]

(v) (a) $1 - \sin\theta = \cos^2\theta$
$\Rightarrow 1 - \sin\theta = 1 - \sin^2\theta$
$\Rightarrow \sin^2\theta - \sin\theta = 0$ [2 marks]

(b) $\Rightarrow \sin\theta(\sin\theta - 1) = 0$
$\Rightarrow \sin\theta = 0 \quad \theta = 0°, 180°$
or $\sin\theta = 1 \quad \theta = 90°$ [2 marks]

(vi) (a) $1 - \cos\theta = \sin^2\theta$
$\Rightarrow 1 - \cos\theta = 1 - \cos^2\theta$
$\Rightarrow \cos^2\theta - \cos\theta = 0$ [2 marks]

(b) $\Rightarrow \cos\theta(\cos\theta - 1) = 0$
$\Rightarrow \cos\theta = 0 \quad \theta = 90°$
or $\cos\theta = 1 \quad \theta = 0°$ [2 marks]

[Total: 22 marks]

2 (i) (a) $\sin^2\theta + \cos\theta = 0$
$\Rightarrow (1 - \cos^2\theta) + \cos\theta = 0$
$\Rightarrow \cos^2\theta - \cos\theta - 1 = 0$ [2 marks]

(b) $\cos\theta = \dfrac{1 \pm \sqrt{1+4}}{2} \qquad a = 1,\ b = -1,\ c = -1$

$= \dfrac{1 \pm \sqrt{5}}{2}$

$\dfrac{1 + \sqrt{5}}{2} > 1$ so no sol^n

$\cos\theta = \dfrac{1 - \sqrt{5}}{2} \Rightarrow \theta = 128.2°$ [1 mark]

(ii) (a) $\cos^2\theta + \sin\theta = 0$
$\Rightarrow (1 - \sin^2\theta) + \sin\theta = 0$
$\Rightarrow \sin^2\theta - \sin\theta - 1 = 0$ [2 marks]

(b) $\sin\theta = \dfrac{1 \pm \sqrt{1+4}}{2}$

As Q2(i)(b) above, this reduces to $\sin\theta = \dfrac{1 - \sqrt{5}}{2} \Rightarrow \theta = -38.2°$
No sol^n in interval $0 \leqslant \theta \leqslant 180°$ [1 mark]

(iii) (a) $2\sin^2\theta + \cos\theta = 0$
$\Rightarrow 2(1 - \cos^2\theta) + \cos\theta = 0$
$\Rightarrow 2\cos^2\theta - \cos\theta - 2 = 0$ [2 marks]

(b) $\cos\theta = \dfrac{1 \pm \sqrt{1+16}}{4}$

$\dfrac{1 + \sqrt{17}}{4} > 1$ so no sol^n

$\cos\theta = \dfrac{1 - \sqrt{17}}{4} \Rightarrow \theta = 141.3°$ [2 marks]

(iv) (a) $4\sin^2\theta - 5\cos\theta + 2 = 0$
$\Rightarrow 4(1 - \cos^2\theta) - 5\cos\theta + 2 = 0$
$\Rightarrow 4\cos^2\theta + 5\cos\theta - 6 = 0$ [2 marks]

(b) $\cos\theta = \dfrac{-5 \pm \sqrt{25 - 4(4)(-6)}}{8}$

$= \dfrac{-5 \pm 11}{8}$ Reject $\dfrac{-5 - 11}{8}$

$\cos\theta = \dfrac{11 - 5}{8} = \dfrac{3}{4} \Rightarrow \theta = 41.4°$ [2 marks]

[Total: 14 marks]

3 **(i)** $\cos\theta = 4\sin\theta \Rightarrow 1 = \dfrac{4\sin\theta}{\cos\theta}$

$\Rightarrow \tan\theta = \dfrac{1}{4}$ [2 marks]

(ii) $\tan\theta = \dfrac{1}{4} \Rightarrow \theta = 14.0°$ [2 marks]

[Total: 4 marks]

4 **(i)** $\tan\theta - \dfrac{2\cos\theta}{\sin\theta} = 0$ $\qquad \tan\theta - \dfrac{2}{\tan\theta} = 0$

$\Rightarrow \quad \tan^2\theta = 2$

$\Rightarrow \quad \tan\theta = \pm\sqrt{2}$

$\tan\theta = \sqrt{2} \Rightarrow \theta = 54.7°,\ \ 54.7° + 180° = 234.7°$

$\tan\theta = -\sqrt{2} \Rightarrow (\theta = -54.7°),\ \ -54.7° + 180° = 125.3°$

$-54.7° + 360° = 305.3°$ [6 marks]

(ii) $\sin\theta + 2\cos\theta = 0 \Rightarrow \tan\theta + 2 = 0$

$\Rightarrow \tan\theta = -2$

$\theta = -63.4°$ (reject) or $-63.4° + 180° = 116.6°$

or $-63.4° + 360° = 296.6°$ [2 marks]

(iii) $2\sin\theta + 3\cos\theta = 0 \Rightarrow \dfrac{2\sin\theta}{3\cos\theta} = -1$

$\Rightarrow \tan\theta = -1.5$

$\Rightarrow \tan\theta = -56.3°$ (not in range), $123.7°$ or $303.7°$ [4 marks]

(iv) $5\cos\theta - 2\sin\theta = 0 \Rightarrow 5 - 2\tan\theta = 0$

$\Rightarrow \tan\theta = 2.5$

$\Rightarrow \theta = 68.2°$ or $180° + 68.2° = 248.2°$ [2 marks]

[Total: 14 marks]

5 **(i)** $\tan^2\theta\cos^3\theta = \dfrac{\sin^2\theta}{\cos^2\theta} \times \cos^3\theta$

$= \sin^2\theta\cos\theta$

$= (1 - \cos^2\theta)\cos\theta$

$= \cos\theta - \cos^3\theta$ [2 marks]

(ii) $\sin\theta(2\sin\theta - \tan\theta) = 2\sin^2\theta - \sin\theta\left(\dfrac{\sin\theta}{\cos\theta}\right)$

$$= 2(1-\cos^2\theta) - \left(\frac{1-\cos^2\theta}{\cos\theta}\right)$$

$$= (1-\cos^2\theta)\left(2 - \frac{1}{\cos\theta}\right)$$

$$= \frac{(1-\cos^2\theta)(2\cos\theta - 1)}{\cos\theta}$$

[3 marks]

[Total: 5 marks]

6 $\dfrac{1}{\cos^2 x} - 1 = \dfrac{1}{\cos^2 x} - \dfrac{\cos^2 x}{\cos^2 x}$

$$= \frac{1-\cos^2 x}{\cos^2 x}$$

$$= \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$$

[2 marks]

7 (i) $(\cos x + \sin x)^2 + (\cos x - \sin x)^2$

$= \cos^2 x + 2\sin x\cos x + \sin^2 x + \cos^2 x - 2\sin x\cos x + \sin^2 x$

$= 2(\cos^2 x + \sin^2 x)$

$= 2$

[2 marks]

(ii) $\left(\dfrac{1}{\tan x} + \dfrac{1}{\sin x}\right)^2 = \left(\dfrac{\cos x}{\sin x} + \dfrac{1}{\sin x}\right)^2$

$$= \frac{(1+\cos x)^2}{\sin^2 x}$$

$$= \frac{(1+\cos x)^2}{1-\cos^2 x} = \frac{(1+\cos x)^2}{(1+\cos x)(1-\cos x)}$$

$$= \frac{1+\cos x}{1-\cos x}$$

[4 marks]

[Total: 6 marks]

8 (i) $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x} = \dfrac{(1-\cos x)+(1+\cos x)}{(1+\cos x)(1-\cos x)}$

$$= \frac{2}{1-\cos^2 x}$$

$$= \frac{2}{\sin^2 x}$$

[2 marks]

(ii) $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x} = 4 \Rightarrow \dfrac{2}{\sin^2 x} = 4$

$$\Rightarrow \sin^2 x = \frac{1}{2}$$

$$\Rightarrow \sin x = \pm\frac{1}{\sqrt{2}}$$

$\sin x = \dfrac{1}{\sqrt{2}} \Rightarrow x = 45°$ or $135°$

$\sin x = -\dfrac{1}{\sqrt{2}} \Rightarrow x = 225°$ or $315°$

[4 marks]

[Total: 6 marks]

9 $\dfrac{1}{\cos x} + \tan^2 x = 5$

$\dfrac{1}{\cos x} + \dfrac{\sin^2 x}{\cos^2 x} = 5$

$\Rightarrow \cos x + \sin^2 x = 5\cos^2 x$
$\Rightarrow 5\cos^2 x - (1 - \cos^2 x) - \cos x = 0$
$\Rightarrow 6\cos^2 x - \cos x - 1 = 0$
$\Rightarrow (3\cos x + 1)(2\cos x - 1) = 0$
$\Rightarrow \cos x = -\dfrac{1}{3}$ or $\cos x = \dfrac{1}{2}$

$\cos x = -\dfrac{1}{3} \Rightarrow x = 109.5°$ or $x = 250.5°$

$\cos x = \dfrac{1}{2} \Rightarrow x = 60°$ or $x = 300°$ [4 marks]

10 $\dfrac{1}{\sin^2 x} - \dfrac{1}{\tan^2 x} = \dfrac{1}{\sin^2 x} - \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{1-\cos^2 x}{\sin^2 x} = \dfrac{\sin^2 x}{\sin^2 x} = 1$ [3 marks]

7 Geometry II

Exercise 7.1 The area of a triangle

1 (i) $\frac{1}{2} \times 49 \times 34 \times \sin 74° = 800.7\,\text{cm}^2$ [2 marks]

(ii) included angle $180° - 81° - 81° = 18°$

$\therefore$ area $= \frac{1}{2} \times 3.5 \times 3.5 \times \sin 18° = 1.9\,\text{cm}^2$ [2 marks]

[Total: 4 marks]

2 (i) $\frac{1}{2} \times 15 \times x \times \sin 44° = 73 \quad \Rightarrow \quad x = \dfrac{73}{\frac{1}{2} \times 15 \times \sin 44°} = 14.$ [2 marks]

(ii) $\frac{1}{2} \times x^2 \times \sin 132° = 56 \quad \Rightarrow \quad x^2 = 150.7 \quad \Rightarrow \quad x = 12.3$ [2 marks]

(iii) $\frac{1}{2} \times 12 \times x \times \sin 71° = 74 \quad \Rightarrow \quad x = \dfrac{74}{\frac{1}{2} \times 12 \times \sin 71°} = 13.0$ [2 marks]

[Total: 6 marks]

3 $\frac{1}{2} \times 9 \times 9 \times \sin 60° = 35.1\,\text{cm}^2$ [2 marks]

4 $\frac{1}{2} \times x^2 \times \sin 60° = 23 \quad \Rightarrow \quad x^2 = \dfrac{23}{\frac{1}{2} \times \sin 60} = 53.1 \quad \Rightarrow \quad 7.3\,\text{cm}$ [2 marks]

5 (i) $\frac{1}{2} \times 12 \times 13 \times \sin x = 45 \quad \Rightarrow \quad \sin x = \frac{15}{26} \quad \Rightarrow \quad x = 35.2°$ or $144.8°$ [3 marks]

(ii) $\frac{1}{2} \times 11 \times 11 \times \sin\theta = 56 \quad \Rightarrow \quad \sin\theta = 0.926 \quad \Rightarrow \quad \theta = 67.8°$ or $112.2°$

$\therefore \quad x = \frac{1}{2}\left(180° - 67.8°\right) = 56.1$ or $x = \frac{1}{2}(180° - 112.2°) = 33.9°$ [4 marks]

[Total: 7 marks]

6 $\frac{1}{2} \times 2x \times (x+3) \times \sin 34° = 22 \quad \Rightarrow \quad x^2 + 3x = \dfrac{22}{\sin 34°} \quad \Rightarrow \quad x^2 + 3x - 39.3 = 0$

$\Rightarrow \quad x = \dfrac{-3 \pm \sqrt{3^2 - 4 \times 1 \times -39.3}}{2 \times 1} = 4.95$ or -7.95 but $x > 0 \quad \therefore \quad x = 4.9$ [4 marks]

7 $w + 2 = \frac{1}{2} \times w \times 8.2 \times \sin 41° \quad \Rightarrow \quad w + 2 = 2.69w \quad \Rightarrow \quad 2 = 1.69w \quad \Rightarrow \quad w = 1.2$ [2 marks]

8 $\frac{1}{2} \times (a+3)^2 \times \sin 22° = 2a + 1 \quad \Rightarrow \quad a^2 - 4.68a + 3.66 = 0$

$\Rightarrow \quad a = \dfrac{4.68 \pm \sqrt{4.68^2 - 4 \times 1 \times 3.66}}{2 \times 1} = 3.7$ or 1.0 [3 marks]

9 A regular hexagon comprises 6 equilateral triangles.

Triangle area $= \frac{1}{2} \times 8 \times 8 \sin 60 = 16\sqrt{3}$

Hexagon area $= 16\sqrt{3} \times 6 = 96\sqrt{3}$ [3 marks]

10 area of each equilateral triangle $= 12\sqrt{3} \div 6 = 2\sqrt{3}$

$\therefore \quad \frac{1}{2} \times x^2 \times \sin 60° = 2\sqrt{3} \quad \Rightarrow \quad x^2 = 8 \quad \Rightarrow \quad x = 2.8$ cm [3 marks]

11 $2 \times \frac{1}{2} \times 12 \times 7 \times \sin 105° = 81.1\,\text{m}^2$ [2 marks]

12 $2 \times \frac{1}{2} \times 8 \times 13 \times \sin\theta = 32 \quad \Rightarrow \quad \sin\theta = \frac{4}{13} \quad \Rightarrow \quad \theta = 17.9°$ and $162.1°$ [3 marks]

Exercise 7.2 The sine rule

1 (i) $\frac{x}{\sin 64°} = \frac{9}{\sin 37°} \quad \Rightarrow \quad x = \frac{9 \times \sin 64°}{\sin 37°} = 13.4$ cm [2 marks]

(ii) $\frac{x}{\sin 36°} = \frac{12}{\sin 88°} \quad \Rightarrow \quad x = \frac{12 \times \sin 36°}{\sin 88°} = 7.1$ cm [2 marks]

[Total: 4 marks]

2 (i) $\frac{y}{\sin 54°} = \frac{13}{\sin 68°} \quad \Rightarrow \quad y = \frac{13 \times \sin 54°}{\sin 68°} = 11.3$ cm [2 marks]

(ii) $\frac{y}{\sin 100°} = \frac{23}{\sin 38°} \quad \Rightarrow \quad y = \frac{23 \times \sin 100°}{\sin 38°} = 36.8$ cm [2 marks]

[Total: 4 marks]

3 (i) $\frac{\sin\theta}{18} = \frac{\sin 38°}{21} \quad \Rightarrow \quad \sin\theta = \frac{18 \times \sin 38°}{21} = 0.528 \quad \Rightarrow \quad \theta = 31.9°$ [3 marks]

(ii) $\frac{\sin\theta}{10} = \frac{\sin 97°}{17} \quad \Rightarrow \quad \sin\theta = \frac{10 \times \sin 97°}{17} = 0.584 \quad \Rightarrow \quad \theta = 35.7°$ [3 marks]

[Total: 6 marks]

4 (i) $\frac{\sin A}{8} = \frac{\sin 37°}{7} \quad \Rightarrow \quad \sin A = \frac{8 \times \sin 37°}{7} = 0.688 \quad \Rightarrow \quad A = 43.5°$ or $136.5°$ [4 marks]

(ii) $B = 180° - 136.5° - 37° = 6.5°$ [2 marks]

[Total: 6 marks]

5 (i) $\frac{PR}{\sin 58°} = \frac{9}{\sin 75°} \quad \Rightarrow \quad PR = \frac{9 \times \sin 58°}{\sin 75°} = 7.9$ cm [2 marks]

(ii) $\frac{1}{2} \times 9 \times 7.90 \times \sin 47° = 26.0\,\text{cm}^2$ [2 marks]

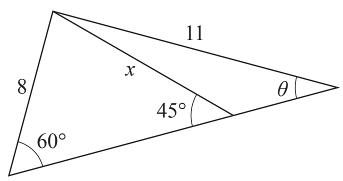
[Total: 4 marks]

6 (i) $\frac{\sin N}{12} = \frac{\sin 51°}{13} \quad \Rightarrow \quad \sin N = \frac{12 \times \sin 51°}{13} = 0.717 \quad \Rightarrow \quad N = 45.8°$ [3 marks]

(ii) $\frac{1}{2} \times 12 \times 13 \times \sin 83.2° = 77.4\ \text{cm}^2$ [2 marks]

[Total: 5 marks]

7



$$\frac{x}{\sin 60^\circ} = \frac{8}{\sin 45^\circ} \quad \Rightarrow \quad x = \frac{8 \times \frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = 4\sqrt{6}$$

$$\frac{\sin\theta}{4\sqrt{6}} = \frac{\sin 135^\circ}{11} \quad \Rightarrow \quad \sin\theta = \frac{4\sqrt{6} \times \frac{\sqrt{2}}{2}}{11} = \frac{4}{11}\sqrt{3}$$ [5 marks]

8 $\frac{XZ}{\sin 70^\circ} = \frac{10}{\sin 40^\circ} \quad \Rightarrow \quad XZ = \frac{10 \times \sin 70^\circ}{\sin 40^\circ} = 14.6 \quad \therefore \quad \frac{\sin\angle XWY}{14.6} = \frac{\sin 140^\circ}{23}$

$\Rightarrow \quad \sin\angle XWY = 0.409 \quad \Rightarrow \quad \angle XWY = 24.1^\circ \quad \therefore \quad \angle ZXW = 15.9^\circ$

Total area $= \frac{1}{2} \times 14.6 \times 23 \times \sin 15.9 = 46.0 \text{ cm}^2$ [7 marks]

9 $\frac{BD}{\sin 45^\circ} = \frac{15}{\sin 100^\circ} \quad \Rightarrow \quad BD = \frac{15 \times \sin 45}{\sin 100^\circ} = 10.$

$\therefore$ area of triangle $BCD = \frac{1}{2} \times 15 \times 10.8 \times \sin 35^\circ = 46.3$

$\therefore$ area of triangle $ABD = 100 - 46.3 = 53.7 \text{ cm}^2$ [5 marks]

10 $\frac{BD}{\sin 30} = \frac{11}{\sin 100} \quad \Rightarrow \quad BD = \frac{11 \sin 30}{\sin 100} = 5.58$

$\frac{\sin\theta}{5.58} = \frac{\sin(30+100)}{12} \quad \Rightarrow \quad \sin\theta = \frac{5.58 \sin 130}{12} \quad \Rightarrow \quad \theta = 20.9^\circ$ (1 d.p.) [4 marks]

Exercise 7.3 The cosine rule

1 **(i)** $x^2 = 12^2 + 10^2 - 2 \times 12 \times 10 \times \cos 48^\circ = 83.4 \quad \Rightarrow \quad x = 9.1 \text{ cm}$ [2 marks]

(ii) $x^2 = 16^2 + 13^2 - 2 \times 16 \times 13 \times \cos 51^\circ = 163.2 \quad \Rightarrow \quad x = 12.8 \text{ m}$ [2 marks]

[Total: 4 marks]

2 **(i)** $\cos\theta = \frac{21^2 + 25^2 - 29^2}{2 \times 21 \times 25} = \frac{3}{14} \quad \Rightarrow \quad \theta = 77.6^\circ$ [2 marks]

(ii) $\cos\theta = \frac{11^2 + 18^2 - 27^2}{2 \times 11 \times 18} = -\frac{71}{99} \quad \Rightarrow \quad \theta = 135.8^\circ$ [2 marks]

[Total: 4 marks]

3 **(i)** The largest angle is opposite the longest side

$\therefore \quad \cos\theta = \frac{9^2 + 12^2 - 13^2}{2 \times 9 \times 12} = \frac{7}{27} \quad \Rightarrow$ largest angle $= 75.0^\circ$ [3 marks]

(ii) The smallest angle is opposite the shortest side

$\therefore \quad \cos\theta = \frac{13^2 + 12^2 - 9^2}{2 \times 13 \times 12} = \frac{29}{39} \quad \Rightarrow$ smallest angle $= 42.0^\circ$ [3 marks]

[Total: 6 marks]

4 **(i)** $PR^2 = 7.3^2 + 8.6^2 - 2 \times 7.3 \times 8.6 \times \cos 56^\circ = 57.0 \quad \Rightarrow \quad PR = 7.6 \text{ cm}$ [2 marks]

(ii) $\frac{1}{2} \times 7.3 \times 8.6 \times \sin 56^\circ = 26.0 \text{ cm}^2$ [2 marks]

[Total: 4 marks]

5 (i) $10^2 = x^2 + 11^2 - 2\times x\times 11\times\cos 60° \quad\Rightarrow\quad 100 = x^2 + 121 - 11x$

$\Rightarrow\quad x^2 - 11x + 21 = 0$ [2 marks]

(ii) $\Rightarrow\quad x = \dfrac{11\pm\sqrt{11^2 - 4\times 1\times 21}}{2\times 1} = \dfrac{11\pm\sqrt{37}}{2} \quad\therefore\quad \text{area} = \dfrac{1}{2}\times 11\times\dfrac{11\pm\sqrt{37}}{2}\times\sin 60°$

$\Rightarrow\quad \text{area} = 40.7\ \text{m}^2 \quad\text{or}\quad 11.7\ \text{m}^2$ [3 marks]

[Total: 5 marks]

6 (i) $\sqrt{7^2 + 7^2 - 2\times 7\times 7\times\cos 110°} = 11.5\ \text{cm}$ [2 marks]

(ii) $\sqrt{7^2 + 7^2 - 2\times 7\times 7\times\cos 70°} = 8.0\,\text{cm}$ [2 marks]

(iii) $\dfrac{1}{2}\times 11.5\times 8.03 = 46.0\,\text{cm}^2$ [2 marks]

[Total: 6 marks]

7 $\sqrt{3^2 + 7^2 - 2\times 3\times 7\times\cos 130°} = 9.2\ \text{km}$ [2 marks]

8 $(8x+1)^2 = (8x)^2 + (x+1)^2 - 2\times 8x\times(x+1)\times\cos 150°$

$\Rightarrow\quad 64x^2 + 16x + 1 = 64x^2 + x^2 + 2x + 1 - 16x(x+1)\times -\dfrac{\sqrt{3}}{2}$

$\Rightarrow\quad 0 = x^2 - 14x + 8x(x+1)\sqrt{3} \quad\Rightarrow\quad x = 0 \quad\text{or}\quad 0 = x - 14 + 8(x+1)\sqrt{3}$

but $8x$ is a side length $\therefore x > 0 \quad\therefore\quad 14 - 8\sqrt{3} = x + 8x\sqrt{3}$

$\Rightarrow\quad x = \dfrac{14 - 8\sqrt{3}}{1 + 8\sqrt{3}} = \dfrac{1}{191}\left(120\sqrt{3} - 206\right)$ [5 marks]

9 $\cos(\text{PQS}) = \dfrac{6^2 + 8^2 - 7^2}{2\times 6\times 8} \quad\Rightarrow$ angle PQS = 57.91 so angle RQS = 122.09

$\text{RS}^2 = 6^2 + 5^2 - 2\times 6\times 5\times\cos(122.09) \quad\Rightarrow\quad \text{RS} = 9.64\,\text{cm}$ (2 d.p.) [4 marks]

Exercise 7.4 Using the sine and cosine rules together

1 (i) $\text{AC}^2 = 12^2 + 14^2 - 2\times 12\times 14\times\cos 60° = 172 \quad\Rightarrow\quad \text{AC} = 13.1\ \text{cm}$ [2 marks]

(ii) $\cos\text{C} = \dfrac{14^2 + 13.1^2 - 12^2}{2\times 14\times 13.1} = \dfrac{4}{\sqrt{43}} \quad\Rightarrow\quad \text{C} = 52.4°$ [2 marks]

(iii) $\dfrac{1}{2}\times 12\times 14\times\sin 60° = 72.7\,\text{cm}^2$ [2 marks]

[Total: 6 marks]

2 (i) $\dfrac{\text{EF}}{\sin 71°} = \dfrac{9.5}{\sin 55°} \quad\Rightarrow\quad \text{EF} = \dfrac{9.5\times\sin 71°}{\sin 55°} = 11.0\,\text{cm}$ [2 marks]

(ii) $\dfrac{\text{DF}}{\sin 54°} = \dfrac{9.5}{\sin 55°} \quad\Rightarrow\quad \text{DF} = \dfrac{9.5\times\sin 54°}{\sin 55°} = 9.4\,\text{cm}$ [2 marks]

(iii) $\dfrac{1}{2}\times 9.5\times 11.0\times\sin 54° = 42.1\,\text{cm}^2$ [2 marks]

[Total: 6 marks]

3 $\cos\theta = \dfrac{7.3^2 + 8.4^2 - 9.3^2}{2\times 7.3\times 8.4} = 0.305 \quad\Rightarrow\quad \theta = 72.3$

$\therefore\quad \text{area} = \dfrac{1}{2}\times 7.3\times 8.4\times\sin 72.3° = 29.2\,\text{cm}^2$ [4 marks]

4 **(i)** $\sqrt{4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 135°} = 9.3\,\text{km}$ [2 marks]

(ii) $\cos\theta = \dfrac{4^2 + 9.27^2 - 6^2}{2 \times 4 \times 9.27} = 0.889 \quad \Rightarrow \quad \theta = 27.2°$

$\therefore$ bearing $= 65 + 27.2 = 092.2°$ [3 marks]

[Total: 5 marks]

5 **(i)** $\sqrt{5.3^2 + 7.2^2 - 2 \times 5.3 \times 7.2 \times \cos 78°} = 8.0\,\text{cm}$ [2 marks]

(ii) $2 \times \dfrac{1}{2} \times 5.3 \times 7.2 \times \sin 78° = 37.3\,\text{cm}^2$ [2 marks]

[Total: 4 marks]

6 $\dfrac{1}{2} \times 6.7 \times 8.3 \times \sin\theta = 23 \quad \Rightarrow \quad \sin\theta = 0.827 \quad \Rightarrow \quad \theta = 55.8°$

$x^2 = 6.7^2 + 8.3^2 - 2 \times 6.7 \times 8.3 \times \cos 55.8° = 51.3 \quad \Rightarrow \quad x = 7.2$ cm [5 marks]

7 **(i)** $4 \times \dfrac{1}{2} \times 3.6 \times 4.65 \times \sin 61° = 29.3\,\text{cm}^2$ [2 marks]

(ii) $\sqrt{3.6^2 + 4.65^2 - 2 \times 3.6 \times 4.65 \times \cos 119°} = 7.1\,\text{cm}$ [2 marks]

[Total: 4 marks]

8 **(i)** $\text{AC} = \sqrt{7^2 + 8^2 - 2 \times 7 \times 8 \times \cos 95°} = 11.1$

$\dfrac{\text{AD}}{\sin 84°} = \dfrac{11.1}{\sin 53°} \quad \Rightarrow \quad \text{AD} = \dfrac{11.1 \times \sin 84°}{\sin 53°} = 13.8$

Total area = triangle ABC + triangle ACD

$= \dfrac{1}{2} \times 8 \times 7 \times \sin 95° + \dfrac{1}{2} \times 11.1 \times 13.8 \times \sin 43° = 27.9 + 52.1 = 80.0\ \text{cm}^2$ [5 marks]

(ii) $\dfrac{\sin \text{B}\hat{\text{A}}\text{C}}{7} = \dfrac{\sin 95°}{11.1} \quad \Rightarrow \quad \sin \text{B}\hat{\text{A}}\text{C} = \dfrac{7 \times \sin 95°}{11.1} \quad \Rightarrow \quad \text{B}\hat{\text{A}}\text{C} = 39.0°$

$\text{BD}^2 = 8^2 + 13.8^2 - 2 \times 8 \times 13.8 \times \cos 82.0° \quad \Rightarrow \quad \text{BD} = 15.0\,\text{m}$ [4 marks]

[Total: 9 marks]

9 Angle QPR = angle PRS = 31° and angle QRP = $180° - (123° + 31°) = 26°$

$\dfrac{\text{PR}}{\sin 123°} = \dfrac{6}{\sin 26°} \quad \Rightarrow \quad \text{PR} = \dfrac{6 \sin 123°}{\sin 26°} = 11.5$

Trapezium area $= \dfrac{1}{2} \times 13 \times 11.5 \times \sin 31° + \dfrac{1}{2} \times 6 \times 11.5 \times \sin 31° = 56.2\,\text{cm}^2$ (1 d.p.) [5 marks]

Exercise 7.5 Problems in three dimensions

1 **(i)** $\text{BD}^2 = 12^2 + 3^2 = 153 \quad \Rightarrow \quad \text{BD} = 12.4$ cm [2 marks]

(ii) $\text{HC}^2 = 12.4^2 + 4^2 = 169 \quad \Rightarrow \quad \text{HC} = 13$ cm [2 marks]

(iii) $\tan \text{C}\hat{\text{A}}\text{F} = \dfrac{4}{\sqrt{153}} \quad \Rightarrow \quad \text{C}\hat{\text{A}}\text{F} = 17.9°$ [2 marks]

[Total: 6 marks]

2 **(i)** $\cos A\hat{B}F = \dfrac{17^2 + 18^2 - 8^2}{2 \times 17 \times 18} = \dfrac{61}{68} \quad \Rightarrow \quad A\hat{B}F = 26.2°$ [2 marks]

(ii) $\cos A\hat{F}B = \dfrac{8^2 + 18^2 - 17^2}{2 \times 8 \times 18} = \dfrac{11}{32} \quad \Rightarrow \quad A\hat{B}F = 69.9°$ [2 marks]

(iii) $BD^2 = 10^2 + 17^2 = 389 \quad \Rightarrow \quad BD = \sqrt{389}$

$BE^2 = 10^2 + 18^2 = 424 \quad \Rightarrow \quad BE = \sqrt{424}$

$\cos D\hat{B}E = \dfrac{389 + 424 - 8^2}{2 \times \sqrt{389} \times \sqrt{424}} = 0.922 \quad \Rightarrow \quad D\hat{B}E = 22.8°$

area BDE $= \dfrac{1}{2} \times \sqrt{389} \times \sqrt{424} \times \sin 22.8° = 78.6\ \text{cm}^2$ [5 marks]

[Total: 9 marks]

3 $\dfrac{1}{3}\pi r^2 h = \dfrac{4}{3}\pi r^3 \quad \Rightarrow \quad h = 4r \quad \Rightarrow \quad \dfrac{h}{r} = 4 \quad \Rightarrow \quad h : r = 4 : 1$ [3 marks]

4 **(i)** Let M be the centre of ABCD

$BM^2 = 3.5^2 + 2^2 = 16.25 \quad \Rightarrow \quad BM = \sqrt{16.25}$

$EM^2 = 8^2 - 16.25 = 47.75 \quad \Rightarrow \quad EM = 6.9\ \text{cm}$ [4 marks]

(ii) $\cos E\hat{A}M = \dfrac{\sqrt{16.25}}{8} \quad \Rightarrow \quad E\hat{A}M = 59.7°$ [2 marks]

(iii) $\tan\theta = \dfrac{6.9}{3.5} \quad \Rightarrow \quad \theta = 63.1°$ [2 marks]

(iv) Let X be the midpoint of AB $\therefore$ $EX^2 = 8^2 - 3.5^2 = 51.75 \quad \Rightarrow \quad EX = 7.19$

$\cos\alpha = \dfrac{7.19^2 + 7.19^2 - 4^2}{2 \times 7.19 \times 7.19} = \dfrac{175}{207} \quad \Rightarrow \quad \alpha = 32.3°$ [4 marks]

[Total: 12 marks]

5 Let X be the midpoint of BC, and let $BC = 2x$

$AX^2 = (2x)^2 - x^2 = 3x^2 \ \therefore\ \tan A\hat{X}D = \dfrac{2x}{x\sqrt{3}} = \dfrac{2}{\sqrt{3}} \quad \Rightarrow \quad A\hat{X}D = 49.1°$ [3 marks]

6 **(i)** $LP = LR = PR$ $\therefore$ LPR is an equilateral triangle

$LR^2 = 10^2 + 10^2 \quad \Rightarrow \quad LR = \sqrt{200}$

area $= \dfrac{1}{2} \times \sqrt{200} \times \sqrt{200} \times \sin 60° = 50\sqrt{3} = 86.6\ \text{cm}^2$ [4 marks]

(ii) Let X be the midpoint of PR

$\tan L\hat{X}Q = \dfrac{10}{\frac{1}{2}\sqrt{200}} = \sqrt{2} \quad \Rightarrow \quad L\hat{X}Q = 54.7°$ [2 marks]

[Total: 6 marks]

7 **(i)** $AC^2 = 12^2 + 9^2 \quad \Rightarrow \quad AC = \sqrt{225} = 15\ \text{cm}$ [1 mark]

(ii) $\tan C\hat{A}G = \dfrac{5}{15} \quad \Rightarrow \quad C\hat{A}G = 18.4°$ [2 marks]

(iii) $AF^2 = 12^2 + 5^2 \quad \Rightarrow \quad AF = \sqrt{169} = 13\ \text{cm}$ [1 mark]

(iv) $\tan F\hat{A}G = \dfrac{9}{13} \quad \Rightarrow \quad F\hat{A}G = 34.7°$ [2 marks]

[Total: 5 marks]

8 **(i)** $\tan C\hat{B}E = \dfrac{CE}{BC} \quad \therefore \quad \tan 15° = \dfrac{CE}{12} \quad \Rightarrow \quad CE = 12 \times \tan 15° = 3.2 \text{ cm}$ [1 mark]

(ii) $\dfrac{1}{2} \times 12 \times 3.2 \times 6 = 115.8 \text{ cm}^3$ [1 mark]

(iii) Two wedges can be put together to form a cuboid 6 cm × 12 cm × 3.2 cm

30 cm ÷ 6 cm = 5; 10 cm ÷ 3.2 cm = 3; 192 cm ÷ 12 cm = 16

So the number of small cuboids is $5 \times 3 \times 16 = 240$.

This leaves a block of dimensions 30 cm × 10 cm × 8 cm

30 cm ÷ 12 cm = 2; 10 cm ÷ 3.2 cm = 3; 8cm ÷ 6 cm = 1

So another $2 \times 3 \times 1 = 6$ cuboids can be cut.

So the number of wedges is $(240 + 6) \times 2 = 492$ [4 marks]

[Total: 6 marks]

9 **(i)** $AC^2 = 8^2 + 8^2 \quad \Rightarrow \quad AC = \sqrt{128}$

$\cos C\hat{A}V = \dfrac{12^2 + \sqrt{128}^2 - 12^2}{2 \times 12 \times \sqrt{128}} = \dfrac{\sqrt{2}}{3} \quad \Rightarrow \quad C\hat{A}V = 61.9°$ [4 marks]

(ii) Let h be the perpendicular height of the pyramid $\therefore \quad h^2 = 12^2 - \left(\dfrac{1}{2}\sqrt{128}\right)^2$

$\Rightarrow \quad h = \sqrt{112} \quad \therefore \quad \tan\theta = \dfrac{\sqrt{112}}{4} = \sqrt{7} \quad \Rightarrow \quad \theta = 69.3°$ [4 marks]

[Total: 8 marks]

10 (i) $AC^2 = 10^2 + 10^2 \quad \Rightarrow \quad AC = \sqrt{200} = 14.1 \text{ cm}$ [1 mark]

(ii) $EG^2 = 15^2 + 15^2 \quad \Rightarrow \quad EG = \sqrt{450} = 21.2 \text{ cm}$ [1 mark]

(iii) $AE^2 = 20^2 + \left(\dfrac{1}{2}(21.2 - 14.1)\right)^2 \quad \Rightarrow \quad AE = \dfrac{5}{2}\sqrt{66} = 20.3 \text{ cm}$ [2 marks]

(iv) $\sin\theta = \dfrac{20}{20.3} \quad \Rightarrow \quad \theta = 80.0°$ [2 marks]

[Total: 6 marks]

11 (i) $\tan\theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta = 26.6°$ [1 mark]

(ii) $\dfrac{1}{2}(2 + 3)2 \times 4 = 20 \text{ m}^3$ [1 mark]

(iii) The rod must be no longer than the distance from the highest point at the back of the shed to the left-hand side of the bottom of the door.

$\therefore$ max length $= \sqrt{1.4^2 + 3^2 + 4^2} = 5.19 = 5.1$ m (1 d.p.) – can only round down [3 marks]

[Total: 5 marks]

12 (i) $\sin 15° = \dfrac{\text{height}}{100} \quad \Rightarrow \quad \text{height} = 100 \times \sin 15° = 25.9 \text{ m}$ [2 marks]

(ii) $\sin 5° = \dfrac{25.9}{\text{distance}} \quad \Rightarrow \quad \text{distance} = \dfrac{25.9}{\sin 5°} = 297.0 \text{ m}$ [3 marks]

[Total: 5 marks]

13 (i) $DF^2 = 10^2 + 25^2 + 2.5^2 \quad \Rightarrow \quad DF = 27.0\text{ m}$ [1 mark]

(ii) $\tan\theta = \dfrac{1.5}{25} \quad \Rightarrow \quad \theta = 3.4°$ [1 mark]

(iii) $FH^2 = 1.5^2 + 10^2 + 25^2 \quad \Rightarrow \quad FH = 26.968$

$\cos D\hat{H}F = \dfrac{1^2 + 26.968^2 - 27.0^2}{2\times 1\times 26.968} = -\dfrac{3}{\sqrt{2909}} \quad \Rightarrow \quad D\hat{H}F = 93.2°$ [4 marks]

(iv) $\tan 3.4° = \dfrac{0.4}{x} \quad \Rightarrow \quad x = \dfrac{0.4}{\tan 3.4°} = \dfrac{20}{3}$

$\therefore$ greatest distance from DH is $\sqrt{10^2 + \left(\dfrac{20}{3}\right)^2} = 12.0\text{ m}$ [1 mark]

(v) $\dfrac{1}{2}(1+2.5)25\times 10 = 437.5\text{ m}^3$ [1 mark]

[Total: 8 marks]

14 (i) The least distance occurs when the camera is directly in front of Jim:

$\therefore$ least distance $= \sqrt{3^2 + 0.95^2} = 3.1\text{ m}$

The greatest distance occurs when the camera is at the end of the track:

$\therefore$ greatest distance $= \sqrt{2^2 + 3^2 + 0.95^2} = 3.7\text{ m}$ [3 marks]

(ii) The greatest angle of elevation occurs when the camera is directly in front of Jim:

$\therefore \quad \tan\theta = \dfrac{0.95}{3} \quad \Rightarrow \quad \theta = 17.6°$

The least angle of elevation occurs when the camera is at the end of the track:

$\therefore \quad \sin\alpha = \dfrac{0.95}{3.7} \quad \Rightarrow \quad \alpha = 14.8°$ [4 marks]

15 (i) From point A: $\tan 5° = \dfrac{80}{x} \quad \Rightarrow \quad x = \dfrac{80}{\tan 5°} = 914.4\text{ m}$

From point B: $\tan 6° = \dfrac{80}{y} \quad \Rightarrow \quad y = \dfrac{80}{\tan 6°} = 761.1\text{ m}$ [3 marks]

(ii) Let D = centre of lighthouse base, and C = closest point to lighthouse

$\cos A\hat{B}D = \dfrac{200^2 + 761.1^2 - 914.4^2}{2\times 200\times 761.1} = -0.712 \quad \Rightarrow \quad A\hat{B}D = 135.4°$

$\therefore \quad C\hat{B}D = 180 - 135.4 = 44.6 \quad \therefore \quad \sin 44.6 = \dfrac{CD}{761.1} \quad \Rightarrow \quad CD = 534.4\text{ m}$ [6 marks]

(iii) $\tan\theta = \dfrac{80}{534.4} \quad \Rightarrow \quad \theta = 8.5°$ [2 marks]

8 Calculus

Exercise 8.1 Differentiation

1 **(i)** $\dfrac{dy}{dx} = 12x^2$ [1 mark]

(ii) $\dfrac{dy}{dx} = 6x$ [1 mark]

(iii) $\dfrac{dy}{dx} = 4$ [1 mark]

(iv) $\dfrac{dy}{dx} = -42x^6$ [1 mark]

(v) $\dfrac{dy}{dx} = -42x^5$ [1 mark]

(vi) $\dfrac{dy}{dx} = 0$ [1 mark]

(vii) $\dfrac{dy}{dx} = x$ [1 mark]

(viii) $\dfrac{dy}{dx} = x^2$ [1 mark]

(ix) $\dfrac{dy}{dx} = x^3$ [1 mark]

[Total: 9 marks]

2 **(i)** $y = 2x^3 + 3x^2 \Rightarrow \dfrac{dy}{dx} = 6x^2 + 6x$ [1 mark]

(ii) $y = 3x^4 + 4x^3 \Rightarrow \dfrac{dy}{dx} = 12x^3 + 12x^2$ [1 mark]

(iii) $y = 4x^5 + 5x^4 \Rightarrow \dfrac{dy}{dx} = 20x^4 + 20x^3$ [1 mark]

(iv) $y = 3x + 5 \Rightarrow \dfrac{dy}{dx} = 3$ [1 mark]

(v) $y = 2x - 4 \Rightarrow \dfrac{dy}{dx} = 2$ [1 mark]

(vi) $y = 5 - 4x \Rightarrow \dfrac{dy}{dx} = -4$ [1 mark]

(vii) $y = 3x^3 + 2 \Rightarrow \dfrac{dy}{dx} = 9x^2$ [1 mark]

(viii) $y = 4x^4 + 4 \Rightarrow \dfrac{dy}{dx} = 16x^3$ [1 mark]

(ix) $y = 5x^5 + 5 \Rightarrow \dfrac{dy}{dx} = 25x^4$ [1 mark]

[Total: 9 marks]

3 **(i)** $y = 2x^4 - 4x^2 - 8x + 8 \quad \Rightarrow \quad \dfrac{dy}{dx} = 8x^3 - 8x - 8$ [1 mark]

(ii) $y = 3x^4 - 4x^3 + 12x - 12 \quad \Rightarrow \quad \dfrac{dy}{dx} = 12x^3 - 12x^2 + 12$ [1 mark]

(iii) $y = 2x^5 + 5x^2 - 10x + 25 \quad \Rightarrow \quad \dfrac{dy}{dx} = 10x^4 + 10x - 10$ [1 mark]

[Total: 3 marks]

4 **(i)** $y = x - \dfrac{2}{x} \quad \Rightarrow \quad y = x - 2x^{-1}$

$\Rightarrow \dfrac{dy}{dx} = 1 + 2x^{-2} = 1 + \dfrac{2}{x^2}$ [2 marks]

(ii) $y = \dfrac{x^2}{2} - \dfrac{2}{x^2} \quad \Rightarrow \quad y = \dfrac{x^2}{2} - 2x^{-2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{2} + 4x^{-3} = x + \dfrac{4}{x^3}$ [2 marks]

(iii) $y = \dfrac{x^3}{3} - \dfrac{3}{x^3} \quad \Rightarrow \quad y = \dfrac{x^3}{3} - 3x^{-3}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2}{3} + 9x^{-4} = x^2 + \dfrac{9}{x^4}$ [2 marks]

(iv) $y = x^3 - x^{-3} \quad \Rightarrow \quad \dfrac{dy}{dx} = 3x^2 + 3x^{-4} = 3x^2 + \dfrac{3}{x^4}$ [2 marks]

(v) $y = 2x^4 - 4x^{-2} \quad \Rightarrow \quad \dfrac{dy}{dx} = 8x^3 + 8x^{-3} = 8x^3 + \dfrac{8}{x^3}$ [2 marks]

(vi) $y = 4x^5 + 5x^{-4} \quad \Rightarrow \quad \dfrac{dy}{dx} = 20x^4 - 20x^{-5} = 20x^4 - \dfrac{20}{x^5}$ [2 marks]

[Total: 12 marks]

5 **(i)** $y = 4x^3 - \dfrac{3}{x^4} \quad \Rightarrow \quad y = 4x^3 - 3x^{-4}$

$\Rightarrow \dfrac{dy}{dx} = 12x^2 + 12x^{-5} = 12x^2 + \dfrac{12}{x^5}$ [2 marks]

(ii) $y = x - \dfrac{1}{x} \quad \Rightarrow \quad y = x - x^{-1}$

$\Rightarrow \dfrac{dy}{dx} = 1 + 1x^{-2} = 1 + \dfrac{1}{x^2}$ [2 marks]

(iii) $y = 2x^2 - \dfrac{1}{2x} \quad \Rightarrow \quad y = 2x^2 - \dfrac{1}{2}x^{-1}$

$\Rightarrow \dfrac{dy}{dx} = 4x + \dfrac{1}{2}x^{-2} = 4x + \dfrac{1}{2x^2}$ [2 marks]

(iv) $y = \dfrac{3}{x^2} + \dfrac{2}{x^3} \quad \Rightarrow \quad y = 3x^{-2} + 2x^{-3}$

$\Rightarrow \dfrac{dy}{dx} = -6x^{-3} - 6x^{-4} = -\dfrac{6}{x^3} - \dfrac{6}{x^4}$ [2 marks]

(v) $y = \frac{2}{5x^2} - \frac{5}{2x^5} \quad \Rightarrow \quad y = \frac{2}{5}x^{-2} - \frac{5}{2}x^{-5}$

$\Rightarrow \frac{dy}{dx} = -\frac{4}{5}x^{-3} + \frac{25}{2}x^{-6} = \frac{-4}{5x^3} + \frac{25}{2x^6}$ [2 marks]

(vi) $y = \frac{4}{x^3} - \frac{3}{x^4} \quad \Rightarrow \quad y = 4x^{-3} - 3x^{-4}$

$\Rightarrow y = -12x^{-4} + 12x^{-5} = \frac{-12}{x^4} + \frac{12}{x^5}$ [2 marks]

[Total: 12 marks]

6 **(i)** $V = 2x \times 3x \times 4x = 24x^3$

$A = 2(2x \times 3x) + 2(2x \times 4x) + 2(3x \times 4x)$

$= 12x^2 + 16x^2 + 24x^2$

$= 52x^2$ [2 marks]

(ii) $V = 24x^3 \quad \Rightarrow \quad \frac{dV}{dx} = 72x^2$

$A = 52x^2 \quad \Rightarrow \quad \frac{dA}{dx} = 104x$ [2 marks]

[Total: 4 marks]

7 **(i)** $y = x^2 - 4x + 3$

$= (x - 1)(x - 3)$

Curve intersects x-axis when $x = 1$ and $x = 3$

$y = x^2 - 4x + 3 \quad \Rightarrow \quad \frac{dy}{dx} = 2x - 4$

At $(1, 0)$, $\frac{dy}{dx} = 2 - 4 = -2$

At $(3, 0)$, $\frac{dy}{dx} = 6 - 4 = 2$ [3 marks]

(ii) Curve intersects y-axis when $x = 0$

At $(0, 3)$ $\frac{dy}{dx} = -4$ [1 mark]

[Total: 4 marks]

8 $y = 2x^3 + 3x^2 - 12x - 9 \quad \Rightarrow \quad \frac{dy}{dx} = 6x^2 + 6x - 12$

$\frac{dy}{dx} = 0$ when $6(x^2 + x - 2) = 0 \quad \Rightarrow \quad (x + 2)(x - 1) = 0$

$\Rightarrow \quad x = -2$ or $x = 1$

$x = -2 \quad \Rightarrow \quad y = 2(-2)^3 + 3(-2)^2 - 12(-2) - 9 = 11$ i.e. $(-2, 11)$

$x = 1 \quad \Rightarrow \quad y = 2(1)^3 + 3(1)^2 - 12(1) - 9 = -16$ i.e. $(1, -16)$ [4 marks]

9 $\frac{ds}{dt} = 5 + 14t = 12 \quad \Rightarrow \quad t = \frac{1}{2}$ so $s = 5\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right)^2 = 4.25\,\text{m}$ [4 marks]

Exercise 8.2 Gradient functions and more complex differentiation

1 (i) $y = x^2(2x - 3) \Rightarrow y = 2x^3 - 3x^2$

$\Rightarrow \frac{dy}{dx} = 6x^2 - 6x$ [2 marks]

(ii) $y = 3x(2x^2 - 5) \Rightarrow y = 6x^3 - 15x$

$\Rightarrow \frac{dy}{dx} = 18x^2 - 15$ [2 marks]

(iii) $y = (x - 2)(x + 5) \Rightarrow y = x^2 + 3x - 10$

$\Rightarrow \frac{dy}{dx} = 2x + 3$ [2 marks]

(iv) $y = (x - 4)(x + 3) \Rightarrow y = x^2 - x - 12$

$\Rightarrow \frac{dy}{dx} = 2x - 1$ [2 marks]

(v) $y = (2x - 1)(3x + 2) \Rightarrow y = 6x^2 + x - 2$

$\Rightarrow \frac{dy}{dx} = 12x + 1$ [2 marks]

(vi) $y = (3x^2 - 2)(2x + 5) \Rightarrow y = 6x^3 + 15x^2 - 4x - 10$

$\Rightarrow \frac{dy}{dx} = 18x^2 + 30x - 4$ [2 marks]

[Total: 12 marks]

2 (i) $(x - 3)(2x^2 + 5) = 2x^3 - 6x^2 + 5x - 15$ [1 mark]

(ii) $y = 2x^3 - 6x^2 + 5x - 15 \Rightarrow \frac{dy}{dx} = 6x^2 - 12x + 5$ [1 mark]

(iii) She has not multiplied out the brackets before differentiating. [1 mark]

[Total: 3 marks]

3 (i) $\frac{4x^3 + 2x}{x} = \frac{4x^3}{x} + \frac{2x}{x} = 4x^2 + 2$ [1 mark]

(ii) $y = 4x^2 + 2 \Rightarrow \frac{dy}{dx} = 8x$ [1 mark]

[Total: 2 marks]

4 (i) $y = \frac{2x^2 + 3x}{5} \Rightarrow y = \frac{2x^2}{5} + \frac{3x}{5}$

$\Rightarrow \frac{dy}{dx} = \frac{4x}{5} + \frac{3}{5}$ [2 marks]

(ii) $y = \frac{2x^3 - 4x^2}{2x} \Rightarrow y = \frac{2x^3}{2x} - \frac{4x^2}{2x} = x^2 - 2x$

$\Rightarrow \frac{dy}{dx} = 2x - 2$ [2 marks]

(iii) $y = \frac{4x^3 - 3x^4}{x^2} \Rightarrow y = \frac{4x^3}{x^2} - \frac{3x^4}{x^2} = 4x - 3x^2$

$\Rightarrow \frac{dy}{dx} = 4 - 6x$ [2 marks]

(iv) $y = \frac{x^3}{3}\left(3x^2 + 2x - 3\right) \quad \Rightarrow \quad y = x^5 + \frac{2}{3}x^4 - x^3$

$\Rightarrow \frac{dy}{dx} = 5x^4 + \frac{8}{3}x^3 - 3x^2$ [2 marks]

(v) $y = x^{\frac{1}{2}}\left(3x^{\frac{3}{2}} + 4x^{\frac{5}{2}}\right) \quad \Rightarrow \quad y = 3x^2 + 4x^3$

$\Rightarrow \frac{dy}{dx} = 6x + 12x^2$ [2 marks]

(vi) $y = 2x^{\frac{3}{2}}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) \quad \Rightarrow y = 2x^2 - 2x$

$\Rightarrow \frac{dy}{dx} = 4x - 2$ [2 marks]

[Total: 12 marks]

5 $y = x^2(2x - 1) \quad \Rightarrow \quad y = 2x^3 - x^2$

$\Rightarrow \frac{dy}{dx} = 6x^2 - 2x$

(i) At $(0, 0)$, $\frac{dy}{dx} = 0$ [1 mark]

(ii) At $(-1, -3)$, $\frac{dy}{dx} = 6(-1)^2 - 2(-1) = 8$ [1 mark]

(iii) At $(2, 12)$, $\frac{dy}{dx} = 6(2)^2 - 2(2) = 20$ [1 mark]

[Total: 3 marks]

6 **(i)** $y = x^2 - 3x \Rightarrow \frac{dy}{dx} = 2x - 3$

When $x = 2$, $\frac{dy}{dx} = 4 - 3 = 1$ [2 marks]

(ii) $y = x^2 - 6x + 10 \Rightarrow \frac{dy}{dx} = 2x - 6$

When $x = 2$, $\frac{dy}{dx} = 4 - 6 = -2$ [2 marks]

(iii) $y = 3x^4 - 2x + 4 \Rightarrow \frac{dy}{dx} = 12x^3 - 2$

When $x = 2$, $\frac{dy}{dx} = 12(8) - 2 = 94$ [2 marks]

[Total: 6 marks]

7 **(i)** $y = 3x^3 - 2x + 4 \Rightarrow \frac{dy}{dx} = 9x^2 - 2$

When $x = -3$, $\frac{dy}{dx} = 9(-3)^2 - 2 = 79$ [2 marks]

(ii) $y = \frac{1}{2}x^2 - 3x - 1 \Rightarrow \frac{dy}{dx} = x - 3$

When $x = -3$, $\frac{dy}{dx} = -3 - 3 = -6$ [2 marks]

(iii) $y = \frac{2}{3}x^3 + \frac{3}{2}x^2 \Rightarrow \frac{dy}{dx} = 2x^2 + 3x$

When $x = -3$, $\frac{dy}{dx} = 2(9) + 3(-3) = 9$ [2 marks]

[Total: 6 marks]

8 $y = \dfrac{2x^2 - 3x^3}{x} \Rightarrow y = \dfrac{2x^2}{x} - \dfrac{3x^3}{x} = 2x - 3x^2$

$\dfrac{dy}{dx} = 2 - 6x$

At $(-1, -5)$, $\dfrac{dy}{dx} = 2 - 6(-1) \Rightarrow \dfrac{dy}{dx} = 8$ [3 marks]

9 $y = 2x\sqrt{x} + \dfrac{2}{\sqrt{x}} \Rightarrow y = 2x \times x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 2x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = 2\left(\dfrac{3}{2}\right)x^{\frac{1}{2}} + 2\left(-\dfrac{1}{2}\right)x^{-\frac{3}{2}} = 3x^{\frac{1}{2}} - x^{-\frac{3}{2}}$

At $(1, 4)$, $\dfrac{dy}{dx} = 3(1)^{\frac{1}{2}} - \left(1^{-\frac{3}{2}}\right) = 2$ [3 marks]

10 $y = 2x^3 - 3x^2 - 36x + 10 \Rightarrow \dfrac{dy}{dx} = 6x^2 - 6x - 36$

$= 6(x^2 - x - 6)$

$= 6(x - 3)(x + 2)$

$\dfrac{dy}{dx} = 0 \Rightarrow x = 3$ or $x = -2$

When $x = 3$, $y = 2(3)^3 - 3(3)^2 - 36(3) + 10 = -71$

When $x = -2$, $y = 2(-2)^3 - 3(-2)^2 - 36(-2) + 10 = 54$

Points are $(-2, 54)$ and $(3, -71)$ [4 marks]

11 $y = (x + 2)(x - 1)(x - 4)$

$= (x + 2)(x^2 - 5x + 4)$

$= x^3 - 3x^2 - 6x + 8$

$\dfrac{dy}{dx} = 3x^2 - 6x - 6$

Intersects x-axis when $x = -2 \Rightarrow \dfrac{dy}{dx} = 3(-2)^2 - 6(-2) - 6$

$= 18$

$x = 1 \Rightarrow \dfrac{dy}{dx} = 3(1)^2 - 6(1) - 6$

$= -9$

$x = 4 \Rightarrow \dfrac{dy}{dx} = 3(4)^2 - 6(4) - 6$

$= 18$

Intersects y-axis when $x = 0 \Rightarrow \dfrac{dy}{dx} = -6$ [6 marks]

12 $\dfrac{dy}{dx} = 3x^2 - 12x + 21 = 9 \Rightarrow x^2 - 4x + 4 = 0 \Rightarrow (x - 2)^2 \Rightarrow x = 2$

When $x = 2$, $y = 2^3 - 6 \times 2^2 + 21 \times 2 = 26$

So the point is $(2, 26)$ [5 marks]

Exercise 8.3 Tangents and normal

1 $y = x^2 - 4 \quad P(1, -3)$

(i) $\dfrac{dy}{dx} = 2x$ [1 mark]

(ii) $P(1, -3) \quad \dfrac{dy}{dx} = 2$ [1 mark]

(iii) Tangent $\quad y + 3 = 2(x - 1)$

$\Rightarrow y + 3 = 2x - 2$

$\Rightarrow y = 2x - 5$ [2 marks]

(iv) Gradient tangent $= 2 \Rightarrow$ Gradient normal $= -\dfrac{1}{2}$

Normal $\; y + 3 = -\dfrac{1}{2}(x - 1)$

$2y + 6 = -x + 1$

$x + 2y + 5 = 0$ [2 marks]

[Total: 6 marks]

2 $y = (x - 1)(x + 2)(x - 3) = (x - 1)(x^2 - x - 6)$

$= x^3 - x^2 - 6x - x^2 + x + 6$

$= x^3 - 2x^2 - 5x + 6$

(i) $\dfrac{dy}{dx} = 3x^2 - 4x - 5$ [2 marks]

(ii) At $(1, 0)$, $\dfrac{dy}{dx} = 3 - 4 - 5 = -6$ Gradient $= -6$ at $(1, 0)$

At $(-2, 0)$, $\dfrac{dy}{dx} = 3(4) - 4(-2) - 5 = 15$ Gradient $= 15$ at $(-2, 0)$

At $(3, 0)$, $\dfrac{dy}{dx} = 3(9) - 4(3) - 5 = 10$ Gradient $= 10$ at $(3, 0)$ [3 marks]

(iii) At $(2, -4)$ $\dfrac{dy}{dx} = 3(2)^2 - 4(2) - 5 = -1$

Tangent $y - (-4) = -1(x - 2)$

$y + 4 = -x + 2 \Rightarrow x + y + 2 = 0$ [2 marks]

(iv) Gradient of tangent $= -1 \Rightarrow$ gradient of normal $= +1$

Equation of normal $y - (-4) = 1(x - 2)$

$\Rightarrow y + 4 = x - 2$

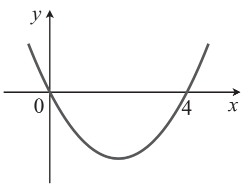
$\Rightarrow y = x - 6$ [2 marks]

[Total: 9 marks]

3 **(i)** $y = x^2 - 4x$

$= x(x - 4)$

U–shaped through $(0, 0)$ and $(4, 0)$



[2 marks]

(ii) When $x = 5$, $y = 5^2 - 4(5)$
$= 5 \Rightarrow (5, 5)$ on curve

$$\frac{dy}{dx} = 2x - 4$$

At $(5, 5)$, $\frac{dy}{dx} = 6$ [3 marks]

[Total: 5 marks]

4 (i) $y = x^3 - 6x^2 + 11x - 6 \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 - 12x + 11$ [1 mark]

(ii) $y = 0$ when $x = 1, 2$ and 3 (When $x = 2$ tangent has negative gradient.)

$$\left.\begin{aligned} &\text{When } x = 1, \frac{dy}{dx} = 3 - 12 + 11 = 2 \\ &\text{When } x = 3, \frac{dy}{dx} = 27 - 36 + 11 = 2 \end{aligned}\right\} \text{tangents at } (1, 0) \text{ and } (3,0) \text{ parallel}$$

[3 marks]

[Total: 4 marks]

5 (i) $y = x^2 - 4 \quad \Rightarrow \quad \frac{dy}{dx} = 2x \quad x = 1 \Rightarrow \frac{dy}{dx} = 2$

$x = -1 \Rightarrow \frac{dy}{dx} = -2$ [2 marks]

(ii) $y = 4 - x^2 \quad \Rightarrow \quad \frac{dy}{dx} = -2x \quad x = 1 \Rightarrow \frac{dy}{dx} = -2$

$x = -1 \Rightarrow \frac{dy}{dx} = 2$ [2 marks]

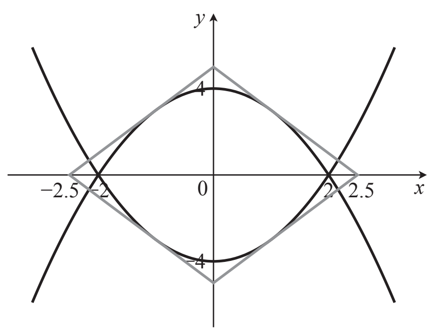
(iii) For $y = x^2 - 4$ At $(1, -3)$ tangent is $y + 3 = 2(x - 1)$
$\Rightarrow y + 3 = 2x - 2$
$\Rightarrow y = 2x - 5$

At $(-1, -3)$ tangent is $y + 3 = -2(x + 1)$
$\Rightarrow y + 3 = -2x - 2$
$\Rightarrow y = -2x - 5$

For $y = 4 - x^2$ At $(1, 3)$ tangent is $y - 3 = -2(x - 1)$
$\Rightarrow y - 3 = -2x + 2$
$\Rightarrow y = -2x + 5$

At $(-1, 3)$ tangent is $y - 3 = 2(x + 1)$
$\Rightarrow y - 3 = 2x + 2$
$\Rightarrow y = 2x + 5$ [6 marks]

(iv) Points of intersection with x-axis are $(-2.5, 0)$ and $(2.5, 0)$

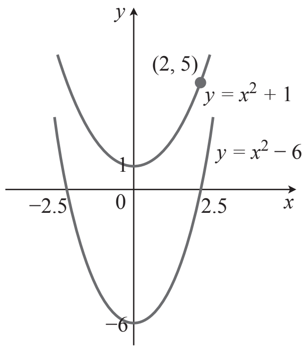


[2 marks]

(v) Quadrilateral is a rhombus. [1 mark]

[Total: 13 marks]

6 (i)



[2 marks]

(ii) $y = x^2 + 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = 2x$

At (2, 5), $\dfrac{dy}{dx} = 4$ [2 marks]

(iii) Using geometry, $y = x^2 + 1$ is the same shape as $y = x^2 - 6$ but moved vertically up through 7 units.

Using calculus, $y = x^2 - 6 \quad \Rightarrow \quad \dfrac{dy}{dx} = 2x$

At (2, −2), $\dfrac{dy}{dx} = 4$ [3 marks]

(iv) Any curve of the form $y = x^2 + c$, where c is constant [2 marks]

[Total: 9 marks]

7 Gradient of normal is $-\dfrac{1}{3}$ so gradient of curve at point of intersection is 3

So $\dfrac{dy}{dx} = 2x + 5 = 3 \quad \Rightarrow \quad x = -1$

So at the point of intersection, $-1 + 3y = 6 \quad \Rightarrow \quad y = \dfrac{7}{3}$

So $\dfrac{7}{3} = (-1)^2 + 5(-1) + a \quad \Rightarrow \quad 7 = 3 - 15 + 3a \quad \Rightarrow \quad 3a = 19 \quad \Rightarrow \quad a = \dfrac{19}{3}$ [7 marks]

8 $f(x) = ax^3 + 2x^2 + b$ Through (2, 0) with gradient −4

(i) (2, 0) on curve $\quad \Rightarrow \quad 0 = a(2)^3 + 2(2)^2 + b$

$\Rightarrow 8a + 8 + b = 0$ [1 mark]

(ii) $f'(x) = 3ax^2 + 4x$ At (2, 0), $f'(2) = -4$

$\Rightarrow -4 = 12a + 8$

$\Rightarrow a = -1$ [2 marks]

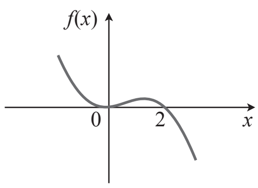
(iii) $a = -1$ Sub into $8a + 8 + b = 0 \Rightarrow b = 0$ [2 marks]

(iv) $f(x) = 2x^2 - x^3$

$= x^2(2 - x) = 0$ when $x = 0$ and $x = 2$

$f(x) = 4x - 3x^2 = 0$ when $x = 0$ and $x = \dfrac{4}{3}$

when $x < 0$, $f'(x)$ −ve



[2 marks]

[Total: 7 marks]

9 $y = x^4 + \dfrac{2}{x^2}$

(i) Replacing x by $(-x)$ gives $y = (-x)^4 + \dfrac{2}{(-x)^2}$

$$= x^4 + \frac{2}{x^2}$$

i.e. replacing x by $(-x)$ gives the same value of y

$\Rightarrow$ symmetrical about the y-axis [2 marks]

(ii) $y = x^4 + 2x^{-2} \quad \Rightarrow \quad \dfrac{dy}{dx} = 4x^3 - 4x^{-3}$

$$\Rightarrow \frac{dy}{dx} = 4x^3 - \frac{4}{x^3}$$

When $\dfrac{dy}{dx} = 0$, $4x^3 = \dfrac{4}{x^3} \quad \Rightarrow \quad x^6 = 1$

$$\Rightarrow x = \pm 1$$

Min turning points are $(-1, 3)$ and $(1, 3)$ [3 marks]

(iii) At $(-1, 3)$ tangent is $y = 3$, normal is $x = -1$
At $(1, 3)$ tangent is $y = 3$, normal is $x = 1$ [4 marks]

(iv) At $(2, 16.5)$ tangent has gradient $4(2)^3 - \dfrac{4}{2^3} = 31.5$

Equation of tangent $(y - 16.5) = 31.5(x - 2)$

$\Rightarrow y - 16.5 = 31.5x - 63$

$\Rightarrow y = 31.5x - 46.5$

At $(2, 16.5)$ normal has gradient $-\dfrac{1}{31.5} = -\dfrac{2}{63}$

Equation of normal $(y - 16.5) = -\dfrac{2}{63}(x - 2)$

$\Rightarrow 63y - 1039.5 = -2x + 4$

$\Rightarrow 2x + 63y = 1043.5$ [4 marks]

(v) At $(-2, 16.5)$ tangent has gradient -31.5 (symmetrical about y-axis)
Equation of tangent $(y - 16.5) = -31.5(x + 2)$

$\Rightarrow y - 16.5 = -31.5x - 63$

$\Rightarrow y = -31.5x - 46.5$

At $(2, 16.5)$ normal has gradient $+\dfrac{2}{63}$

Equation of normal $(y - 16.5) = \dfrac{2}{63}(x + 2)$

$\Rightarrow 63y - 1039.5 = 2x + 4$

$\Rightarrow 2x - 63y = 1043.5$ [4 marks]

[Total: 17 marks]

Exercise 8.4 Increasing and decreasing functions

1 **(i)** $y = 2x^2 - 4 \Rightarrow \frac{dy}{dx} = 4x$

Increasing for $\frac{dy}{dx} > 0 \Rightarrow x > 0$ [2 marks]

(ii) $y = 3x - 4 \Rightarrow \frac{dy}{dx} = 3$

Increasing for all x [2 marks]

(iii) $y = x^2 - 4x + 5 \Rightarrow \frac{dy}{dx} = 2x - 4$

Increasing for $2x - 4 > 0 \Rightarrow x > 2$ [2 marks]

(iv) $y = x^2 + 5x \Rightarrow \frac{dy}{dx} = 2x + 5$

Increasing for $2x + 5 > 0 \Rightarrow x > -2.5$ [2 marks]

(v) $y = 2x^2 - 4x + 3 \Rightarrow \frac{dy}{dx} = 4x - 4$

Increasing for $4x - 4 > 0 \Rightarrow x > 1$ [2 marks]

(vi) $y = (x+2)(x-4) = x^2 - 2x - 8$

$\Rightarrow \frac{dy}{dx} = 2x - 2$

Increasing for $2x - 2 > 0 \Rightarrow x > 1$ [2 marks]

(vii) $y = 2x^3 - 3x^2 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x$

$= 6x(x-1)$

Increasing for $6x(x-1) > 0 \Rightarrow x < 0$ or $x > 1$ [2 marks]

(viii) $y = 3x^3 - 6x + 5 \Rightarrow \frac{dy}{dx} = 9x^2 - 6$

$= \left(3x - \sqrt{6}\right)\left(3x + \sqrt{6}\right)$

Increasing for $x < -\frac{\sqrt{6}}{3}$ or $x > \frac{\sqrt{6}}{3}$ [2 marks]

[Total: 16 marks]

2 **(i)** $y = 3x^2 - 3 \Rightarrow \frac{dy}{dx} = 6x$

Decreasing for $x < 0$ [2 marks]

(ii) $y = x^2 - 4x + 5 \Rightarrow \frac{dy}{dx} = 2x - 4$

Decreasing for $x < 2$ [2 marks]

(iii) $y = 2x(x-4) = 2x^2 - 8x \Rightarrow \frac{dy}{dx} = 4x - 8$

Decreasing for $4x - 8 < 0 \Rightarrow x < 2$ [2 marks]

(iv) $y = \frac{x}{2} - x^2 \Rightarrow \frac{dy}{dx} = \frac{1}{2} - 2x$

Decreasing for $\frac{1}{2} - 2x < 0 \Rightarrow x > \frac{1}{4}$ [2 marks]

(v) $y=(4+x)(x-3)=x^2+x-12 \Rightarrow \dfrac{dy}{dx}=2x+1$

Decreasing for $2x+1<0 \Rightarrow x<-\dfrac{1}{2}$ [2 marks]

(vi) $y=5-x^3 \Rightarrow \dfrac{dy}{dx}=-3x^2$

Decreasing for all x [2 marks]

(vii) $y=(2x-3)^2=4x^2-12x+9 \Rightarrow \dfrac{dy}{dx}=8x-12$

Decreasing for $8x-12<0 \Rightarrow x<1.5$ [2 marks]

(viii) $y=6+3x-x^3 \Rightarrow \dfrac{dy}{dx}=3-3x^2$

Decreasing for $3-3x^2<0 \Rightarrow x^2>1$

$\Rightarrow x<-1$ or $x>1$ [2 marks]

[Total: 16 marks]

3 $y=x^3+3x^2+6x+4 \Rightarrow \dfrac{dy}{dx}=3x^2+6x+6$

$= 3(x^2+2x+2)$

$= 3[(x+1)^2+1]$

>0 for all x

$\Rightarrow$ function increasing [3 marks]

4 $y=5-4x-x^3 \Rightarrow \dfrac{dy}{dx}=-4-3x^2$

$= -(3x^2+4)$

<0 for all x

$\Rightarrow$ function decreasing [3 marks]

5 **(i)** $y=1+4x+\dfrac{1}{x} \Rightarrow \dfrac{dy}{dx}=4-\dfrac{1}{x^2}$

$\Rightarrow \dfrac{dy}{dx}=0$ when $x=\pm\dfrac{1}{2}$

(a) Increasing if $4>\dfrac{1}{x^2} \Rightarrow x<-\dfrac{1}{2}$ or $x>\dfrac{1}{2}$

(b) Decreasing if $4<\dfrac{1}{x^2} \Rightarrow -\dfrac{1}{2}<x<\dfrac{1}{2}$ with $x\neq 0$ (not defined) [4 marks]

(ii) $y=2x^3-3x^2-36x+9 \Rightarrow \dfrac{dy}{dx}=6x^2-6x-36$

$= 6(x^2-x-6)$

$= 6(x-3)(x+2)$

$\Rightarrow \dfrac{dy}{dx}=0$ when $x=-2,\ x=3$

(a) Increasing if $x<-2$ or $x>3$

(b) Decreasing if $-2<x<3$ [4 marks]

(iii) $y = x^3 - x \Rightarrow \dfrac{dy}{dx} = 3x^2 - 1$

$\Rightarrow \dfrac{dy}{dx} = 0$ when $x = \pm\dfrac{1}{\sqrt{3}}$

(a) Increasing if $x < -\dfrac{1}{\sqrt{3}}$ or $x > \dfrac{1}{\sqrt{3}}$

(b) Decreasing if $-\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$ [4 marks]

(iv) $y = x^4 - 16 \Rightarrow \dfrac{dy}{dx} = 4x^3$

(a) Increasing if $x > 0$

(b) Decreasing if $x < 0$ [2 marks]

[Total: 14 marks]

6 $y = (x - a)^3$
$= x^3 - 3ax^2 + 3a^2x - a^3$

$\dfrac{dy}{dx} = 3x^2 - 6ax + 3a^2$

$= 3(x^2 - 2ax^2 + a^2)$
$= 3(x - a)^2$

$\dfrac{dy}{dx} \geqslant 0$ for all a, so increasing [3 marks]

7 **(i)** $y = x^3 - 15x^2 + 63x - 10$

$\dfrac{dy}{dx} = 3x^2 - 30x + 63$

$= 3(x^2 - 10x + 21)$
$= 3(x - 3)(x - 7)$

$\dfrac{dy}{dx} < 0$ if $3 < x < 7$, so decreasing for $3 < x < 7$ [3 marks]

(ii) Increasing for $x < 3$ or $x > 7$ [1 mark]

[Total: 4 marks]

8 $P = 1000 - 4t - t^2$

(i) $t = 0 \Rightarrow P = 1000$ [1 mark]

(ii) $\dfrac{dP}{dt} = -4 - 2t \Rightarrow$ rate of decrease is $4 + 2t$ [2 marks]

(iii) When $t = 6$, $P = 1000 - 24 - 36 = 940$

Decreased by $60 = \dfrac{60}{1000} \times 100\% = 6\%$ [3 marks]

[Total: 6 marks]

9 $P = t^3 - t^2 + 120t + 8000$
t = years after 2018

(i) In 2025, $t = 7$, $P = 7^3 - 7^2 + 120(7) + 8000$
$= 9134$ [1 mark]

(ii) Rate of growth $\frac{dP}{dt} = 3t^2 - 2t + 120$

In 2025, $t = 7$, $\frac{dP}{dt} = 3(49) - 2(7) + 120$

$= 253$ per annum [2 marks]

(iii) $P > 10\,000 \Rightarrow t^3 - t^2 + 120t + 8000 > 10\,000$

$\Rightarrow t^3 - t^2 + 120t > 2000$

Check $t = 8$ LHS $= 8^3 - 8^2 + 120(8)$ (you know $t = 7$ gives 9232)

$= 1408$

$t = 9$ LHS $= 9^3 - 9^2 + 120(9)$

$= 1728$

$t = 10$ LHS $= 10^3 - 10^2 + 120(10)$

$= 2100$

P is the figure at start of year, so 9728 at start of 2027, 10 100 at start of 2028, so population first exceeds 10 000 at some point in 2027 [3 marks]

[Total: 6 marks]

10 $f'(x) = 2x - \frac{16}{x^2} < 0 \Rightarrow x^3 < 8 \Rightarrow x < 2$ [4 marks]

Exercise 8.5 Second derivatives

1 (i) $y = 2x^3 - 3x^2 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x$

$\Rightarrow \frac{d^2y}{dx^2} = 12x - 6$ [2 marks]

(ii) $y = 5x - 4 \Rightarrow \frac{dy}{dx} = 5$

$\Rightarrow \frac{d^2y}{dx^2} = 0$ [2 marks]

(iii) $y = 4x - x^4 \Rightarrow \frac{dy}{dx} = 4 - 4x^3$

$\Rightarrow \frac{d^2y}{dx^2} = -12x^2$ [2 marks]

[Total: 6 marks]

2 (i) $y = 2x^4 - 3x^2 - 5x + 2 \Rightarrow \frac{dy}{dx} = 8x^3 - 6x - 5$

$\Rightarrow \frac{d^2y}{dx^2} = 24x^2 - 6$ [2 marks]

(ii) $y = x^6 - 6x^2 + 3x + 1 \Rightarrow \frac{dy}{dx} = 6x^5 - 12x + 3$

$\Rightarrow \frac{d^2y}{dx^2} = 30x^4 - 12$ [2 marks]

(iii) $y = 2x^3 + 4x^2 - 7x - 2 \Rightarrow \frac{dy}{dx} = 8x^2 + 8x - 7$

$\Rightarrow \frac{d^2y}{dx^2} = 16x + 8$ [2 marks]

[Total: 6 marks]

3 **(i)** $y = (2x - 3)(3x - 2)$

$= 6x^2 - 13x + 6$

$\frac{dy}{dx} = 12x - 13 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 12$ [2 marks]

(ii) $y = (x^2 - 1)(x + 1)$

$= x^3 + x^2 - x - 1$

$\frac{dy}{dx} = 3x^2 + 2x - 1 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 6x + 2$ [2 marks]

(iii) $y = (4x - 3)^3$

$= (4x - 3)(16x^2 - 24x + 9)$

$= 64x^3 - 144x^2 + 108x - 27$

$\frac{dy}{dx} = 192x^2 - 288x + 108 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 384x - 288$ [3 marks]

[Total: 7 marks]

4 **(i)** $y = 5(4x - 1)(x^2 + 2)$

$= 5(4x^3 - x^2 + 8x - 2)$

$= 20x^3 - 5x^2 + 40x - 10$

$\frac{dy}{dx} = 60x^2 - 10x + 40 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 120x - 10$ [3 marks]

(ii) $y = 3x(2x - 5)(2x + 3)$

$= 3x(4x^2 - 4x - 15)$

$= 12x^3 - 12x^2 - 45x$

$\frac{dy}{dx} = 36x^2 - 24x - 45 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 72x - 24$ [3 marks]

(iii) $y = 5x(3x - 2)^2$

$= 5x(9x^2 - 12x + 4)$

$= 45x^3 - 60x^2 + 20x$

$\frac{dy}{dx} = 135x^2 - 120x + 20 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 270x - 120$ [3 marks]

[Total: 9 marks]

5 $x + y = 20 \quad P = xy = 96$

(i) $y = 20 - x$ [1 mark]

(ii) $P = x(20 - x) = 20x - x^2$ [1 mark]

(iii) $\frac{dy}{dx} = -1, \quad \frac{dP}{dx} = 20 - 2x$ [2 marks]

(iv) Rate of change of $\frac{dP}{dx} = \frac{d^2P}{dx^2} = -2$ [1 mark]

[Total: 5 marks]

6 $y = (x - 3)(2x^2 + 3x - 4)$

$= 2x^3 - 3x^2 - 13x + 12$

(i) $\frac{dy}{dx} = 6x^2 - 6x - 13, \quad \frac{d^2y}{dx^2} = 12x - 6$ [3 marks]

(ii) At $(-2, 10)$ $\frac{dy}{dx} = 6(-2)^2 - 6(-2) - 13$

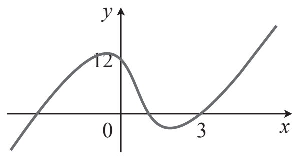
$= 23$ positive so $/$

At (0, 12) $\frac{dy}{dx} = -13$ negative so \

At (3, 0) $\frac{dy}{dx} = 6(3)^2 - 6(3) - 13$

$= 23$ positive so / [3 marks]

(iii) $x = 0 \Rightarrow y = 12$



[2 marks]

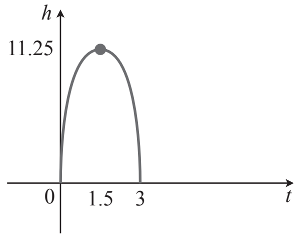
[Total: 8 marks]

7 $h = 15t - 5t^2$

(i) $\frac{dh}{dt} = 15 - 10t$ $\frac{d^2h}{dt^2} = -10$ [2 marks]

(ii) When $\frac{dh}{dt} = 0$, stone is instantaneously at rest at its highest point [1 mark]

(iii)



$h = 5t(3 - t)$ [2 marks]

[Total: 5 marks]

8 (i) $y = x^4 - 9x^2$
$= x^2(x^2 - 9)$
$= x^2(x + 3)(x - 3)$

Meets x-axis at $(-3, 0)$, $(0, 0)$ and $(3, 0)$.

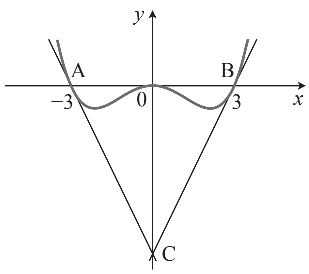
$\frac{dy}{dx} = 4x^3 - 18x$

At $(-3, 0)$ $\frac{dy}{dx} = 4(-3)^3 - 18(-3) = -54$

At $(0, 0)$ $\frac{dy}{dx} = 0$

At $(3, 0)$ $\frac{dy}{dx} = 4(3)^3 - 18(3) = 54$ [4 marks]

(ii)



[2 marks]

(iii) AB is $y = 0$

AC has equation $(y - 0) = -54(x + 3) \quad \Rightarrow \quad y = -54x - 162$

BC has equation $(y - 0) = 54(x - 3) \quad \Rightarrow \quad y = 54x - 162$

Points of intersection are A(−3, 0), B(3, 0) (from diagram (ii))

and C(0, −162) from equation of AC and BC. [4 marks]

(iv) Triangle has AB = 6, OC = 162

$\Rightarrow$ area $= \frac{1}{2} \times 6 \times 162$

$= 486$ units^2 [2 marks]

[Total: 12 marks]

9 $y = 2x^4 - 5x^2 - 3 \quad \Rightarrow \quad \frac{dy}{dx} = 8x^3 - 10x \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 24x^2 - 10$

$24x^2 - 10 = 86 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

When $x = 2$, $y = 2 \times 2^4 - 5 \times 2^2 - 3 = 9$, and when $x = -2$, $y = 2 \times (-2)^4 - 5 \times (-2)^2 - 3 = 9$

The intersection points are (2, 9) and (−2, 9) [4 marks]

Exercise 8.6 Stationary points and applications of maxima and minima

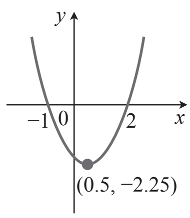
1 (i) (a) $y = x^2 - x - 2 \quad \Rightarrow \quad \frac{dy}{dx} = 2x - 1$

$\frac{dy}{dx} = 0$ when $x = 0.5$ [1 mark]

(b) (c) $\frac{d^2y}{dx^2} = 2 \Rightarrow$ min turning point [1 mark]

(d) $x = 0.5 \Rightarrow y = 0.5^2 - 0.5 - 2 = -2.25$ [1 mark]

(e)



[2 marks]

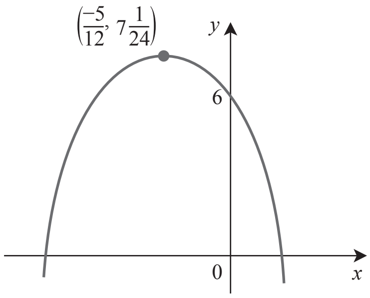
(ii) (a) $y = 6 - 5x - 6x^2 \quad \Rightarrow \quad \frac{dy}{dx} = -5 - 12x$

$\frac{dy}{dx} = 0$ when $x = \frac{-5}{12}$ [1 mark]

(b) (c) $\frac{d^2y}{dx^2} = -12 \Rightarrow$ max turning point [1 mark]

(d) $x = \frac{-5}{12} \quad \Rightarrow \quad y = 6 - 5\left(\frac{-5}{12}\right) - 6\left(\frac{-5}{12}\right)^2 = 7\frac{1}{24}$ [1 mark]

(e)



[2 marks]

(iii) **(a)** $y = x^3 - 3x \Rightarrow \frac{dy}{dx} = 3x^2 - 3 = 0$ for $x \pm 1$ [1 mark]

(b) $\frac{d^2y}{dx^2} = 6x \quad \frac{d^2y}{dx^2} = +6$ for $x = 1$

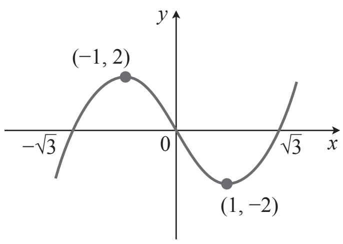
$= -6$ for $x = -1$ [2 marks]

(c) $\frac{d^2y}{dx^2} = 6x \Rightarrow$ max at $x = -1$, min at $x = 1$ [1 mark]

(d) $x = -1 \Rightarrow y = (-1)^3 - 3(-1) = 2 \quad (-1, 2)$

$x = 1 \Rightarrow y = (1)^3 - 3(1) = -2 \quad (1, -2)$ [2 marks]

(e)



[2 marks]

(iv) **(a)** $y = x^3 + 3x^2 - 24x - 7 \Rightarrow \frac{dy}{dx} = 3x^2 + 6x - 24$

$= 3(x^2 + 2x - 8)$

$= 3(x + 4)(x - 2)$

$= 0$ for $x = -4, x = 2$ [2 marks]

(b) $\frac{d^2y}{dx^2} = 6x + 6$

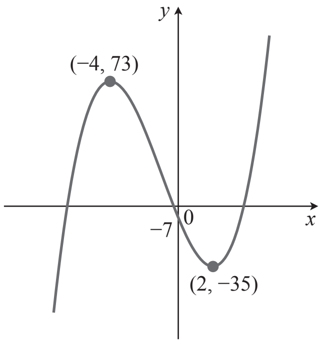
At $x = -4 \quad \frac{d^2y}{dx^2} = -18$

At $x = 2 \quad \frac{d^2y}{dx^2} = 18$ [2 marks]

(c) $x = -4 \Rightarrow$ max turning point $\quad x = 2 \Rightarrow$ min turning point [1 mark]

(d) $x = -4 \Rightarrow y = 73 \quad x = 2 \Rightarrow y = -35$ [2 marks]

(e)



[2 marks]

(v) **(a)** $y = (x+1)^2(x-1)^2 = (x^2-1)^2$

$= x^4 - 2x^2 + 1$

$$\frac{dy}{dx} = 4x^3 - 4x \quad = 0 \text{ when } 4x(x^2-1) = 0$$

$$4x(x+1)(x-1) = 0$$

$$x = -1 \text{ or } x = 0 \text{ or } x = 1$$

[3 marks]

(b) $\frac{d^2y}{dx^2} = 12x^2 - 4 \quad x = -1 \Rightarrow \frac{d^2y}{dx^2} = 8$

$$x = 0 \Rightarrow \frac{d^2y}{dx^2} = -4$$

$$x = 1 \Rightarrow \frac{d^2y}{dx^2} = 8$$

[2 marks]

(c) $x = -1 \Rightarrow$ min turning point

$x = 0 \Rightarrow$ max turning point

$x = 1 \Rightarrow$ min turning point

[2 marks]

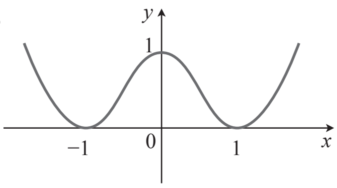
(d) $x = -1 \Rightarrow y = 0$

$x = 0 \Rightarrow y = 1$

$x = 1 \Rightarrow y = 0$

[2 marks]

(e)



[2 marks]

[Total: 38 marks]

2 $y = x^2 - ax + b$ min at $(2, -3)$

$\frac{dy}{dx} = 2x - a = 0$ when $x = 2 \Rightarrow a = 4$

$\frac{d^2y}{dx^2} = 2$ confirming min turning point

Through $(2, -3) \Rightarrow -3 = (2)^2 - 4(2) + b$

$-3 = -4 + b$

$b = 1$

[3 marks]

8

Calculus

3 $y = x^4 - 8x^2$

(i) $\frac{dy}{dx} = 4x^3 - 16x, \quad \frac{d^2y}{dx^2} = 12x^2 - 16$ [2 marks]

(ii) When $\frac{dy}{dx} = 0, \quad 4x(x^2 - 4) = 0$

$4x(x + 2)(x - 2) = 0$

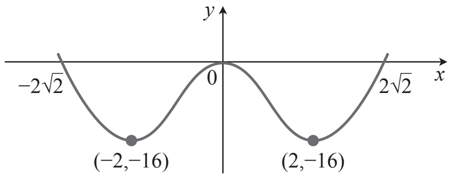
stationary points $x = 0, \frac{d^2y}{dx^2} = -16 \Rightarrow \text{max } (0, 0)$

$x = -2, \frac{d^2y}{dx^2} = 32 \Rightarrow \text{min } (-2, -16)$

$x = 2, \quad \frac{d^2y}{dx^2} = 32 \Rightarrow \text{min } (2, -16)$ [3 marks]

(iii) $y = 0 \quad \Rightarrow \quad x^4 - 8x^2 = 0$

$\Rightarrow \quad x^2(x^2 - 8) = 0 \quad \Rightarrow \quad x = 0, \pm 2\sqrt{2}$



[4 marks]

[Total: 9 marks]

4 $y = 4x^3 - 16x$

(i) $\frac{dy}{dx} = 12x^2 - 16, \quad \frac{d^2y}{dx^2} = 24x$ [2 marks]

(ii) When $\frac{dy}{dx} = 0, 12x^2 - 16 = 0 \Rightarrow 4(3x^2 - 4) = 0$

$\Rightarrow 3x^2 = 4$

$\Rightarrow x = \pm\frac{2}{\sqrt{3}}$

$x = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad y = 4\left(\frac{8}{3\sqrt{3}}\right) - 16\left(\frac{2}{\sqrt{3}}\right)$

$= \frac{-64\sqrt{3}}{9}$

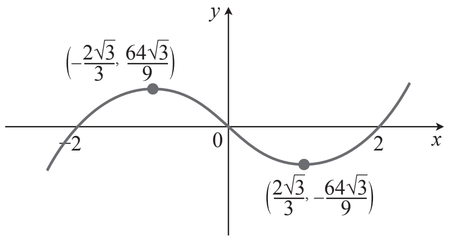
$x = \frac{2}{\sqrt{3}} \Rightarrow \frac{d^2y}{dx^2} \text{ +ve} \Rightarrow \text{min at } \left(\frac{2\sqrt{3}}{3}, -\frac{64\sqrt{3}}{9}\right)$

$x = -\frac{2}{\sqrt{3}} \Rightarrow y = 4\left(\frac{-8}{3\sqrt{3}}\right) - 16\left(-\frac{2}{\sqrt{3}}\right)$

$= \frac{+64\sqrt{3}}{9}$

$x = -\frac{2}{\sqrt{3}} \Rightarrow \frac{d^2y}{dx^2} \text{ −ve} \quad \Rightarrow \quad \text{max at } \left(\frac{-2\sqrt{3}}{3}, \frac{64\sqrt{3}}{9}\right)$ [5 marks]

(iii)



$y = 0 \Rightarrow 4x(x^2 - 4) = 0$

$\Rightarrow x = 0$ or $x = \pm 2$ [2 marks]

[Total: 9 marks]

5 (i) $y = (x - 1)^2(x - 4)$

$= (x^2 - 2x + 1)(x - 4)$

$= x^3 - 6x^2 + 9x - 4$

$\frac{dy}{dx} = 3x^2 - 12x + 9 \quad \frac{d^2y}{dx^2} = 6x - 12$ [3 marks]

(ii) $\frac{dy}{dx} = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$

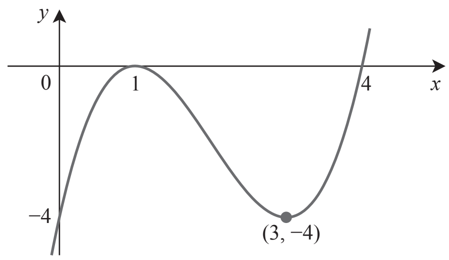
$\Rightarrow 3(x - 1)(x - 3) = 0$

$\Rightarrow x = 1$ or $x = 3$

$x = 1 \Rightarrow \frac{d^2y}{dx^2} = -6 \Rightarrow$ max at $(1, 0)$

$x = 3 \Rightarrow \frac{d^2y}{dx^2} = +6 \Rightarrow$ min at $(3, -4)$ [3 marks]

(iii)



[2 marks]

[Total: 8 marks]

6 $y = ax^2 + bx + c$ Through $(0, 3)$ $3 = c$

(i) $\frac{dy}{dx} = 2ax + b = 0$ when $x = \frac{-b}{2a}$

max at $(1, 4) \Rightarrow$ through $(1, 4)$

$\Rightarrow 4 = a + b + 3$

$$\left.\begin{aligned} &\Rightarrow a + b = 1 \\ \text{max at } (1, 4) &\Rightarrow 1 = \frac{-b}{2a} \\ &\Rightarrow 2a + b = 0 \end{aligned}\right\} a = -1, b = 2$$

Curve $y = 3 + 2x - x^2$ [1 mark]

(ii) $\frac{d^2y}{dx^2} = 2a = -2 \Rightarrow$ stationary point is max [4 marks]

[Total: 5 marks]

7 **(i)** $x + y = 12 \Rightarrow y = 12 - x$ [1 mark]

(ii) $S = x^2 + y^2 = x^2 + (12 - x)^2$
$\Rightarrow S = 2x^2 - 24x + 144$ [1 mark]

(iii) $\dfrac{\mathrm{d}S}{\mathrm{d}x} = 4x - 24 \quad \dfrac{\mathrm{d}^2S}{\mathrm{d}x^2} = 4$ [2 marks]

(iv) Turning point when $\dfrac{\mathrm{d}S}{\mathrm{d}x} = 0 \Rightarrow x = 6; \dfrac{\mathrm{d}^2S}{\mathrm{d}x^2} = 4 \therefore$ min

$x + y = 12 \Rightarrow y = 6$

Value of $S = x^2 + y^2 = 72$ [2 marks]

[Total: 6 marks]

8 **(i)** Surface area $= 2\pi rh + \pi r^2 = 250\pi$
$\Rightarrow 2rh + r^2 = 250$

$\Rightarrow h = \dfrac{250 - r^2}{2r}$ [2 marks]

(ii) $V = \pi r^2 h = \pi r^2\left(\dfrac{250 - r^2}{2r}\right)$

$= \pi r\left(125 - \dfrac{r^2}{2}\right)\left(= \dfrac{\pi r}{2}(250 - r^2)\right)$ [2 marks]

(iii) $V = 125\pi r - \dfrac{\pi r^3}{2} \Rightarrow \dfrac{\mathrm{d}V}{\mathrm{d}r} = 125\pi - \dfrac{3\pi r^2}{2}$

$\dfrac{\mathrm{d}^2V}{\mathrm{d}r^2} = -3\pi r$ [2 marks]

(iv) $\dfrac{\mathrm{d}^2V}{\mathrm{d}r^2} < 0 \Rightarrow$ max when $125\pi - \dfrac{3\pi r^2}{2} = 0$

$\Rightarrow r^2 = \dfrac{250}{3}$

$r = \dfrac{5\sqrt{30}}{3}, h = \dfrac{250 - r^2}{2r} = \dfrac{5\sqrt{30}}{3}$ [2 marks]

[Total: 8 marks]

9 **(i)** $V = x(48 - 2x)^2$
$= 4x^3 - 192x^2 + 2304x$ [3 marks]

(ii) $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 384x + 2304$
$= 12(x^2 - 32x + 192)$
$= 12(x - 8)(x - 24)$
$= 0$ when $x = 8$ or $x = 24$ (reject – too large)

$\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24x - 384$
$= -192$ when $x = 8 \therefore$ max

Max capacity $= 8(48 - 16)^2$
$= 8192\,\mathrm{cm}^2$ [2 marks]

[Total: 5 marks]

10 (i) Amount of wire for square = $(56 - 8x)$ cm
$\Rightarrow$ each side is $(14 - 2x)$ cm
Area = $(14 - 2x)^2$
$= (196 - 56x + 4x^2)\,\text{cm}^2$ [2 marks]

(ii) Area of rectangle = $3x \times x = 3x^2\,\text{cm}^2$
$\Rightarrow$ total area $A = 7x^2 - 56x + 196\,\text{cm}^2$ [3 marks]

(iii) $\dfrac{dA}{dx} = 14x - 56$
$= 0$ when $x = 4$
$\dfrac{d^2A}{dx^2} = 14 \Rightarrow$ min. A when $x = 4$ [2 marks]

(iv) min. $A = 7(16) - 56(4) + 196$
$= 84\,\text{cm}^2$ [1 mark]

[Total: 8 marks]

11 (i) Base of box is $(120 - 2x)$ cm by $(75 - 2x)$ cm
Volume = $x(120 - 2x)(75 - 2x)$
$= x(9000 - 240x - 150x + 4x^2)$
$V = 4x^3 - 390x^2 + 9000x$ [2 marks]

(ii) $\dfrac{dV}{dx} = 12x^2 - 780x + 9000$
$= 12(x^2 - 65x + 750)$
$= 12(x - 15)(x - 50)$
$= 0$ when $x = 15$ or $x = 50$ (not viable since $75 - 2x$ would be negative)

$\dfrac{d^2V}{dx^2} = 24x - 780$

$x = 15 \Rightarrow \dfrac{d^2V}{dx^2} = -420$

$\Rightarrow x = 15$ is a maximum [4 marks]

(iii) Max volume = $4(15)^3 - 390(15)^2 + 9000(15)$
$= 60\,750\,\text{cm}^3$
(Alternatively, substitute into $x(120 - 2x)(75 - 2x)$
$= 15(90)(45)$
$= 60\,750\,\text{cm}^3$) [1 mark]

[Total: 7 marks]

12 (i) $\dfrac{dy}{dx} = 2x - \dfrac{a}{x^2}$ When $x = 2$, $2 \times 2 - \dfrac{a}{2^2} = 0 \Rightarrow a = 16$ [4 marks]

(ii) When $x = 2$, $9 = 2^2 + \dfrac{16}{2} + b \Rightarrow b = -3$ [2 marks]

(iii) $\dfrac{d^2y}{dx^2} = 2 + \dfrac{32}{x^3}$ When $x = 2$, $\dfrac{d^2y}{dx^2} = 2 + \dfrac{32}{2^3} = 6 > 0 \Rightarrow$ the point is a minimum [3 marks]

[Total: 9 marks]

9 Matrices

Exercise 9.1 Multiplying matrices

1 **(i)** $2\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -5 & -1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2\times 2-5 & 2\times -3-1 \\ 2\times 1+3 & 2\times 4+6 \end{bmatrix} = \begin{bmatrix} -1 & -7 \\ 5 & 14 \end{bmatrix}$ [1 mark]

(ii) $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -5 & -1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2\times -5 + -3\times 3 & 2\times -1 + -3\times 6 \\ 1\times -5 + 4\times 3 & 1\times -1 + 4\times 6 \end{bmatrix} = \begin{bmatrix} -19 & -20 \\ 7 & 23 \end{bmatrix}$ [2 marks]

(iii) $\begin{bmatrix} -5 & -1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -5\times 2 + -1\times 1 & -5\times -3 + -1\times 4 \\ 3\times 2 + 6\times 1 & 3\times -3 + 6\times 4 \end{bmatrix} = \begin{bmatrix} -11 & 11 \\ 12 & 15 \end{bmatrix}$ [2 marks]

(iv) $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2\times 2 + -3\times 1 & 2\times -3 + -3\times 4 \\ 1\times 2 + 4\times 1 & 1\times -3 + 4\times 4 \end{bmatrix} = \begin{bmatrix} 1 & -18 \\ 6 & 13 \end{bmatrix}$ [2 marks]

[Total: 7 marks]

2 **(i)** $-2p + 14 = 3 \quad \Rightarrow \quad -2p = -11 \quad \Rightarrow \quad p = 5.5$

$-6 + 35 = q \quad \Rightarrow \quad q = 29$ [3 marks]

(ii) $-20 + 6m = 8 \quad \Rightarrow \quad 6m = 28 \quad \Rightarrow \quad m = \frac{14}{3}$

$4n + 2m = 3 \quad \therefore \quad 4n + 2\times\frac{14}{3} = 3 \quad \Rightarrow \quad 4n = -\frac{19}{3} \quad \Rightarrow \quad n = -\frac{19}{12}$ [3 marks]

[Total: 6 marks]

3 $r\begin{bmatrix} 5 & 3 \\ 3 & -4 \end{bmatrix} + t\begin{bmatrix} 2 & 4 \\ 6 & -9 \end{bmatrix} = \begin{bmatrix} m & 6 \\ 0 & n \end{bmatrix} \quad \therefore \quad 3r + 4t = 6 \text{ and } 3r + 6t = 0$

subtracting: $6t - 4t = 0 - 6 \quad \Rightarrow \quad 2t = -6 \quad \Rightarrow \quad t = -3$

$\therefore \quad 3r + 6\times -3 = 0 \quad \Rightarrow \quad 3r = 18 \quad \Rightarrow \quad r = 6$

$5r + 2t = m \quad \therefore \quad m = 5\times 6 + 2\times -3 = 24$

$-4r - 9t = n \quad \therefore \quad n = -4\times 6 - 9\times -3 = 3$ [6 marks]

4 **(i)** $3 + 4a = 9 \quad \Rightarrow \quad 4a = 6 \quad \Rightarrow \quad a = 1.5$ [2 marks]

(ii) $7b + 3 = 17 \quad \Rightarrow \quad 7b = 14 \quad \Rightarrow \quad b = 2$ [2 marks]

[Total: 4 marks]

5 **(i)** $3x + y = 13$ and $2x - 4y = 18$ [2 marks]

(ii) $2x - 4(13 - 3x) = 18 \quad \Rightarrow \quad 2x - 52 + 12x = 18 \quad \Rightarrow \quad 14x = 70$

$\Rightarrow \quad x = 5 \quad \therefore \quad y = 13 - 3\times 5 = -2$ [3 marks]

[Total: 5 marks]

6 **(i)** $2y + 15 = -9 \quad \Rightarrow \quad 2y = -24 \quad \Rightarrow \quad y = -12$

$10 - 4x = x \quad \Rightarrow \quad 5x = 10 \quad \Rightarrow \quad x = 2$

$10y + 5x = w \quad \therefore \quad w = 10\times -12 + 5\times 2 = -110$ [5 marks]

(ii) $20 + 6p = -6 \quad \Rightarrow \quad 6p = -26 \quad \Rightarrow \quad p = -\frac{13}{3}$

$15 + pq = 9 \quad \therefore \quad 15 - \frac{13}{3}q = 9 \quad \Rightarrow \quad 45 - 13q = 27 \quad \Rightarrow \quad q = \frac{18}{13}$

$-12 + 2q = r \quad \therefore \quad r = -12 + 2\times\frac{18}{13} = -\frac{120}{13}$ [5 marks]

[Total: 10 marks]

7 $m-12=-m^2$ **and** $m^2+6=5m$

$\Rightarrow m^2+m-12=0$ **and** $m^2-5m+6=0$

$\Rightarrow (m+4)(m-3)=0$ **and** $(m-3)(m-2)=0$

$\Rightarrow m=-4$ or $m=3$ **and** $m=3$ or $m=2$

$\therefore m=3$ [6 marks]

8 **(i)** $p\begin{bmatrix} p-1 & 4 \\ 0 & p-q \end{bmatrix}+\begin{bmatrix} 2p-q & 1 \\ 2p & 5q \end{bmatrix}=\begin{bmatrix} 10 & 3p+2q \\ 2p & 13 \end{bmatrix}$

$\therefore\ p(p-1)+2p-q=10$ ①

and $4p+1=3p+2q$ ②

and $p(p-q)+5q=13$ ③ [3 marks]

(ii) ① $\Rightarrow\ p^2-p+2p-10=q \quad\Rightarrow\quad q=p^2+p-10$

substitute $q=p^2+p-10$ into ② $\therefore\quad p+1=2(p^2+p-10)$

$\Rightarrow 2p^2+p-21=0 \quad\Rightarrow\quad (2p+7)(p-3)=0 \quad\Rightarrow\quad p=-\dfrac{7}{2}$ or 3

If $p=-\dfrac{7}{2}$ then $q=\left(-\dfrac{7}{2}\right)^2-\dfrac{7}{2}-10=-\dfrac{5}{4}$

Substitute $p=-\dfrac{7}{2}$ and $q=-\dfrac{5}{4}$ into ③ $-\dfrac{7}{2}\times\left(-\dfrac{7}{2}+\dfrac{5}{4}\right)+5\times-\dfrac{5}{4}=\dfrac{13}{8}\neq 13$

$\therefore\quad p\neq-\dfrac{7}{2}$

If $p=3$ then $q=3^2+3-10=2$

Substitute $p=3$ and $q=2$ into ③ $3\times(3-2)+5\times 2=13$

$\therefore\quad p=3$ and $q=2$ [5 marks]

[Total: 8 marks]

9 $\begin{bmatrix} 3 & 2 \\ -1 & n \end{bmatrix}\begin{bmatrix} 3 & m \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 3 & m \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & n \end{bmatrix}$

$\therefore\quad 9+2=9-m \quad\Rightarrow\quad m=-2$

and $3m+0=6+mn \quad\therefore\quad -6=6-2n \quad\Rightarrow\quad 2n=12 \quad\Rightarrow\quad n=6$ [5 marks]

10 $\begin{bmatrix} -5 & 4 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & 4 \end{bmatrix}=\begin{bmatrix} a & b \\ c & 4 \end{bmatrix}\begin{bmatrix} -5 & 4 \\ 6 & 1 \end{bmatrix}$

$-5b+16=4a+b \quad\Rightarrow\quad 8=2a+3b \quad\Rightarrow\quad b=\dfrac{1}{3}(8-2a)$

$6a+c=-5c+24 \quad\Rightarrow\quad a+c=4 \quad\Rightarrow\quad c=4-a$ [4 marks]

11 **(i)** $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 7 & -3 \\ 0 & 4 \end{bmatrix}=\begin{bmatrix} 14 & -2 \\ 35 & -3 \end{bmatrix}$ [2 marks]

(ii) $\begin{bmatrix} 7 & -3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -3 & 6 \\ 4 & 2 \end{bmatrix}=\begin{bmatrix} -33 & 36 \\ 16 & 8 \end{bmatrix}$ [2 marks]

(iii) $(\mathbf{AB})\mathbf{C}=\begin{bmatrix} 14 & -2 \\ 35 & -3 \end{bmatrix}\begin{bmatrix} -3 & 6 \\ 4 & 2 \end{bmatrix}=\begin{bmatrix} -50 & 80 \\ -117 & 204 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} -33 & 36 \\ 16 & 8 \end{bmatrix}=\mathbf{A}(\mathbf{BC})$

[2 marks]

[Total: 6 marks]

12 $2a+3c=1$ and $a+2c=0$ $\therefore$ $2\times-2c+3c=1$ $\Rightarrow$ $c=-1$ $\therefore$ $a=2$

$2b+3d=0$ and $b+2d=1$ $\therefore$ $2(1-2d)+3d=0$ $\Rightarrow$ $2-4d+3d=0$

$\Rightarrow$ $2-d=0$ $\Rightarrow$ $d=2$ $\therefore$ $b=1-2\times2=-3$ [6 marks]

13 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$4a+3b=1$ and $3a+2b=0$ $\Rightarrow$ $b=-\dfrac{3a}{2}$ $\therefore$ $4a+3\times-\dfrac{3a}{2}=1$

$\Rightarrow$ $-\dfrac{1}{2}a=1$ $\Rightarrow$ $a=-2$ $\therefore$ $b=-\dfrac{3\times-2}{2}=3$

$3c+2d=1$ and $4c+3d=0$ $\Rightarrow$ $d=-\dfrac{4c}{3}$ $\therefore$ $3c+2\times-\dfrac{4c}{3}=1$

$\Rightarrow$ $\dfrac{c}{3}=1$ $\Rightarrow$ $c=3$ $\therefore$ $d=-\dfrac{4\times3}{3}=-4$

$\therefore$ $\mathbf{M}=\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ [7 marks]

14 Let $\mathbf{P}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ $a+b=0$ and $c+d=0$

$b=-a$ and $d=-c$ so **P** can be any matrix of the form $\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}$, for example $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ [3 marks]

Exercise 9.2 Transformations and matrices

1 (i) $\begin{bmatrix} 2 & 6 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}=\begin{bmatrix} 36 \\ 22 \end{bmatrix}$ $\therefore$ image $=(36,\ 22)$ [2 marks]

(ii) $\begin{bmatrix} 2 & 6 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix}=\begin{bmatrix} 20 \\ 22 \end{bmatrix}$ $\therefore$ image $=(20,\ 22)$ [2 marks]

[Total: 4 marks]

2 $\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} -4a+2b \\ 3a+b \end{bmatrix}$ $\therefore$ image $=(2b-4a,\ 3a+b)$ [2 marks]

3 (i) $\begin{bmatrix} a & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\therefore$ $a+9=5$ $\Rightarrow$ $a=-4$ [2 marks]

(ii) $\begin{bmatrix} 4 & a \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\therefore$ $4+3a=5$ $\Rightarrow$ $a=\dfrac{1}{3}$ [2 marks]

(iii) $\begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\therefore$ $-1+3a=2$ $\Rightarrow$ $a=1$ [2 marks]

[Total: 6 marks]

4 (i) $\begin{bmatrix} a & b \\ 2a & -b \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\therefore$ $a+2b=3$ and $2a-2b=4$ [2 marks]

(ii) adding: $a+2a=3+4$ $\Rightarrow$ $3a=7$ $\Rightarrow$ $a=\dfrac{7}{3}$

$\therefore$ $\dfrac{7}{3}+2b=3$ $\Rightarrow$ $2b=\dfrac{2}{3}$ $\Rightarrow$ $b=\dfrac{1}{3}$ [3 marks]

[Total: 5 marks]

5 (i) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ -2 & 6 \end{bmatrix}$ [2 marks]

(ii) $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 1 & 2 \end{bmatrix}$ [2 marks]

(iii) $2\mathbf{M} = \begin{bmatrix} 5 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow \mathbf{M} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$ [2 marks]

[Total: 6 marks]

6 $\mathbf{M} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 3 & 6 \end{bmatrix} \Rightarrow \mathbf{M} = \begin{bmatrix} 3 & -1 \\ 3 & 5 \end{bmatrix}$ [2 marks]

7 (i) $3c - 21 = 0 \Rightarrow c = 7$ [2 marks]

(ii) $5d - 21 = -1 \Rightarrow d = 4$

$-15 + 3e = 0 \Rightarrow e = 5$ [3 marks]

[Total: 5 marks]

8 (i) $\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \therefore \; 2x + y = 4 \text{ and } -4x + 3y = 2$

$\therefore \; -4x + 3(4 - 2x) = 2 \Rightarrow -4x + 12 - 6x = 2 \Rightarrow 10 = 10x$

$\Rightarrow x = 1 \quad \therefore \; y = 4 - 2 \times 1 = 2 \quad \therefore \; \text{D} = (1, 2)$ [5 marks]

(ii) $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \therefore \; x + 2y = 4 \text{ and } 4x + 9y = 2$

$\therefore \; 4(4 - 2y) + 9y = 2 \Rightarrow 16 - 8y + 9y = 2 \Rightarrow y = -14$

$\therefore \; x = 4 - 2 \times -14 = 32 \quad \therefore \; \text{D} = (32, -14)$ [5 marks]

[Total: 10 marks]

9 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore$ $(0, 0)$ is transformed to the point $(0, 0)$ [2 marks]

10 (i) $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \therefore \; 2x + 3y = x \text{ and } y = y$

$\therefore x + 3y = 0$ or any equivalent equation [1 mark]

(ii) e.g. $(3, -1)$ or any point which satisfies $x + 3y = 0$ [1 mark]

[Total: 2 marks]

11 $\begin{bmatrix} a & b \\ -b & 2a \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow a - 2b = 1 \text{ and } -b - 4a = -2$

So $b + 4(2b + 1) = 2 \Rightarrow 9b = -2 \Rightarrow b = -\frac{2}{9}$ and so $a = 2\left(-\frac{2}{9}\right) + 1 = \frac{5}{9}$ [5 marks]

Exercise 9.3 Transformations of the unit square

1 (i) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \therefore$ matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ [1 mark]

(ii) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \therefore$ matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ [1 mark]

(iii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ [1 mark]

(iv) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}-1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$ [1 mark]

[Total: 4 marks]

2 **(i)** $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}-1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ [1 mark]

(ii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ [1 mark]

(iii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}0\\-1\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ [1 mark]

[Total: 3 marks]

3 **(i)** $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}7\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}0\\7\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$ [1 mark]

(ii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}\frac{1}{3}\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}0\\\frac{1}{3}\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}\frac{1}{3} & 0\\0 & \frac{1}{3}\end{bmatrix}$ [1 mark]

(iii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}-4\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}0\\-4\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}-4 & 0\\0 & -4\end{bmatrix}$ [1 mark]

[Total: 3 marks]

4 **(i)** $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}3\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}0\\3\end{bmatrix}$ is an enlargement, scale factor 3, centre $(0, 0)$ [2 marks]

(ii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}1\\0\end{bmatrix}$ is a reflection in the line $y = x$ [2 marks]

(iii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}-1\\0\end{bmatrix}$ is a rotation of 90° about $(0, 0)$ [2 marks]

(iv) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}0\\-1\end{bmatrix}$ is a rotation of 180° about $(0, 0)$
or an enlargement, scale factor -1, centre $(0, 0)$ [2 marks]

(v) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}1\\0\end{bmatrix}$ is a rotation of 270° about $(0, 0)$
or a clockwise rotation of 90° about $(0, 0)$ [2 marks]

(vi) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}0\\1\end{bmatrix}$ is a reflection in the y-axis [2 marks]

[Total: 12 marks]

5 **(i)** $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}-1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ [1 mark]

(ii) $\begin{bmatrix}1\\0\end{bmatrix}\rightarrow\begin{bmatrix}0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\begin{bmatrix}1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ [1 mark]

(iii) $\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}-1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$ [1 mark]

(iv) $\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}0\\2\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ [1 mark]

[Total: 4 marks]

6 An enlargement of scale factor 6 increases the area by a scale factor of 6^2
$\therefore$ the new area is $1\times6^2=36$ [1 mark]

7 An enlargement of scale factor a increases the area by a scale factor of a^2
$\therefore$ the new area is $1\times a^2=a^2$ [1 mark]

8 **(i)** $\overrightarrow{OP'}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a\\c\end{bmatrix}$ $\Rightarrow$ $P'=(a,\ c)$

$\overrightarrow{OQ'}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}a+b\\c+d\end{bmatrix}$ $\Rightarrow$ $Q'=(a+b,\ c+d)$

$\overrightarrow{OR'}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}b\\d\end{bmatrix}$ $\Rightarrow$ $R'=(b,\ d)$ [3 marks]

(ii) $\overrightarrow{P'Q'}=\overrightarrow{OQ'}-\overrightarrow{OP'}=\begin{pmatrix}a+b\\c+d\end{pmatrix}-\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}b\\d\end{pmatrix}$ [2 marks]

(iii) $\overrightarrow{OR'}=\overrightarrow{P'Q'}$ $\therefore$ lines OR and PQ are equal in length and parallel

$\therefore$ OP′Q′R′ is a parallelogram [2 marks]

[Total: 7 marks]

9 A rotation of 90° must be repeated four times for a shape to return to its starting point.
So n could be any multiple of 4 [2 marks]

Exercise 9.4 Combining transformations

1 **(i)** $\begin{bmatrix}-2&4\\3&-5\end{bmatrix}\begin{bmatrix}3&1\\2&0\end{bmatrix}=\begin{bmatrix}2&-2\\-1&3\end{bmatrix}$ [2 marks]

(ii) $\begin{bmatrix}2&-2\\-1&3\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}-4\\8\end{bmatrix}$ $\therefore$ the image is $(-4,\ 8)$ [2 marks]

[Total: 4 marks]

2 **(i)** $\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ [1 mark]

(ii) $\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}0\\-1\end{bmatrix}$ $\therefore$ matrix is $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ [1 mark]

(iii) $\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ [2 marks]

(iv) $\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix}$ is a rotation of 270° about $(0,\ 0)$
or a clockwise rotation of 90° about $(0,\ 0)$ [2 marks]

[Total: 6 marks]

3 **(i)** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ [2 marks]

(ii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $\therefore$ the image is $(2, -4)$ [2 marks]

[Total: 4 marks]

4 **(i)** $\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$ [1 mark]

(ii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ [1 mark]

(iii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 7 & 0 \end{bmatrix}$ [2 marks]

(iv) No change to area due to reflection.
The enlargement increases the area by a factor of 7^2
$\therefore$ the new area is $1 \times 7^2 = 49$ [1 mark]

[Total: 5 marks]

5 **(i)** $\begin{bmatrix} 3 & 6 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 39 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -111 & 171 \\ 2 & -18 \end{bmatrix}$ [3 marks]

(ii) $\begin{bmatrix} -111 & 171 \\ 2 & -18 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -393 \\ 22 \end{bmatrix}$ $\therefore$ the image is $(-393, 22)$ [2 marks]

[Total: 5 marks]

6 **(i)** $\begin{bmatrix} -5 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -6 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -10 & 30 \\ 13 & 0 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -10 & -10 \\ 52 & 91 \end{bmatrix}$ [3 marks]

(ii) $\begin{bmatrix} -10 & -10 \\ 52 & 91 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -50 \\ 377 \end{bmatrix}$ $\therefore$ $-10x - 10y = -50$ and $52x + 91y = 377$

$\Rightarrow \quad x + y = 5 \quad \Rightarrow \quad y = 5 - x \quad \therefore \quad 52x + 91(5 - x) = 377$

$\Rightarrow \quad 52x + 455 - 91x = 377 \quad \Rightarrow \quad 78 = 39x \quad \Rightarrow \quad x = 2$

$\therefore \quad y = 5 - 2 = 3 \quad \therefore \quad \mathrm{A} = (2, 3)$ [5 marks]

[Total: 8 marks]

7 Reflection in the line $y = -x$ followed by reflection in the x-axis

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} =$ rotation of 90° about the origin [2 marks]

8 Rotation of 180° about the origin $= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ [1 mark]

9 Enlargement scale factor $\frac{1}{k} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$ [1 mark]

10 The repeat of a reflection returns a point to its original position $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ [1 mark]

11 The rotation must be 90° (either anticlockwise or clockwise) or 180°

So $\mathrm{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ or $\mathrm{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ [2 marks]